Ergodic Capacity Analysis for Ubiquitous Cooperative Networks Employing Amplify-and-Forward Relaying

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***Abstract*—This paper studies the ergodic capacity for ubiquitous cooperative networks employing amplify-and-forward relaying in Rayleigh fading. A general *asymmetric* channel model is considered, in which the average signal-to-noise ratios associated with different wireless channels are generally unequal. We derive an *exact* expression of the ergodic capacity in a single-integral-form, which serves as a benchmark for ubiquitous cooperative networks. To the best of our knowledge, this exact expression has not been reported in the literature. To evaluate this integral more efficiently, we develop a hybrid Gaussian quadrature expression in *closed-form*, which has a high relative accuracy. Finally, it is demonstrated that the obtained analytical results overlap the simulation curves, while the existing bounds are loose in various scenarios.**

***Keywords*–Amplify-and-forward; ergodic capacity; Rayleigh fading; relaying.**

I. INTRODUCTION

Cooperative networks for ubiquitous communications have gained considerable attention in the last decade, due to their great potential to combat fading impairments [1]–[4]. The main idea of cooperative communications is that several geographically distributed wireless terminals, including the source and relay(s), collaborate with one another to form a virtual multi-antenna array, which enables *distributed* spatial diversity. Amplify-and-Forward (AF) is one of the most popular relaying protocols, in which the relay simply amplifies its received signal and forwards it to the destination [4]–[6].

A. Ergodic Capacity

The fading environment manifests itself in *ergodic* fading when the channel coherence time is (much) shorter than the codeword length, due to the usage of sufficiently long codewords and/or high mobility of wireless terminals [7]. In an ergodic fading scenario, each codeword typically spans *many* coherence time intervals, giving rise to rapid fading fluctuations *within* each codeword. The codeword is long enough to average out the randomness of fading, rendering Shannon capacity a deterministic constant independent of the instantaneous fading state. Mathematically, Shannon capacity of ergodic channels, a.k.a., ergodic capacity, is equal to the maximum achievable rate averaged over the fading distribution, which depends only on the fading statistics [7], [8]. Ergodic capacity is a fundamental information-theoretic performance measure, which captures the maximum rate of reliable communications under ergodic fading [7].

B. Related Work

It is of paramount importance to characterize the ergodic capacity, which serves as a benchmark for practical communication systems. However, due to the nonlinear expression of the end-to-end Signal-to-Noise Ratio (SNR) in AF relaying, only few works have studied the *exact* ergodic capacity for AF networks [9][10]. Specifically, the exact analysis of the ergodic capacity was limited *only* to a *symmetric* single-relay network *without* the the direct source-destination link [9, eq. (11)]. The major limitation of the *symmetric* channel setting is that the average received SNRs of the first- and the second-hop channels must be identical in Rayleigh fading. However, this does not necessarily hold in practice due to large-scale path-loss and shadowing effects. Furthermore, Fan *et al.* derived an *exact* expression of the ergodic capacity for a *multi*-relay network without the direct link where only a *single*-relay with the maximum second-hop SNR is allowed to assist the communication. Since the selection of the relay in [10][11] neglects the first-hop channel conditions, the achievable diversity order is *always* equal to one in Rayleigh fading, irrespective of the number of relays. That is, from a diversity point of view, the relaying scheme in [10][11] achieves the same performance as in a single-relay network with no direct link, which makes the exact analysis by Fan *et al.* [10] valid only for a single-relay network without the direct link.

In addition to the exact analysis, other works have been devoted to finding bounds or approximations of the ergodic capacity. Specifically, various upper bounds were obtained using the geometric-mean [12, eq. (8)], harmonic-mean [13, eq. (18)], and Jensen's inequality [12, eq. (5)], [14, eq. (10)]. In addition, series expansions of the ergodic capacity were developed in [9, eq. (4)], [12, eq. (17)], and [15, eq. (9)]. These bounds/approximations, unfortunately, suffer from low accuracy and/or high computational complexity. To the best of our knowledge, an *exact* expression of the ergodic capacity has not been reported for a general *asymmetric* AF cooperative network *with* the direct link, which motivates our work.

C. Contribution

To the best of our knowledge, the *exact* analytical expression of the ergodic capacity for a single-relay network *with* the direct link has not been reported in the literature, which motivates this work. In this paper, we carry out *exact* analysis under the general *asymmetric* channel setting where different channels have generally unequal average SNRs. The exact

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expression of the ergodic capacity is derived in a single integral form for the single-relay network with the direct link. To numerically evaluate this expression, we further develop a hybrid Gaussian quadrature rule in *closed-form*, which achieves a high *relative* accuracy of 10^{-6}, and thus, is very suitable for real-time evaluations. These obtained expressions serve as a useful tool to analytically evaluate the performance limits of piratical AF relaying systems.

The remainder of the paper is organized as follows. Section II introduces the system model. Section III conducts the ergodic capacity analysis. Section IV validates the capacity analysis by simulations. Finally, Section V concludes the paper.

Notation: We use $x \triangleq y$ to denote that x, by definition, equals y. $\mathbb{E}(\cdot)$, $\ln(\cdot)$, and $\log_2(\cdot)$ denote the expectation, natural logarithm, and base-2 logarithm, respectively. For a random variable X, $f_X(\cdot)$ and $F_X(\cdot)$ are the probability density function (PDF) and cumulative distribution function (CDF), respectively. Finally, $X \sim \mathcal{CN}(\mu, \sigma^2)$ means that X is a circularly symmetric complex Gaussian random variable with mean μ and variance σ^2.

II. System Model

Consider a cooperative network composed of three single-antenna terminals: a source S, a relay R, and a destination D, where the direct S-D link exists. Let h_{sd}, h_{sr}, and h_{rd} denote flat fading gains for the S-D, S-R, and R-D links, respectively. The fading gains are assumed independent and modeled as $h_{ij} \sim \mathcal{CN}(0, \Omega_{ij})$, $ij \in \{sd, sr, rd\}$. The additive white Gaussian noises (AWGNs) associated with h_{ij}'s follow the distribution of $\mathcal{CN}(0, \sigma_{ij}^2)$, $ij \in \{sd, sr, rd\}$. Let E_s and E_r denote the transmit powers at S and R, respectively. The *instantaneous* SNRs of the S-D, S-R, and R-D links are respectively denoted $\gamma_{sd} \triangleq \frac{E_s|h_{sd}|^2}{\sigma_{sd}^2}$, $\gamma_{sr} \triangleq \frac{E_s|h_{sr}|^2}{\sigma_{sr}^2}$, and $\gamma_{rd} \triangleq \frac{E_r|h_{rd}|^2}{\sigma_{rd}^2}$, and the corresponding *average* SNRs are $\bar{\gamma}_{sd} \triangleq \frac{E_s\Omega_{sd}}{\sigma_{sd}^2}$, $\bar{\gamma}_{sr} \triangleq \frac{E_s\Omega_{sr}}{\sigma_{sr}^2}$, and $\bar{\gamma}_{rd} \triangleq \frac{E_r\Omega_{rd}}{\sigma_{rd}^2}$. We focus on a general *asymmetric* channel setting where $\bar{\gamma}_{sd}$, $\bar{\gamma}_{sr}$, and $\bar{\gamma}_{rd}$ are *unequal* in general, which makes our analysis practical.

III. Exact Analysis of Ergodic Capacity

We consider the CSI-assisted orthogonal AF relaying with a maximum ratio combiner at the destination, yielding the end-to-end received SNR at the destination as follows [1], [2], [4]

$$\gamma_{\text{AF}} \triangleq \gamma_{sd} + \frac{\gamma_{sr}\gamma_{rd}}{\gamma_{sr} + \gamma_{rd} + 1}. \tag{1}$$

The ergodic capacity is thus given by

$$C_{\text{AF}} \triangleq \frac{1}{2}\mathbb{E}\big\{\log_2(1 + \gamma_{\text{AF}})\big\}, \tag{2}$$

where the pre-log factor $1/2$ accounts for the orthogonal relaying.

Theorem 1: The ergodic capacity for a single-relay AF network with the direct link is given by

$$C_{\text{AF}} = \frac{1}{2\ln(2)}\left\{e^{\frac{1}{\bar{\gamma}_{sd}}}E_1\left(\frac{1}{\bar{\gamma}_{sd}}\right) + \frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}}I\right\}, \tag{3}$$

where

$$I \triangleq \int_0^\infty \phi(x)dx. \tag{4}$$

The integrand of (4), $\phi(x)$, is defined as follows:

$$\begin{aligned}\phi(x) \triangleq& \exp\left(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}}x + \frac{1}{\bar{\gamma}_{sd}}\right)E_1\left(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}}x + \frac{1}{\bar{\gamma}_{sd}}\right)\\ &\times\sqrt{x\left(x + \frac{2}{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\right)}K_1\left(\sqrt{x\left(x + \frac{2}{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\right)}\right)\\ &\times\exp\left(-\frac{\bar{\gamma}_{sr} + \bar{\gamma}_{rd}}{2\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}x\right),\end{aligned} \tag{5}$$

where $E_1(\cdot)$ and $K_1(\cdot)$ denote the exponential integral function [16, eq. (15.1.1)] and the first-order modified Bessel function of the second kind [16, eq. (9.6.11)], respectively.

Proof: See Appendix A. ■

At the first glance, the exact expression of ergodic capacity in (3) involving the single-integral of (4) is complicated. In fact, the integrand of (5), $\phi(x)$, is a well-behaved function for $x > 0$. First of all, $\exp\big(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}}x + \frac{1}{\bar{\gamma}_{sd}}\big)E_1\big(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}}x + \frac{1}{\bar{\gamma}_{sd}}\big)$ and $\sqrt{x(x + 2/\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}})}K_1\big(\sqrt{x(x + 2/\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}})}\big)$ in $\phi(x)$ are both monotonically decreasing for $x > 0$. This is easily justified as follows. For $x > 0$, we have $\big[e^xE_1(x)\big]' = \frac{xe^xE_1(x)-1}{x} < \frac{1}{x}\big(\frac{x+1}{x+2} - 1\big) < 0$ [17, eq. (6.8.2)] and $[xK_1(x)]' = -xK_0(x) < 0$ [18, eq. (8.486.14)]. Thus, $e^xE_1(x)$ and $xK_1(x)$ are monotonically decreasing functions for $x > 0$. Furthermore, $\exp\big(-\frac{\bar{\gamma}_{sr}+\bar{\gamma}_{rd}}{2\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}x\big)$ in $\phi(x)$ decays exponentially fast for $x > 0$. Therefore, the multiplication of these terms, namely $\phi(x)$ in (5), is a monotonically decreasing and exponentially decaying function of x for $x > 0$. Furthermore, $\phi(x)$ is a smooth function of x, which has derivatives of any order. These properties enables efficient numerical calculation of (3), detailed as follows.

It is well-known that the Gaussian-Laguerre quadrature is extremely accurate for large $x > 0$ with integrand of the form $e^{-x}g(x)$ for some function $g(\cdot)$, and the composite Gaussian-Legendre quadrature is very effective for finite integrals [19, Ch. 3]. To exploit the benefits of both quadrature rules, we rewrite the integral of (3) into two sub-integrals

$$I = \int_0^\tau \phi(x)dx + \int_\tau^\infty \phi(x)dx, \tag{6}$$

where the first (finite) integral over $[0, \tau]$, $\tau > 0$, is readily computed using the composite Gaussian-Legendre quadrature [19, eq. (3.3.1)], and the second (semi-infinite) integral over $[\tau, \infty)$ can be accurately evaluated by the Gaussian-Laguerre quadrature [16, eq. (25.4.45)]. As a result, we propose a *hybrid Gaussian quadrature* to compute the integral (3) as follows:

$$\begin{aligned}I =& \frac{\tau}{2M}\sum_{j=1}^{M}\sum_{i=1}^{N_1}w_{1,i}\phi\Big[\frac{\tau}{2M}(t_{1,i} + 2j - 1)\Big]\\ &+ \sum_{i=1}^{N_2}w_{2,i}e^{t_{2,i}}\phi(t_{2,i} + \tau) + \mathcal{R},\end{aligned} \tag{7}$$

where $\tau > 0$ is the integral limit chosen for Gaussian-Legendre quadrature, M represents the number of subintervals considered in the composite Gaussian-Legendre quadrature [19],

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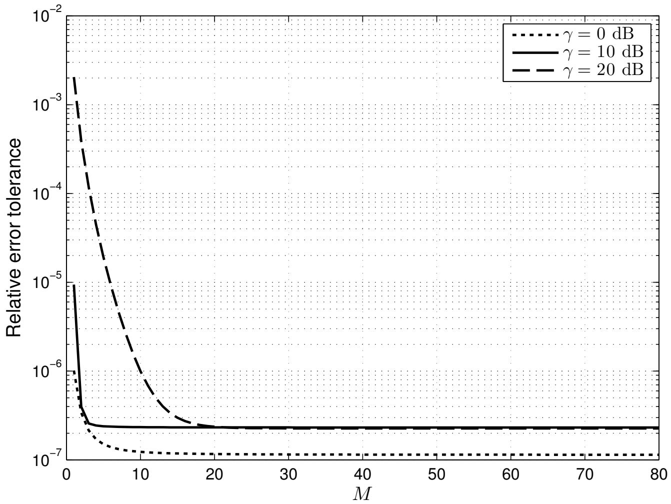

Figure 1. Relative error tolerance, $|\mathcal{R}/I|$, versus M for $d_r = 0.5$ and $\gamma = 0, 10, 20$ dB, where $\tau = 1$, $N_1 = N_2 = 15$.

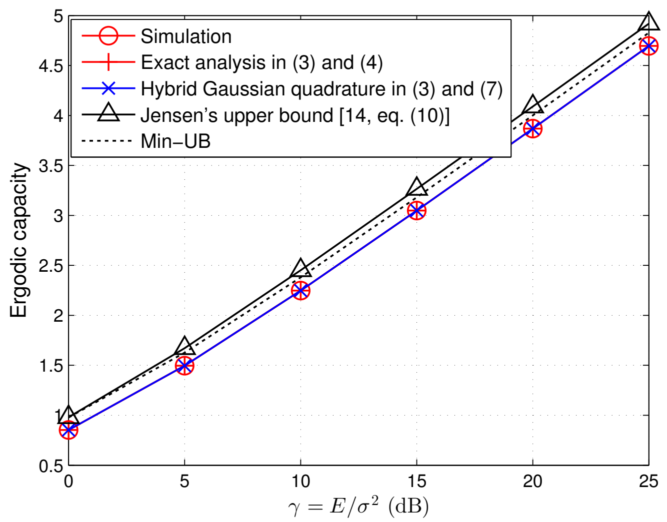

Figure 2. Ergodic capacity of single-relay AF network with the direct link under asymmetric channel setting where $d_r = 0.6$.

and $\mathcal{R}$ is the remainder. Also, $w_{1,i}$ and $t_{1,i}$, $i = 1, \cdots, N_1$, are, respectively, the weights and zeros of the Legendre polynomial of order N_1 [16, Table 25.4], and $w_{2,i}$ and $t_{2,i}$, $i = 1, \cdots, N_2$, are, respectively, the weights and zeros of the Laguerre polynomial of order N_2 [16, Table 25.9]. It is obvious that the accuracy of (7) is dependent on τ, M, N_1, and N_2. In general, any specific accuracy can be achieved by *simultaneously* increasing M, N_1, and N_2 [19], for any τ. In practice, however, using small M, N_1, and N_2 to achieve the target accuracy for a specific τ is desirable. The choice of parameters are discussed in detail in the next section.

IV. Numerical Results

Consider a line S-R-D model in which R is located in the straight line between S and D. Let d_r denote the distance between S and R. The path loss model for typical urban environments is adopted, i.e., $\Omega_{sd} = 1$, $\Omega_{sr} = d_r^{-4}$, and

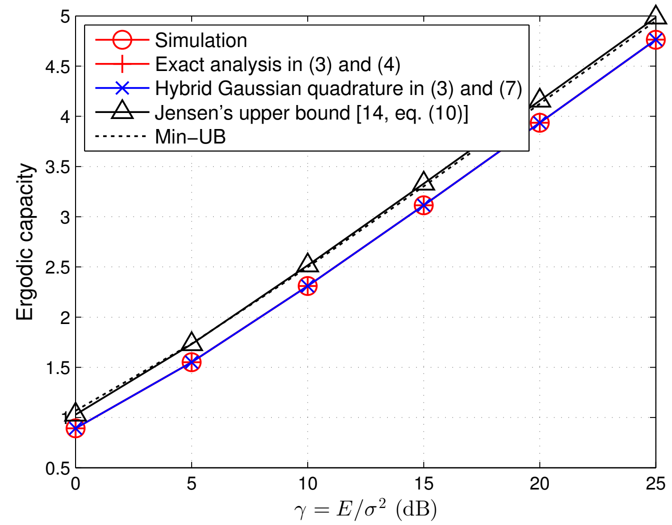

Figure 3. Ergodic capacity of single-relay AF network with the direct link under symmetric channel setting where $d_r = 0.5$.

$\Omega_{rd} = (1 - d_r)^{-4}$ [8]. We set $E_s = E_r = \frac{E}{2}$, where E is the total power consumed in the whole network. The variances of AWGNs are set to be identical, i.e., $\sigma_{ij}^2 = \sigma^2$ for $ij \in \{sd, sk, kd\}_{k=1}^K$. By varying the source-relay distance d_r, the link SNRs $\bar{\gamma}_{sd}$, $\bar{\gamma}_{sr}$, and $\bar{\gamma}_{rd}$ are unequal in general, constituting a general *asymmetric* network. The ratio of the total transmission power E to the AWGN variance, $\gamma \triangleq E/\sigma^2$, is referred to as the network SNR.

A. Choice of Parameters in (7)

The parameters τ, M, N_1, and N_2 in (7) need to be chosen appropriately to achieve a desirable accuracy. In practice, τ is chosen such that $\phi(x) < 0.1$ for $x \geq \tau$, and thus, the approximation error of the (second) integral over $[\tau, \infty)$ in (6) can be made negligible by using a reasonably small value of N_2, e.g., $N_2 = 15$ [19]. Since $\phi(x) < 0.0249$ for all $x \geq 1$, we choose $\tau = 1$ and $N_2 = 15$. Then, the overall accuracy of (7) mainly depends on the first (finite) integral over $[0, \tau]$ in (6). It is possible to increase the accuracy of this integral by simply increasing M for any fixed value of N_1. Thus, by setting $\tau = 1$ and $N_1 = N_2 = 15$, the overall accuracy of (7) is solely dependent on the parameter M. The *relative* error tolerance, i.e., $|\mathcal{R}/I|$, versus M is illustrated in Fig. 1 for $\gamma = 0, 10, 20$ dB, where $d_r = 0.5$. As a benchmark, we compute the integral I of (4) using MATLAB *quadgk* function at a relative error tolerance of 10^{-10}, which is considered as the "exact" value. The difference between this exact value and that computed using (7) is the *absolute* error tolerance $\mathcal{R}$. It is clearly seen that for $\tau = 1$, $N_1 = N_2 = 15$, and $M = 20$, the *relative* error tolerance, $|\mathcal{R}/I|$, is smaller than 10^{-6}. Therefore, we suggest that $\tau = 1$, $N_1 = N_2 = 15$, and $M = 20$ constitutes a suitable choice in (7) to yield an accurate and efficient approximation for (4).

B. Ergodic Capacity Evaluation

For a single-relay AF network *with* the direct link, the following results are compared: i) proposed "exact" result

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$$A = \frac{2}{\bar{\gamma}_{sd}} e^{\frac{1}{\bar{\gamma}_{sd}}} \Bigg\{ \int_0^\infty E_1\Big(\frac{1+z}{\bar{\gamma}_{sd}}\Big) e^{(\frac{1}{\bar{\gamma}_{sd}} - \frac{1}{\bar{\gamma}_{sr}} - \frac{1}{\bar{\gamma}_{rd}})z} \sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg) dz - \Bigg[\underbrace{E_1\Big(\frac{1+z}{\bar{\gamma}_{sd}}\Big) \int_0^z e^{(\frac{1}{\bar{\gamma}_{sd}} - \frac{1}{\bar{\gamma}_{sr}} - \frac{1}{\bar{\gamma}_{rd}})x} \sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg) dx}_{\triangleq B(z)} \Bigg]_{z=0}^{\infty} \Bigg\} \tag{A.4}$$

$$= \frac{2}{\bar{\gamma}_{sd}} e^{\frac{1}{\bar{\gamma}_{sd}}} \int_0^\infty E_1\Big(\frac{1+z}{\bar{\gamma}_{sd}}\Big) e^{(\frac{1}{\bar{\gamma}_{sd}} - \frac{1}{\bar{\gamma}_{sr}} - \frac{1}{\bar{\gamma}_{rd}})z} \sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg) dz. \tag{A.5}$$

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by (3) and (4), where (4) is computed using *quadgk* at a relative error tolerance of 10^{-10}; ii) proposed hybrid Gaussian quadrature using (7) in (3), where $\tau = 1$, $N_1 = N_2 = 15$, and $M = 20$; iii) Jensen's upper bound [14, eq. (10)]; and iv) "Min-UB" which refers to the widely-used upper bound using the minimum of the two-hop SNRs, i.e., $\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{sr}+\gamma_{rd}+1} \leq \min(\gamma_{sr}, \gamma_{rd})$. The comparison is performed for an *asymmetric* network with $d_r = 0.6$ in Fig. 2 and a symmetric network with $d_r = 0.5$ in Fig. 3. We observe that the ergodic capacity computed using the hybrid Gaussian quadrature in (7) is in excellent agreement with the exact value of (3). This validates once again the accuracy and efficiency of the proposed hybrid Gaussian quadrature. Furthermore, it is clearly seen that Jensen's upper bound and the Min-UB are loose upper bounds, throughout the whole SNR range. Indeed, the loose bounds/approximations highlight the usefulness of the obtained exact expression.

V. Conclusion

We analyzed the ergodic capacity of AF relaying for ubiquitous cooperative networks in Rayleigh fading where the average SNRs of different wireless channels are unequal in general. For the single-relay case with the direct link, we derived the ergodic capacity in an exact single-integral-form, which serves as a benchmark for AF relaying systems. To facilitate evaluation of this exact expression, we derived a hybrid Gaussian quadrature rule in closed-form, which is extremely accurate and easy to compute.

An interesting extension of this work is to study the *exact* ergodic capacity for general *multi*-relay AF networks, which will be addressed in our further work.

Appendix A
Proof of Theorem 1

Let $\gamma_r \triangleq \frac{\gamma_{sr}\gamma_{rd}}{\gamma_{sr}+\gamma_{rd}+1}$, and thus, $\gamma_{\text{AF}} = \gamma_{sd} + \gamma_r$. It follows from [20, eq. (11)] that the CDF of γ_r is

$$F_{\gamma_r}(z) = 1 - 2e^{-\left(\frac{1}{\bar{\gamma}_{sr}} + \frac{1}{\bar{\gamma}_{rd}}\right)z} \sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{z(z+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg),$$

for $z > 0$. For independent Rayleigh fading channels, γ_{sd}, γ_{sr}, and γ_{rd} are *independent* exponential random variables with means $\bar{\gamma}_{sd}$, $\bar{\gamma}_{sr}$, and $\bar{\gamma}_{rd}$, respectively, implying that γ_{sd} and γ_r are also independent. Thus, the CDF of γ_{AF}, $F_{\gamma_{\text{AF}}}(z) = \int_0^z F_{\gamma_r}(x) f_{\gamma_{sd}}(z-x)dx$, is evaluated to

$$F_{\gamma_{\text{AF}}}(z) = 1 - e^{-\frac{z}{\bar{\gamma}_{sd}}} - \frac{2}{\bar{\gamma}_{sd}} e^{-\frac{z}{\bar{\gamma}_{sd}}} \int_0^z e^{\left(\frac{1}{\bar{\gamma}_{sd}} - \frac{1}{\bar{\gamma}_{sr}} - \frac{1}{\bar{\gamma}_{rd}}\right)x} \times \sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg) dx. \tag{A.1}$$

Since $I_{\text{AF}} = \frac{1}{2}\log_2(1+\gamma_{\text{AF}}) > 0$, we have

$$\mathbb{E}\{I_{\text{AF}}\} = \int_0^\infty \big[1 - F_{I_{\text{AF}}}(z)\big] dz = \frac{1}{2\ln(2)} \int_0^\infty \frac{1 - F_{\gamma_{\text{AF}}}(z)}{1+z} dz, \tag{A.2}$$

where (A.2) follows from $F_{I_{\text{AF}}}(z) = F_{\gamma_{\text{AF}}}(2^{2z}-1)$ and the change of variable $2^{2z} - 1 \to z$. Substituting (A.1) into (A.2) and using [18, eq. (3.352.4)], we obtain

$$\mathbb{E}\{I_{\text{AF}}\} = \frac{1}{2\ln(2)} \Big\{ A + e^{\frac{1}{\bar{\gamma}_{sd}}} E_1\Big(\frac{1}{\bar{\gamma}_{sd}}\Big) \Big\}, \tag{A.3}$$

where $A \triangleq \frac{2}{\bar{\gamma}_{sd}} \int_0^\infty \frac{e^{-\frac{z}{\bar{\gamma}_{sd}}}}{1+z} \int_0^z e^{\left(\frac{1}{\bar{\gamma}_{sd}} - \frac{1}{\bar{\gamma}_{sr}} - \frac{1}{\bar{\gamma}_{rd}}\right)x} K_1\big(2\sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\big) \sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} dx dz$. Using [16, eq. (5.1.24)] and [16, eq. (5.1.26)], we have $\big[E_1\big(\frac{1+z}{\bar{\gamma}_{sd}}\big)\big]' = -\frac{1}{1+z}\exp\big(-\frac{1+z}{\bar{\gamma}_{sd}}\big)$. Thus, A is evaluated in (A.4) and (A.5) at the top of this page, where (A.5) follows by the following property

$$\lim_{z \to z_0} B(z) = 0, \text{ for } z_0 = 0, \infty. \tag{A.6}$$

This equality holds for $z_0 = 0$ because $\lim_{z\to 0} E_1\big(\frac{1+z}{\bar{\gamma}_{sd}}\big) = E_1\big(\frac{1}{\bar{\gamma}_{sd}}\big) < \infty$. To prove the equality of (A.6) for $z_0 = \infty$, we first obtain a trivial lower bound $B_l(z) = 0$ and an upper bound $B_u(z)$ for $B(z)$ as follows:

$$B(z) \leq e^{\frac{z}{\bar{\gamma}_{sd}}} E_1\Big(\frac{1+z}{\bar{\gamma}_{sd}}\Big) \int_0^z e^{-(\frac{1}{\bar{\gamma}_{sr}} + \frac{1}{\bar{\gamma}_{rd}})x} \times \sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} K_1\Bigg(2\sqrt{\frac{x(x+1)}{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\Bigg) dx$$

$$= \frac{1}{2} e^{-\frac{1}{\bar{\gamma}_{sd}}} \mathbb{E}\{\gamma_r\} \lim_{z\to\infty} \Big\{ e^{\frac{1+z}{\bar{\gamma}_{sd}}} E_1\Big(\frac{1+z}{\bar{\gamma}_{sd}}\Big) \Big\} \tag{A.7}$$

$$\triangleq B_u(z),$$

where (A.7) follows by the fact that $\mathbb{E}\{\gamma_r\} = \int_0^\infty [1 - F_{\gamma_r}(x)]dx$. Since $\frac{1}{2}\ln(1+\frac{2}{x}) < e^x E_1(x) < \ln(1+\frac{1}{x})$ [16, eq. (5.1.20)], $\lim_{x\to\infty}\{\frac{1}{2}\ln(1+\frac{2}{x})\} = 0$, and

$\lim_{x\to\infty}\{\ln(1+\frac{1}{x})\} = 0$, using Squeeze Theorem [21], we have $\lim_{x\to\infty}\{e^x E_1(x)\} = 0$. This implies that $\lim_{z\to\infty}\left\{e^{\frac{1+z}{\bar{\gamma}_{sd}}} E_1\left(\frac{1+z}{\bar{\gamma}_{sd}}\right)\right\} = 0$. Since $0 < \gamma_r < \min(\gamma_{sr}, \gamma_{rd})$, we have $0 < \mathbb{E}\{\gamma_r\} < \mathbb{E}\{\min(\gamma_{sr}, \gamma_{rd})\} < \min(\bar{\gamma}_{sr}, \bar{\gamma}_{rd}) < \infty$. By definition of (A.7), we have $\lim_{z\to\infty} B_u(z) = 0$. Thus, for $z \geq 0$ we have $B_l(z) \leq B(z) \leq B_u(z)$, where $\lim_{z\to\infty} B_l(z) = \lim_{z\to\infty} B_u(z) = 0$. Using the Squeeze Theorem [21], we have $\lim_{z\to\infty} B(z) = 0$, which consequently proves (A.5). Applying change of variable $\frac{2z}{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} \to x$ in (A.5), we have

$$A = \frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}} \int_0^\infty \exp\left(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}} x + \frac{1}{\bar{\gamma}_{sd}}\right) \times E_1\left(\frac{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}{2\bar{\gamma}_{sd}} x + \frac{1}{\bar{\gamma}_{sd}}\right) \exp\left(-\frac{\bar{\gamma}_{sr}+\bar{\gamma}_{rd}}{2\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}} x\right) \times \sqrt{x\left(x + \frac{2}{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\right)} K_1\left(\sqrt{x\left(x + \frac{2}{\sqrt{\bar{\gamma}_{sr}\bar{\gamma}_{rd}}}\right)}\right) dx.$$

Finally, substituting the above A into (A.3) yields (3).

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