The Sampling Theorem for Finite Duration Signals

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Abstract

The Shannon's sampling theorem was derived using the assumption that the signals must exist over infinite time interval. But all of our applications are based on finite time intervals. The objective of this research is to correct this inconsistency. In this paper we show where and how this infinite time assumption was used in the derivation of the original sampling theorem and then we extend the results to finite time case. Our research shows that higher sample rate is necessary to recover finite duration signals. This paper validates, with detailed theory, the common industrial practice of higher sample rate. We use the infinite dimensionality property of function space as the basis of our theories. A graphical example illustrates the problem and the solution.

Keywords: Sampling methods, Communication, Linear system, Wavelet transform, Modulation.

1. Objective

This paper is an extended version of [1]. It provides more details of the theories and presents many related ideas including the re-sampling process. The objective of this paper is to extend the original sampling theorem [2] to finite duration signals. It is shown here that the proof of the Shannon's sampling theorem assumed that the signal must exist for infinite time. This assumption came because the proof used Fourier transform theory which in turn uses infinite time. We give a new proof that does not require infinite time assumption and as a result of elimination of this assumption we get a new theory.

Our research shows that more you sample more information you get about the signal when your signal measurement window is finite. We provide some theoretical analysis to justify our results. A very fundamental and well known concept in mathematics, infinite dimensionality of function space, is used as a basis of our research. Thus the main focus of the paper is on sampling theorem and on the number of samples. Since the result establishes a new view in signal processing, we apply the result to few other areas like signal reconstruction and up-down sampling.

In engineering practice most of the applications use two to four times the Nyquist sample rate. In audio engineering much higher rate is used [3]. So the results of this paper are not new ideas in the practical world. However, this engineering practice also points out that there is something wrong somewhere in our theory. There is also this (mis)conception that higher sample rate provides redundant information. Therefore we examine the core issues and assumptions behind the original theory of [2], make some changes, and provide a theoretical proof of the high sample rate concept. It should be noted that the theory in [2] is not wrong, we are only changing one of the assumptions that is more meaningful in the present technology.

Besides sampling theorem, another objective is to highlight the infinite time assumption behind the existing theories. This infinite time assumption is not practical in engineering. Thus we emphasize the infeasibility of the approaches based on transfer function and Fourier transform. All of them use infinite time assumption. In the past many engineers have rejected these approaches because they are useful for only Linear Time Invariant (LTI) systems. Now we have another reason – the infinite time assumption. Interestingly enough, we show that LTI systems do not exist in engineering.

This research leads us to realize that the concept of finite time duration of signals is the backbone of all our engineering systems. Therefore we need to do something about it, i.e., we should start a research in reducing these inconsistencies between the theory and the practice. Eventually, if we can successfully provide a new direction, then our technology will be more predictable and reliable. We may get significant product quality improvements. It may also be possible to reduce waste and thus help to create a greener technology [4].

In this paper our objective is not really to make a big jump in this new research on finite time direction but occasionally we have touched upon the various related topics, problems, and solutions. We believe that this is an important area of investigation even in mathematics. It should be noted though that all time domain approaches are closer to finite time reality. However unless we create or change some basic engineering definitions all our theories will remain somewhat inconsistent and unsatisfactory.

During the publication process of this research many colleagues and reviewers have made many comments and questions on the subject of this paper. We have tried to include our answers to many of them. As a result, the paper got little bit defocused from its original goal and the contents got diluted. We hope the integration of all these subjects still maintains some coherency and novelty.

The contents of this paper can be described using the following high level summary. We first show, in Section 2, that infinite time assumption is not really needed in engineering. Then we present a new modulation method, in Section 3, and show how we encountered this infinite time issue in a practical engineering problem. To solve the problem over finite time and to provide its theoretical foundation we discuss in details the concept of infinite dimensionality of function space in Section 4. Using this infinite dimensionality concept, in Section 5, we show that finite rate sample representations actually converge to the original function as rate increases to infinity. In Section 6, we provide new proofs of the original sampling theorem and provide a numerical example in Section 7. We also discuss briefly using a numerical example, in Section 8, how approaches based on analytical expressions rather than samples can help to resample a finite duration signal. Finally, in Sections 9 and 10, we discuss the nonlinear nature of engineering systems and explain why time domain approach with high sample rates is more meaningful.

2. Infinite Time

In this section we show that the assumption of infinite time duration for signals is not practical and is not necessary for our theories. In real life and in all our engineering systems we use signals of finite time durations only. Intuitively this finite duration concept may not be quite obvious though. Ordinarily we know that all our engineering systems run continuously for days, months, and years. Traffic light signaling systems, GPS satellite transmitters, long distance air flights etc. are some common examples of systems of infinite time durations. Then why do we talk about finite duration signals? The confusions will be cleared when we think little bit and examine the internal design principles, the architecture of our technology, and the theory behind our algorithms. Originally we never thought that this question will be asked, but it was, and therefore we look here, at the implementations, for an explanation.

The computer based embedded engineering applications run under basically two kinds of operating systems (OS). One of these OS uses periodic approaches. In these systems the OS has only one interrupt that is produced at a fixed rate by a timer counter. Here the same application runs periodically, at the rate of this interrupt, and executes a fixed algorithm over and over again on input signals of fixed and finite time duration. As an example, in digital communication engineering, these signals are usually the symbols of same fixed duration representing the digital data and the algorithm is the bit recovery process. Every time a symbol comes, the algorithm recovers the bits from the symbol and then goes back to process the next arriving symbol.

Many core devices of an airplane, carrying passengers, are called flight critical systems. Similarly there are life critical systems, like pacemaker implanted inside human body. It is a very strict requirement that all flight critical and life critical systems have only one interrupt. This requirement is mainly used to keep the software simple and very deterministic. They all, as explained before, repeat the same periodic process of finite duration, but run practically for infinite time.

The other kind of applications is based on the real time multi-tasking operating systems (RTOS). This OS is required for systems with more than one interrupts which normally appear at asynchronous and non-periodic rate. When you have more than one interrupts, you need to decide which one to process first. This leads to the concept of priority or assignment of some kind of importance to each interrupt and an algorithm to select them. The software that does this work is nothing but the RTOS. Thus RTOS is essentially an efficient interrupt handling algorithm.

These RTOS based embedded applications are designed as a finite state machine. We are not going to present a theory of RTOS here. So to avoid confusions we do not try to distinguish among threads, tasks, processes, and states etc. We refer to all of these concepts as tasks, that is, we ignore all details below the level of tasks, in this paper. These tasks are executed according to the arrival of interrupts and the design of the application software. The total application algorithm is still fixed and finite but the work load is distributed among these finite numbers of tasks. The execution time of each task is finite also. These tasks process the incoming signals of finite time and produce the required output of finite size.

An example will illustrate it better. A digital communication receiver can be designed to have many tasks – signal processing task, bit recovery task, error correcting task etc. They can be interconnected by data buffers, operating system calls, and application functions. All these tasks together, implement a finite state machine, execute a finite duration algorithm, and process a finite size data buffer. These data buffers are originated from the samples of the finite duration signals representing the symbols.

We should point out that there are systems which are combinations or variants of these two basic concepts. Most commercial RTOS provide many or all of these capabilities. Thus although all of the engineering systems run continuously for all time, all of them are run under the above two basic OS environment. Or in other words for all practical engineering designs the signal availability windows, the measurement windows, and the processing windows are all of finite time. For more details of real time embedded system design principles see many standard text books, for example [5, pp73-88].

The signals may exist theoretically or mathematically for infinite time but in this paper none of our theories, derivations, and assumptions will use that infinite time interval assumption.

In the next section we describe the concept of a new digital communication scheme [6][7] to demonstrate the need for high sample rate. This scheme will also give the details of how finite time analysis can be used in our engineering systems.

3. Motivation

Almost all existing communication systems use sinusoidal functions as symbols for carrying digital data. But a sine function has only three parameters, amplitude, frequency, and phase. Therefore you can only transmit at most three parameters per symbol interval. That is a very inefficient use of symbol time. If instead we use general purpose functions then we can carry very large amount of information, thus significantly increasing the information content per symbol time. However, as we show below, these general purpose functions will require a large number of samples over its symbol time, and hence a high sample rate, to represent them precisely. We present a new digital communication system, called function modulation (fm) [6], to introduce the application of non-sinusoidal functions and the need for a new sampling theorem.

Figure 1 shows an fm transmitter. The left hand side (LHS) vertical box shows four bits, as example, that will be transmitted using one symbol, s(t), shown in the right hand side (RHS) graph. Each bit location in the LHS box is represented by a graph or a general function. These functions, called bit functions, are combined by an algorithm to produce the RHS graph or function. A very simple example of the algorithm may be to add all the bit functions for which the bit values are ones and ignore whose bit values are zeroes. We call this algorithm a 0-1 addition algorithm. Since the bits in the LHS vertical box are continuously changing after every symbol time, the symbol s(t), $t \in [0,T]$, is also continuously changing.

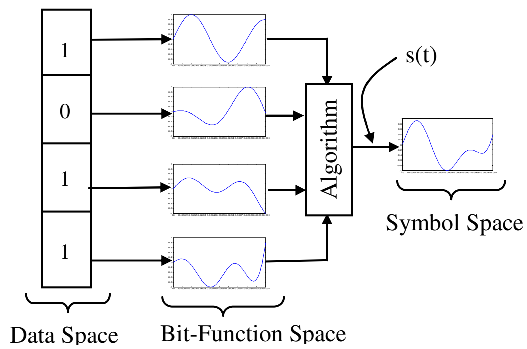


Figure 1. fm Transmitter

For this 0-1 addition algorithm we can write:

$$s(t) = d_1 g_1(t) + d_2 g_2(t) + \ldots + d_M g_M(t) \qquad (1)$$

where $d_i \in \{0,1\}$ are the bit values. If we select $\{g_i(t), t\in[0,T], i=1\ldots M\}$ as a set of independent bit functions then we will be able to recover the bits if we know s(t). Here M is the number of bits to be transmitted using one symbol. This process of recovery of $\{d_i\}$ from s(t) will require very precise knowledge of s(t). That can be achieved only by providing large number of samples for s(t) and for each member of $\{g_i(t)\}$. Note that in (1) s(t), $\{g_i(t)\}$, and $\{d_i\}$are all known quantities. In a later section we highlight the similarity of expression (1) with Fourier series and its consequences.

The functions used in fm are not defined over infinite time interval; they are defined only over the symbol time, which are usually very small, of the order of microseconds or milliseconds, and should not be considered as infinite time intervals. The Nyquist rate will provide very few samples on these small intervals and will not enable us to reconstruct them correctly. We use these general classes of functions to represent digital data, because they have higher capacity to represent information compared to simple sine wave functions. Modern Digital Signal Processors (DSP) are ideally suited to handle them also. The DSP technology, high speed and high resolution Analog to Digital Converters (ADC), along with the analytical functions are quite capable of handling powerful design methods, which cannot be implemented using hardware based concepts like voltage controlled oscillators or phase lock loops etc.

The fm receivers are more complex than the fm transmitters. Mainly because the objective of the fm receiver is to decompose the received functions into the component bit functions that were used to create the symbol at the transmitter. The decomposition process is usually more complex than the composition process. However, if the bit functions are orthogonal then the decomposition process is very trivial [7]. Figure 2 shows the block diagram of a fm receiver based on orthogonal bit functions. Note however that the transmitter design is the same whether we use orthogonal or non-orthogonal bit functions.

In Figure 2, the functions $\{g_i(t)\}$are assumed to be orthogonal bit functions. They are M in number, the

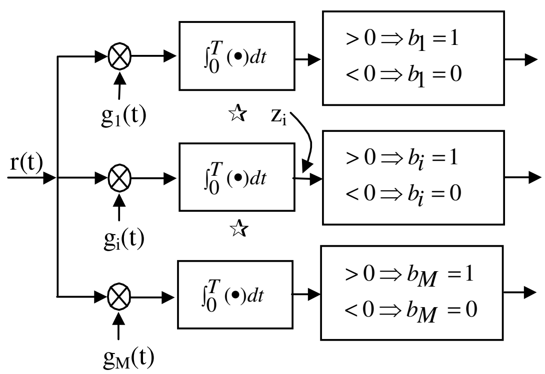


Figure 2. fm Receiver for orthogonal functions

number of bits to be transmitted using one symbol. The received symbol r(t) is generated by the corresponding fm transmitter using 0-1 addition algorithm as shown in (1). We can write the symbol r(t) using the following relation:

$$r(t) = g_1(t)x_1 + g_2(t)x_2 + .. + g_M(t)x_M + w(t) \quad (2)$$

Where the set $\{x_i\}$ in (2) represents the unknown bit values and w(t) is the additive white Gaussian noise term. If the functions in $\{g_i(t)\}$ are orthogonal then we can get the estimate for $\{x_i\}$ using the following integration (3). Here w_i is the projection of the noise on the i-th orthogonal function.

$$x_i = \int_0^T r(t)g_i(t)dt + w_i \quad (3)$$

The integration process (3) is shown in the Figure 2, along with the thresholds for detecting the bits.

Unlike similar figures in communication text books, here all parallel paths produce bit data. Therefore all integrations in all paths must be very precisely performed. This process will also require large volume of samples as well as powerful numerical integration methods, preferably based on analytical approaches. The receiver for non-orthogonal functions [6] is more complex and also demands large sample rate. Later we point out that the fm method is essentially an implementation of the concept behind the finite term Fourier series.

The fm design provides a method for using general purpose functions for digital data communication. General functions can carry more information than sinusoidal functions. We highlight in many different ways the well known fact that any general continuous function defined over any finite time interval has infinite dimension and therefore can carry infinite amount of information. Intuitively this concept and its consequences in communication engineering may not be very clear, so we describe it in many details beginning with the following section. We also show that to extract this infinite information content we have to sample the functions, theoretically, at infinite rate. Thus the motivation for this research is to establish the theoretical foundation for the function modulation method. The engineering foundation of the fm method over real time voice band telephone line has already been demonstrated and presented [6].

4. Infinite Dimensionality

We will use the following basic notations and definitions in our paper. Consider the space $L_2[a,b]$ of all real valued measurable functions defined over the finite, closed, and real interval [a,b]. We assume that the following Lebesgue integral is bounded:

$$\int_a^b |f(t)|^2 \, dt < \infty \,, \ \ \forall f \in L_2[a,b] \quad (4)$$

And then we define the norm:

$$\|f\| = \left[\int_a^b |f(t)|^2 \, dt\right]^{1/2} , \ \forall f \in L_2[a,b] \quad (5)$$

Measurable functions form an equivalence class, in the sense that each function in this class has the same integral value. Two such functions in the same equivalent class differ on some countable discrete set whose measure is zero thus without affecting the integral value. We can always find a continuous function that can represent this equivalent class [8, pp418-427] in the sense of $L_2[a,b]$ norm. Thus for all engineering purposes we can think about continuous functions only [9, pp27-28].

The space $L_2[a,b]$ is a complete space. This completeness property ensures that every convergent sequence $\{f_n\}$ converges to a function f that belongs to $L_2[a,b]$ space. That is, $L_2[a,b]$ includes all the limit points. The norm (5) is used to prove the convergence because it embeds the concept of distance between the elements of a sequence.

We also define the inner product for $L_2[a,b]$ as:

$$\langle f,g\rangle = \int_a^b f(t)g(t)dt \quad \forall f,g \in L_2[a,b] \qquad (6)$$

For $L_2[a,b]$ the integral in (6) exists and therefore the inner product is well defined. In a finite dimensional vector space the inner product of two vectors u = $\{u_i\}$ and v = $\{v_i\}$ is defined by

$$u'v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Which is similar to (6) when you take the limit of the approximation given by the following:

$$\langle f,g\rangle = \int_a^b f(t)g(t)dt$$
$$\approx \Delta t[f(t_1)g(t_1) + f(t_2)g(t_2) + \cdots + f(t_n)g(t_n)]$$

Thus a function space is very intimately linked with the concept of finite dimensional linear vector space when we look at it as nothing but a collection of infinite samples.

Under the above conditions the function space, $L_2[a,b]$, is a Hilbert space. Hilbert space is defined as a complete inner product space. The inner product comes from the definition (6) and the completeness from the norm (5). The inner product helps to introduce the concept of orthogonality in the function space. We also define the distance between two functions in $L_2[a,b]$ space by:

$$d(f,g) = \|f-g\| = \left[\int_a^b |f(t)-g(t)|^2\,dt\right]^{1/2} \qquad (7)$$

The metric d in (7) defines the mean square distance between any two functions in $L_2[a,b]$.

One very important property of the Hilbert space [9, pp31-32] related to the communication theory, is that it contains a countable set of orthonormal basis functions. Let $\{\varphi_n(t), n = 1.2..., t \in [a,b]\}$ be such a set of basis functions. Then the following is true:

$$\langle \varphi_m, \varphi_n\rangle = \delta_{mn} = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n \end{cases} \qquad (8)$$

Also for any f ϵ $L_2[a,b]$ we have the Fourier series

$$f(t) = \sum_{n=1}^{\infty} a_n\varphi_n(t), \quad \forall t \in [a,b] \qquad (9)$$

The above expression (9) really means that for any given ε > 0 there exists an N such that

$$\left\| f(t) - \sum_{n=1}^{k} a_n\varphi_n(t)\right\| < \varepsilon, \ \forall k > N \qquad (10)$$

In this context we should also mention that the coefficients in (9) can be obtained using the following expression:

$$a_n = \int_a^b f(t)\varphi_n(t)dt\,, \ n = 1,2,\ldots \qquad (11)$$

In this paper we will consider only the continuous functions and their Riemann integrability. Riemann integration is the normal integration process we use in our basic calculus. We note that the continuous functions are measurable functions and the Riemann integrable functions are also Lebesgue integrable. For continuous functions the values for these two integrals are also same. Thus the Hilbert space theory (4-11) and the associated concepts will still remain applicable to our problems. We should point out though that the space of continuous functions is not complete for the $L_2[a,b]$ norm defined by (5). That means, there exists a sequence of continuous functions that does not converge to a continuous function under the $L_2[a,b]$ norm. However it will converge to a measurable function under L_2 norm, that is, in the mean.

Equality (9) happens only for infinite number of terms. Otherwise, the Fourier representation in (10) is only approximate for any finite number of terms. In this paper ε in (10) will be called as the measure of approximation or accuracy estimate in representing a continuous function. The Hilbert space theory ensures the existence of N in (10) for a given ε. The existence of such a countably infinite number of orthonormal basis functions (8) proves that the function space is an infinite dimensional vector space. This dimensionality does not depend on the length of the interval [a,b]. Even for a very small interval, like symbol time, or an infinite interval, a function is always an infinite dimensional vector. The components of this vector are the coefficients of (9).

Hilbert space theory shows that a function can be represented by equation (9). The coefficients in (9) carry the information about a function. Since there are infinite numbers of coefficients, a function carries an infinite amount of information. Our digital communication theory will be significantly richer if we can use even a very small portion of this infinite information content of a function. The function modulation approach provides a frame work for such a system. The fm scheme essentially implements equation (9), for fm transmitter, where the coefficients used are zero or one instead of any real number. Similarly Figure 2, the fm receiver for orthogonal functions, implements expression (11). For an fm receiver (11) will produce zero or one as the values for $\{a_n\}$. It is clear that if we can find a band limited set of orthogonal functions then equations (9) and (11) will allow us to create a fm system with almost unlimited capacity [4].

It is not necessary to have orthonormal basis functions for demonstrating that the function space is infinite dimensional. The collection of all polynomial functions {t^n, n=1,2,..}is linearly independent over the interval [a,b] and their number is also countable infinity. These polynomials can be used to represent any analytic function, i.e., a function that has all derivatives. Using Taylor's series we can express such a f(t) at t as:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (t-c)^n \tag{12}$$

around the neighborhood of any point c. Thus the above polynomial set is also a basis set for the function space. Therefore using the infinite Taylor series expression (12), we prove that a function is an infinite dimensional vector over a finite interval. Here the information content of the function is defined by the derivative coefficients and the polynomial functions. Expression (12) also shows that the information content of a general function is infinity.

The above two theories prove that the dimension of the function space is infinity. The number of such functions in this function space is also infinity, actually uncountable infinity. This is illustrated using the following logic. Consider any coefficient in the right hand side of (9). You will get a new function every time you change that coefficient. Since that coefficient can be adjusted to any value in the interval [0,1] you get a continuum of functions. Thus the cardinality of the function space is uncountable infinity whereas the dimensionality is countable infinity.

We say that to represent a function accurately over any interval we need two sets of information: (A) An infinite set of basis functions, not necessarily orthogonal, like the ones given by (8) and (B) An infinite set of coefficients in the infinite series expression for the function, similar to (9). That is, these two sets completely define the information content in a mathematical function. In most cases the basis set described in (A) will remain fixed. We will distinguish functions only by their coefficients described in (B). Each function will have different coefficients in its expression for (9).

We normally represent vectors as rows or columns with components as real numbers. As an example, a three dimensional vector has three real components. Similarly an n dimensional vector has n real components. Along that line an infinite dimensional vector will have infinite number components. We can represent a function by an infinite dimensional vector by selecting the coefficients of (9) as components of this vector. In this sense every function is an infinite dimensional vector. We will show later that the samples of a function can also be used to represent these components of an infinite dimensional vector. Thus these samples will bring this mathematics to engineering, because the ADCs can produce these samples.

We now show that a band limited function is also infinite dimensional and therefore carries infinite amount of information. Consider a band limited function f(t), with bandwidth [-W,+W]. Then f(t) is given by the following inverse Fourier Transform [2]:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(w)e^{iwt}dw$$

$$= \frac{1}{2\pi}\int_{-2\pi W}^{+2\pi W} F(w)e^{iwt}dw \tag{13}$$

In (13) t is defined for all time in (–∞,+∞). But the frequency w is defined only over [–W,+W], and it can take any value: integer, rational, or irrational frequencies, within that range.

The second line in expression (13) shows that the band limited function f(t) has uncountably infinite number of frequencies. That is, f(t) is created using infinite number of frequencies and therefore is an infinite dimensional vector. This is true even when we consider a small interval of time for the function f(t). In that small interval the function still has all the infinite frequency components corresponding to the points in [–W,+W]. This is another way of showing that a band limited function is an infinite dimensional vector over a finite measurement window.

We have been talking about countable and uncountable infinities. To refresh our memory, countable infinity is the number of elements in the set of integers {1,2,…} and the uncountable infinity is the number of points in the interval [0,1]. The set of rational numbers is also countable. Clearly uncountable infinity is larger than the countable infinity. However, one interesting fact is that any real number can be represented as a limit of a sequence of rational numbers. This fact is mathematically stated as the set of rational numbers is a dense set in the set of real numbers [8, pp43-45]. Therefore when we talk about uncountable infinity we can in many cases think in terms of countable infinity also. The relationship between measurable functions and continuous functions are similar as mentioned before.

We point out here that a constant function f(t) = C, as an element of function space, is also an infinite dimensional vector. The only difference is that all sample values are same. In terms of Taylor series the coefficients for a constant function are {C,0,0,….}, which is an infinite dimensional vector.

The infinite dimensionality idea of a function can be understood in another very interesting way.

Consider the real line interval [0,1]. We know that it has uncountably infinite number of points. If we stretch this line to [0,2] we will still have all these uncountable number of points inside it. Now if we bend it and twist it all the points will still be there but the line will now become a function, a graph, in the two dimensional plane. Thus a function has uncountably infinite number of points. Every sample you take will have different coordinates and therefore different information. Therefore we can prove that a function can be exactly represented by this infinite number of samples and that more samples you take over this finite interval better will be the representation of the function.

The definition of dimension should be clearly pointed out. The dimension of a vector space is the number of basis vectors of the space. For function space, both Fourier series (9) and the Taylor series (12) show, that the number of basis vectors is countable infinity. Therefore the dimension of the function space is countable infinity. Any element of this vector space will also have the same number of components in its representation as a vector. Therefore the total number of components in a vector is also called the dimension of the vector.

We now show that samples can also be used to represent this infinite dimensionality. We prove that it is theoretically necessary to sample a function that is defined over finite time interval, infinite number of times, to extract all the information from the function.

5. Sample Convergence

Let f(t) be a continuous function defined over the real time interval [a,b]. Assume that we divide this finite time interval [a,b] into n > 1 equal parts using equally spaced points $\{t_1, t_2, .. t_n, t_{n+1}\}$. Where $t_1 = a$ and $t_{n+1} = b$. Use the following notations to represent the t-subintervals

$$\Delta t_i = \begin{cases} [t_i, t_{i+1}), & i = 1,2,..,n-1 \\ [t_n, t_{n+1}], & i = n \end{cases}$$

Define the characteristic functions:

$$X_i(t) = \begin{cases} 1, & t \in \Delta t_i \\ 0, & t \notin \Delta t_i \end{cases} \quad i = 1,2,..,n \qquad (14)$$

In this case the characteristic functions, $X_i(t)$ are orthonormal over the interval [a,b] with respect to the inner product on L_2 [a,b], because

$$X_i(t)X_j(t) = 0,\ i \neq j,\ \forall t \in [a,b] \qquad (15)$$

Also define the simple functions as:

$$f_n(t) = \sum_{i=1}^{n} f(t_i)X_i(t) \quad \forall t \in [a,b] \qquad (16)$$

Here $f(t_i)$ is the sampled value of the function f(t) at time $t = t_i$ that is, at the beginning of each sample interval Δt. It is easy to visualize that $f_n(t)$ is a sequence of discrete step functions over n. Expression (16) is an approximate Fourier series representation of f(t) over [a,b]. This representation uses the samples of the function f(t) at equal intervals, $f_n(t)$ uses n number of samples.

We show that this approximate representation (16) improves and approaches f(t) as we increase the value of n. Which will essentially prove that more you sample more information you get about the function.

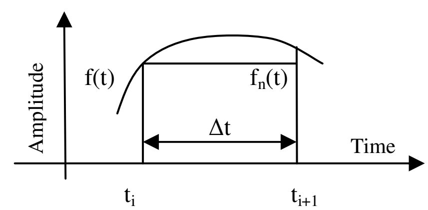


Figure 3. Simple function approximation

Thus the higher sample rate is meaningful and does not produce any redundant information. The following theorem is quite intuitive; its proof is also very simple. However, its consequence is very profound in the field of digital signal processing and in communication engineering.

Theorem 1: $f_n \to f$ in L_2[a,b] as $n \to \infty$.

First consider Figure 3, where we show the simple function $f_n(t)$ and the original continuous function f(t), between two consecutive sample points on the time line. It is geometrically obvious that the maximum error between the two functions reduces as the interval Δt reduces. Mathematically, consider the error

$$\Delta y_n = \max_t |f(t) - f_n(t)|,\ \forall t \in [a,b] \qquad (17)$$

It is then clear that $\{\Delta y_n\}$ is a monotonically decreasing sequence of n since the function f(t) is continuous over the closed interval [a,b]. Therefore, given any ε>0 we can find an N such that $\Delta y_n \leq \varepsilon/\sqrt{(b-a)}$ for all n > N. We can also verify that

$$\|f - f_n\| = \left[\int_a^b |f(t) - f_n(t)|^2 dt\right]^{1/2}$$

$$= \left[\int_a^b |f(t) - \sum_{i=1}^{n} f(t_i)X_i(t)|^2\, dt\right]^{1/2} \qquad (18)$$

Since $X_i(t) = 1$ for all t we can rewrite the above expression without affecting the integral as

$$= \left[\int_a^b |\sum_{i=1}^{n} f(t)X_i(t) - \sum_{i=1}^{n} f(t_i)X_i(t)|^2\, dt\right]^{1/2}$$

Rearranging the terms we can write

$$= \left[\int_a^b \left[\sum_{i=1}^n \left(f(t) - f(t_i)\right) X_i(t)\right]^2 dt\right]^{1/2}$$

Now performing the squaring operation, noting that (15) holds, we can write the above as:

$$= \left[\int_a^b \left[\sum_{i=1}^n [f(t) - f(t_i)]^2 X_i^2(t)\right] dt\right]^{1/2}$$

$$\leq \left[\int_a^b \left[\sum_{i=1}^n [\Delta y_n]^2 X_i^2(t)\right] dt\right]^{1/2}$$

$$= \left[\Delta y_n^2 \int_a^b \left(\sum_{i=1}^n X_i^2\,(t)\right) dt\right]^{1/2}$$

$$= [\Delta y_n^2 (b-a)]^{1/2} = \sqrt{(b-a)}\Delta y_n \leq \varepsilon$$

Thus from (18) we see that $\forall n \geq N$

$$\| f(t) - \sum_{i=1}^n f(t_i) X_i(t) \| \leq \epsilon, \forall t \in [a,b] \qquad (19)$$

Which means:

$$f(t) = \sum_{i=1}^{\infty} f(t_i) X_i(t)\,,\ \forall t \in [a,b] \qquad (20)$$

This concludes the proof of Theorem 1.

The expression (20) says that a function is an infinite dimensional vector and can be correctly represented by all the infinite samples, while the expression (19) can be used for approximate representation with accuracy given by ε. Essentially Theorem 1 proves that infinite sample rate is necessary to represent a continuous function correctly over a finite time interval. Another important interpretation of Theorem 1 is that the information content of a function is available in the samples of the function. Thus the amount of information these general purpose functions can carry is actually infinity. A communication system can be designed to extract a large amount of information from this infinite content. The fm system uses such a general class of function and can be used to carry more information than conventional designs.

It is clear that the Theorem 1 does not depend on the bandwidth of the function f(t). However, for any given ε>0 the number N will depend on the bandwidth.

Theorem 1 is similar to the one described for measurable functions in [8, pp389-391]. But the coefficients are not the sampled values in that theorem. For measurable functions, samples are usually taken on the y-axis. Another proof can be found in [10, pp247-257] where the Bernstein polynomial has been used instead of the characteristic function. Although Bernstein polynomial functions are not orthogonal, like the characteristic functions used in Theorem 1, but they are defined over finite and closed interval, occasionally know as functions with compact support. We will see that it has an important consequence when we reconstruct the function using the samples.

Theorem 1 shows that the approximating functions (16) converge in the mean, because we have used the L_2 norm. In engineering we normally like pointwise, that is point by point convergence. We say that a sequence of functions, like $\{f_n\}$ in (16), converges uniformly to a function f on the closed interval [a,b] if for every ε > 0 there exists an N > 0, depending only on ε and not on t in the interval [a,b], such that $|f_n(t) - f(t)| < \varepsilon$ for all n >N, and for all t in [a,b]. It should be pointed out that uniform convergence implies pointwise convergence.

Since the function f(t) is continuous over a closed interval, the sequence (16) is bounded, because the sample values are bounded. Therefore if we show that the supremum or the maximum of (16) converges then obviously the sequence will converge uniformly also [8, pp308-311]. That is we have to show that

$$\sup_{a \leq t \leq b} |f(t) - f_n(t)| < \varepsilon\,,\ \forall n > N \qquad (21)$$

Incidentally, the supremum and the maximum are the same thing for continuous functions over closed interval and also the difference really does not matter for engineering problems. Over every small interval Δt_i we can write

$$|f(t) - f_n(t)| = |f(t) X_i(t) - f(t_i) X_i(t)|$$

$$= |f(t) - f(t_i)| X_i(t)$$

According to the mean value theorem, there exists a c_i within every interval Δt_i such that

$$f(t) = f(t_i) + (t - t_i) f'(c_i)$$

Therefore using (16) we can write

$$|f(t) - f_n(t)| = \sum_{i=1}^n |f(t) - f(t_i)| X_i(t) \qquad (22)$$

$$= \sum_{i=1}^n |(t - t_i) f'(c_i)| X_i(t)$$

$$= \sum_{i=1}^n |t - t_i| |f'(c_i)|\, X_i(t)$$

$$\leq \Delta t\, M\ \sum_{i=1}^n X_i(t) = \frac{b-a}{n} M \qquad (23)$$

Where, M is the upper bound of the derivative of f(t) on the entire interval [a,b]. Since f(t) is a continuous function over a closed interval, M always exists. So in the above proof we have assumed that the function is differentiable. Thus the right hand side of (23) is independent of t and therefore is an uniform bound for all t for the left hand side of (22). Thus we can see that the difference expressed in the left side of (22) goes to zero as n goes to infinity. This shows that (20) is true for all t, thus proving the uniform convergence. The above derivation also proves the intuitive assertion made in (17).

We have shown that the approximations generated by sample values of the original function f(t) converges to the original function. As it converges the number of samples increases, because that is the way we constructed the approximating function $f_n(t)$. Since the approximations improve, the $f_n(t)$ improves, confirming that more samples are better and does not generate redundant information. These infinite samples collectively define the function. A complete description of the function can only be obtained by these infinite numbers of samples, which can be considered as components of an infinite dimensional vector. The above basic tools now can help us to give theoretical proofs of sampling theorem.

6. Sampling Theorem

Consider the simple sinusoidal function

$$s(t) = A \sin (2 \pi f t + \theta) \qquad (24)$$

and assume that it is defined only for one period for simplicity, although not necessary. We can think of this sinusoidal function as the highest frequency component of a band limited signal. So if we can recover this function by sampling it then we will be able to recover the entire original signal, because the other components of the band limited signal are changing slowly at lower frequencies. We try now to determine how many samples we need to recover the sine function.

We can see from the above expression (24) that a sinusoidal function can be completely specified by three parameters A, f, and θ. That is we can express a sine function as a three dimensional vector:

$$s = [A, f, \theta] \qquad (25)$$

However (25) is very misleading. There is a major hidden assumption; that the parameters of (25) are related by the sine function. Therefore more precise representation of (25) should be:

$$s = [A, f, \theta, \text{“sine”}] \qquad (26)$$

The word sine in (26) means the Taylor's series, which has an infinite number of coefficients. Therefore when we say (25) we really mean (26) and that the sine function, as usual, is really an infinite dimensional vector.

We can use the following three equations to solve for the three unknown parameters, A, f, and θ of a sinusoidal function:

$$\begin{aligned} s_1 &= A \sin (2 \pi f t_1 + \theta) \\ s_2 &= A \sin (2 \pi f t_2 + \theta) \\ s_3 &= A \sin (2 \pi f t_3 + \theta) \end{aligned} \qquad (27)$$

where t_1 , t_2 , t_3 are sample times and s_1 , s_2 , s_3 are corresponding sample values. Again a correct representation in terms of samples would be

$$s = [(s_1,t_1), (s_2,t_2), (s_3,t_3), \text{“sine”}]$$

Hence with sinusoidal assumption, a sine function can be completely specified by three samples. The above analysis gives a simple proof of the sampling theorem. We can now state the well known result:

Theorem 2: A sinusoidal function, with the assumption of sinusoidality, can be completely specified by three non-zero samples of the function taken at any three points in its period.

From (27) we see that if we assume sinusoidality then more than three samples, or higher than Nyquist rate, will give redundant information. However without sinsoidality assumptions more samples we take more information we get, as is done in common engineering practice. It should be pointed out that Shannon's sampling theorem assumes sinusoidality. Because it is derived using the concept of bandwidth, which is defined using Fourier series or transform, and which in turn uses sinusoidal functions.

Theorem 2 says that the sampling theorem should be stated as $f_s > 2f_m$ instead of $f_s \geq 2f_m$ that is, the equality should be replaced by strict inequality. Here, f_m is the signal bandwidth, and f_s is the sampling frequency. There are some engineering books [11, p63] that mention strict inequality.

Shannon states his sampling theorem [2, p448] in the following way: "If a function f(t) contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced 1/2 W seconds apart". The proof in [2] is very simple and runs along the following lines. See also [12, p271]. A band limited function f(t) can be written as in (13). Substituting t = n/(2W) in (13) we get the following expression:

$$f\left(\frac{n}{2W}\right) = \frac{1}{2\pi}\int_{-2\pi W}^{+2\pi W} F(w)e^{iw\frac{n}{2W}}\,dw \qquad (28)$$

Then the paper [2] makes the following comments: "On the left are the values of f(t) at the sampling points. The integral on the right will be recognized as essentially the nth coefficient in a Fourier-series expansion of the function F(w), taking the interval –W to +W as a fundamental period. This means that the values of the samples f(n/2W) determine the Fourier coefficients in the series expansion of F(W)." It then continues "Thus they determine F(w), since F(w) is zero for frequencies greater than W, and for lower frequencies F(w) is determined if its Fourier coefficients are determined".

Thus the idea behind Shannon's proof is that from the samples of f(t) we reconstruct the unknown Fourier transform F(w) using (28). Then from this known F(w) we can find f(t) using (13) for all time t. One important feature of the above proof is that it requires that the function needs to exist for infinite time, because only then you get all the infinite samples from (28). We show that his proof can be extended to reconstruct functions over any finite interval with any degree of accuracy by increasing the sample rate. The idea behind the proof is similar, we construct F(w) from the samples of f(t).

In this proof we use the principles behind the numerical inversion of Laplace transform method as described in [13, p359]. Let F(w) be the unknown band limited Fourier transform, defined over [-W,+W]. Let the measurement window for the function f(t) be [0,T], where T is finite and not necessarily a large number. Divide the frequency interval 2W into K smaller equal sub-intervals of width Δw with equally spaced points $\{w_j\}$ and assume that the set of samples $\{F(w_j)\}$ is constant but unknown over that j-th interval. Then we can express the integration in (13) approximately as:

$$f(t) \approx \frac{1}{2\pi}(\Delta w)\sum_{j=1}^{K} e^{itw_j}\, F(w_j) \qquad (29)$$

The right hand side of (29) is a linear equation in $\{F(w_j)\}$, which is unknown. Now we can also divide the interval [0,T] into K equal parts with equally spaced points $\{t_j\}$ and let the corresponding known sample values be $\{f(t_j)\}$. Then if we repeat the expression (29) for each sample point t_j we get K simultaneous equations in the K unknown variables $\{F(w_j)\}$as shown below:

$$\begin{bmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_K) \end{bmatrix} = \frac{\Delta w}{2\pi} \begin{bmatrix} e^{it_1w_1} & e^{it_1w_2} \ldots e^{it_1w_K} \\ e^{it_2w_1} & e^{it_2w_2} \ldots e^{it_2w_K} \\ & \vdots \\ e^{it_Kw_1} & e^{it_Kw_2} \ldots e^{it_Kw_K} \end{bmatrix} \begin{bmatrix} F(w_1) \\ F(w_2) \\ \vdots \\ F(w_K) \end{bmatrix} \qquad (30)$$

These equations are independent because the exponential functions in (29) are independent.

We recall that a set of functions $G(t) = \{g_i(t),\ i = 1 \ldots M, t \in [0,T]\}$ is called dependent over the interval if there exists constants c_i, not all zero, such that

$$g_1(t)c_1 + g_2(t)c_2 + \cdots + g_M(t)c_M = 0$$

for all t in [0,T]. If not, then it is independent [14, pp177-181]. The above expression is a linear combination of functions. Here the coefficients $\{c_i,\ i = 1 \ldots M\}$ are all real numbers. It essentially says that one function cannot be constructed using other functions.

Therefore we can solve (30) for $\{F(w_j)\}$. Theorem 1 ensures that the sets $\{F(w_j)\}$ and $\{f(t_j)\}$ can be selected to achieve any level of accuracy requirements in (13) for either f(t) or F(w). For convenience we assume that the number of terms K in (29) is equal to $Tkf_s = 2kWT$. Here f_s is the Nyquist sample rate and k > 1. We state the following new sampling theorem.

Theorem 3: Let f(t) be a band limited function with bandwidth restricted to [-W,+W] and be available over the finite measurement window [0,T]. Then given any accuracy estimate ε >0, there exists a constant k>1 such that 2kWT equally spaced samples of f(t) over [0,T] will completely specify the Fourier transform F(w) of f(t) with the given accuracy ε. This F(w) can then be used to find f(t) for all time t.

Note that we did not say that the function does not exist over the entire real line. We only said that our measurement window is finite. What happens to the function beyond the finite interval is not needed for our analysis. The main point of our paper is that we do not need to be concerned with the existence of our signals over the entire real line.

We have given, as in (30), a very general numerical method of solving an integral equation. The method can be applied also to the case when f(t) in the left hand side of the equation (13) is unknown. The equation (13) itself can be generalized too. A well known [15] generalization is given below:

$$f(t) = \int_a^b K(t,w)F(w)dw \qquad (31)$$

Here we have replaced the sinusoidal function by the kernel function K(t,w). Expression (31) represents a relationship between frequency components with the time functions for finite duration signals. In that sense it gives the bandwidth information of any given function f(t). It may even be possible to solve the equation (31) analytically over finite time and frequency ranges for some specific kernel functions. More we research in this very practical finite duration engineering problem better will be our definitions and theories and they will be closer to reality. We point out here that there is a need for extending the definition of bandwidth of a function from infinite time to finite time.

In a sense Shannon's sampling theorem gives a sufficient condition. That is, if we sample at twice the bandwidth rate and collect all the infinite number of samples then we can recover the function. We point out that this is not a necessary condition. That is, his theorem does not say that if T is finite then we cannot recover the function accurately by sampling it. We have confirmed this idea in the above proof of Theorem 3. That is if T is finite we have to sample at infinite rate to get all the infinite number of samples. Or in other words more we sample more information we get. This is because a function is an infinite dimensional vector and therefore it can be correctly specified only if we get all the infinite number of samples.

Shannon proves his sampling theorem in another way [2]. Any continuous function can be expressed using the Hilbert space based Fourier expression (9). He has used the expression (9) for a band limited function f(t), defined over infinite time interval. He has shown that if we use

$$\varphi_n(t) = \frac{sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)]} \tag{32}$$

Then the coefficients of (9) can be written as

$$a_n = f(n/f_s) \tag{33}$$

Thus the function f(t) can be expressed as:

$$f(t) = \sum_{n=-\infty}^{\infty} f(n/f_s)\frac{sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)]} \tag{34}$$

Here f_s = 2W, where W is the bandwidth of the function f(t). The set $\{\varphi_n\}$ in (32) is orthogonal only over $(-\infty,+\infty)$.

Observe that the above is very similar to our proof of theorem 1. Shannon used sinc functions as the orthogonal basis functions, whereas in our theorem 1 we used rectangular pulses as the orthogonal basis functions. We know that the sinc function is the Fourier transform of the rectangular pulse. Only difference is that the sinc functions require infinite time interval.

We make the following observations about (34):

1. The representation (34) is exact only when infinite time interval and infinite terms are considered.
2. If we truncate to finite time interval then the functions φ_n in (32) will no longer be orthogonal, and therefore will not form a basis set, and consequently will not be able to represent the function f(t) correctly.
3. If in addition we consider only finite number of terms of the series in (34) then more errors will be created because we are not considering all the basis functions. We will only be considering a subspace of the entire function space.

We prove again, that by increasing the sample rate we can get any desired approximation of f(t), over any finite time interval [0,T], using the same sinc functions of (32). From calculus we know that the following limit holds:

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0 \tag{35}$$

Assume that f_s is the Nyquist sampling frequency, i.e., f_s = 2W. Let us sample the signal at k times the Nyquist rate. Here k>1 is any real number. Then using (35), we can show that given any T and a small $\delta > 0$, there exists an N such that

$$\left|\frac{sin(\pi k f_s t)}{\pi k f_s t}\right| < \delta, \forall k > N, \forall t \geq T \tag{36}$$

Thus these orthogonal functions (32) substantially go to zero outside any finite interval [0,T] for large enough sampling rate and still maintain their orthogonality property, substantially, over [0,T]. Therefore, for a given band limited function f(t), with signal capture time limited to the finite window [0,T], we can always find a high enough sample rate, kf_s so that given any ε>0 the following will be true:

$$\left\| f(t) - \sum_{n=0}^{K} f(\frac{n}{kf_s})\frac{sin\{\pi k f_s[t-(n/kf_s)]\}}{\pi k f_s[t-(n/kf_s)]}\right\| < \varepsilon \tag{37}$$

$$\forall k > N, \forall t \in [0,T]$$

The number of functions in the above series (37) is now K, which is equal to the number of samples over the period [0,T]. Thus K= kf_sT = 2kWT. As k increases the number of sinc functions increases and the distance between the consecutive sinc functions reduces thus giving higher sample rate. See the numerical example given below. The original proof [2] for (32-34) still remains valid as we increase the sample rate. That is, the sinc functions in (32) still remain orthogonal. It can be shown using the original method that the coefficients in (33) remain valid and represent the sample values. Of course, the original proof requires the infinite time interval assumption. Thus the system still satisfies the Hilbert Space theory expressed by (4-11) making expression (37) justified. Now we can state the following new sampling theorem.

Theorem 4: Let f(t) be a band limited function with bandwidth restricted to [-W,+W] and available over the finite measurement window [0,T]. Then given any accuracy estimate ε there exists k>1 such that 2kWT equally spaced samples of f(t) over [0,T] along with their sinc functions, will completely specify the function f(t) for all t in [0,T] at the given accuracy.

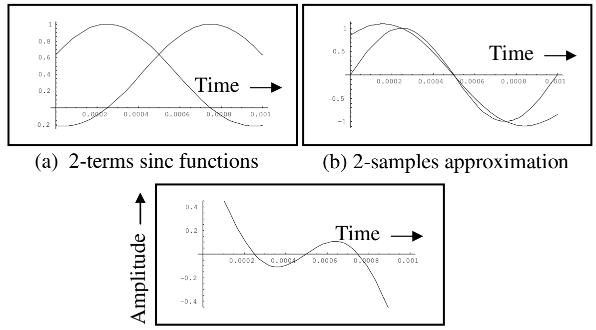
We should point out, like in Theorem 2, that if we assume infinite time interval then faster than the Nyquist rate will also not give redundant information. This concept is also easily seen from the Fourier series expression (9). To solve for the coefficients of (9) we need infinite number of samples to form a set of simultaneous equations similar to (30). As we increase the sample rate the solution of (30) will only become better, that is, the resolution of the coefficients will increase and the unknown function will also get better approximations.

For finite time assumption higher sampling rate is necessary to achieve the desired accuracy. The reason is same; the concept of infinite dimensionality must be maintained over finite time interval. That can be achieved only by higher sample rate. We also repeat, if you know the analytical expression then the number of samples must be equal to the number of unknown parameters of the analytical expression. This case does not depend on the time interval.

A lot of research work has been performed on the Shannon's sampling theorem paper [2]. Somehow the attention got focused on the WT factor, now well known as the dimensionality theorem. It appears that people have [16][17] assumed that T is constant and finite, which is not true. Shannon said in his paper [2] many times that T will go to infinite value in the limit. No one, it seems, have ever thought about the finite duration issue. This is probably because of the presence of infinite time in the Fourier transform theory. The paper [15] gives a good summary of the developments around sampling theorem during the first thirty years after the publication of [2]. Interestingly [15] talks briefly about finite duration time functions, but the sampling theorem is presented for the frequency samples, that is, over Fourier domain which is of infinite duration on the frequency axis. Now we give a numerical example to show how higher rate samples actually improves the function reconstruction.

7. A Numerical Example

We illustrate the effect of sample rate on the reconstruction of functions. Since every function can be considered as a Fourier series of sinusoidal harmonics, we take one sine wave and analyze it. This sine function may be considered as the highest frequency component of the original band limited signal. The Nyquist rate would be twice the bandwidth, that is, in this case twice the frequency of the sine wave. We are considering only one period, and therefore the Nyquist rate will give only two samples of the signal during the finite interval of its



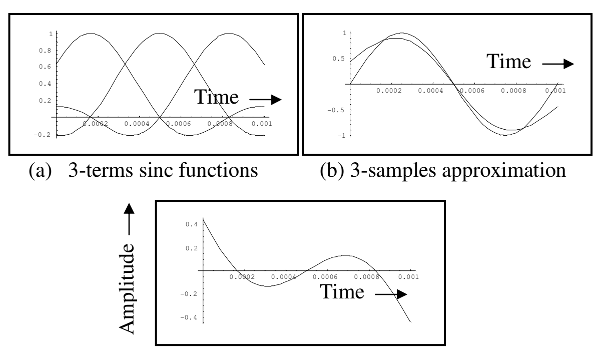
(a) 2-terms sinc functions (b) 2-samples approximation

(c) 2-samples error

Figure 4. Reconstruction using two samples

period. We are also assuming that we do not know the analytical expression of the sine wave that we are trying to reconstruct.

In this example we use the sinc functions of Shannon's theory, equation (37), to reconstruct the signal from the samples. In Figures 4-6 x-axis represents the angles of the sine function in terms of degrees or samples or time. The y-axis represents the amplitude of the sine function. Figure 4 shows the reconstruction process using two samples of the signal. In part (a) we show the sample locations and the corresponding sinc functions over the interval of one period. In part (b) we show how the construction formula (37) reproduces the function. Part (c) shows the error between the reconstructed sine function and the original sine function. We can see that the reconstruction really did not work well with two samples. Thus for finite time interval signals this process of recovering the function using expression (37) and the Nyquist rate provides very poor results.



(a) 3-terms sinc functions (b) 3-samples approximation

(c) 3-samples error

Figure 5. Reconstruction using three samples

Therefore in all engineering application we cannot use the Nyquist rate.

In Figure 5 we repeat the process described for Figure 4 with three samples. In part (b) of Figure 5 we still have significant errors. The error is prominent at the two edges because the sinc functions do not stop at the end, they continue up to infinity. This is where Bernstein polynomials, with compact support, will fit better [10]. Usually Bernstein polynomial converges very slowly and thus will require large number of samples. We mention about the Bernstein polynomial because it has some good analytical qualities [10, pp247-258]. However when we have large number of samples we may have many other better options for reconstructing the functions as described in the next section.

Figure 6 shows how the reconstruction process improves when we substantially increase the sample rate to six samples per period or three times the Nyquist rate. In this figure we still use the sinc function approach, i.e., expression (37). Notice that the errors at the edges are also reduced. This is because, as we increase the sample rate the sinc functions become narrower as predicted by the expression (36) and major part of the functions remain inside the signal interval.

The graphs show that the error decreases as we increase the sample rate as predicted by the new sampling theory and infinite dimensionality of the function space. It is clear from the examples that, for the same number of samples, a different recovery function, instead of sinc function, will give different result. In the next section we discuss this different reconstruction approach.

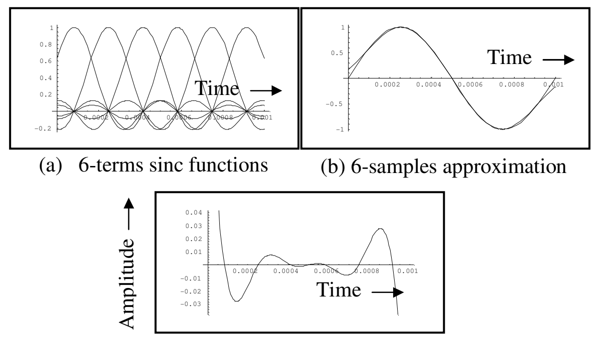
8. Re-sampling and Reconstruction

In many applications in engineering we may require different sampling rates mainly to control the computation time of the processor. The theory presented in this paper essentially says that, sample as fast as you can. That will give you maximum amount of information about the signal you want to process. However, we may not be able to select the desired Analog to Digital Converters (ADC) to sample the signal at the rate we want, because of many reasons. The two most important of them are the cost and the power required by the ADC chips.

After we have all the samples, collected at the highest feasible sample rate, then to reduce the sample rate we can simply drop few samples. This approach will maintain the quality of the signal representation. This is can be justified from the expression (30). Note that the Fourier series expression (9) can also be converted into a form given by (30). Normal decimation process changes the bandwidth thus losing the accuracy.

However, if we want to increase the sample rate, after collecting all the samples from the ADC, then we have to interpolate the samples and resample the analytical expression, thus obtained, using the new higher rate.

We emphasize the idea of using the analytical expression for the received signal in our algorithms. Instead of focusing on the samples we should focus on the mathematical expression and on the design of the algorithms around that mathematical expression. If we can achieve that then the number of samples will play a very minimal role. We will still have the complete expression of the signal even when we use very few samples.



(a) 6-terms sinc functions (b) 6-samples approximation

6-samples error

Figure 6. Reconstruction using six samples

Since in this paper we are dealing with finite time intervals, we do not yet have the proper definition of bandwidth. We also cannot use the conventional filters because they use transfer function, which uses Laplace transform, and which in turn is defined only over infinite time interval. Classical linear system theory is inappropriate for finite time interval problems. Note that the normal decimation and interpolation methods used in digital signal processing techniques [18, pp303-316] cannot be used also, because they use bandwidth related and transfer function based filters. Thus all of our analysis must rely on the time domain approaches.

We have described several analytical approaches for signal definition from the samples. Any method based on Fourier series (9), similar to (37), can be used as an analytical expression. Once you have the expression you can generate any number of samples from that expression. Approximation theory [10], a time domain approach, is also very rich in the area of interpolation using analytical expressions and can be used for re-sampling. This analytical expression approach requires that we use the entire batch of data. This batch allows us to see the entire life history of the system. This history can be more effective in signal processing than a sample by sample approach.

We want to add another layer of information to our signal processing approach. All of the above methods assume that we do not have any total system level information about the origin of the signal we are trying to reconstruct. More specifically, in digital communication receiver, for example, we know how the received signal was constructed at the transmitter. We can use that information to reconstruct an analytic expression from the samples at the receiver and then go for re-sampling, if necessary. This will produce more realistic results than straight forward application of approximation theories to the samples.

We give a numerical example to illustrate this global or system level concept of re-sampling and reconstruction of signal analysis. In function modulation [6] for example, at the transmitter, we used linear combination of a set of sinusoidal frequencies. We also used some constraints on the coefficients of this linear combination. The purpose of these constraints was to control the bandwidth (defined using the conventional sense) of the stream of concatenated symbols existing over infinite time. At the receiver we may not be able to use these constraints, but definitely we can interpolate using these specific frequency signals and achieve a higher level of accuracy in the reconstruction process.

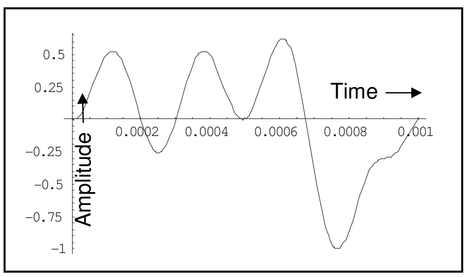


Figure 7. Transmitted symbol for fm

In our experiment, we transmitted the signal shown in Figure 7. We used the voice band telephone line to transmit the signal. The signal sample rate both at the transmitter and at the receiver was 16 kHz at 16 bits resolution. The telephone companies sample the voice signals at 8 kHz rate and at 8 bits resolution. This sample rate difference or some other unknown reasons distorted the received signal very significantly as shown in the Figure 8. As we can see from the figures the received signal has two positive peaks as opposed to three positive peaks in the transmitted signal. As if the second trough of the transmitted signal got folded up in the received signal.

It is clear that the conventional signal recovery methods, that use local concepts, no matter how many samples we take, cannot bring the received signal back to the transmitted form. However, a global approach or a systems approach, where we use the knowledge of the entire system can definitely help. We used the same sine wave frequencies of the transmitter, to interpolate the samples at the receiver. Here, of course,

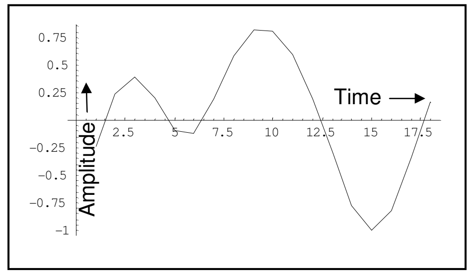


Figure 8. Received symbol for fm

the high sample rate played an important role in the least square interpolation method. The details of the signal processing, is quite involved, and is not given here. The large sample rate and the systems approach helped us to bring the received signal back to a shape that is very close to the transmitted signal, as shown in Figure 9, which allowed us to detect the bits correctly. Thus we can see that a total system level or global approach in signal processing can perform miracles.

At this end, we repeat again that all of the existing theories that are based on infinite time assumptions

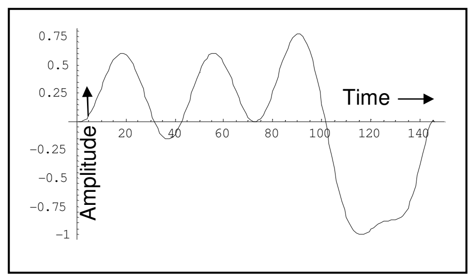


Figure 9. Best fit of the received symbol

should be carefully reviewed, redefined, and recreated for their finite time applications. We do not want any infinite time assumptions behind any of our finite time applications. That is not theoretically correct. More research will be required to generate analytical results for finite time systems. We should design our theories based on the engineering constraints. Our technology has advanced significantly. We can now use many mathematical theories that we could not use before.

Besides infinite time assumption most of our theories assume linearity also. We point out here in the next section how linearity concepts are deeply embedded in all our theories thus ignoring some basic engineering constraints. It is to be noted that the original sampling theorem used linearity assumption also, because it was based on Fourier theory.

9. Linearity Assumptions

All of our engineering systems are nonlinear because of two very important reasons. We briefly discuss them here to point out in another section that the transform methods that are based on linearity assumptions cannot be effectively used in engineering problems. Also we want to raise the importance of this nonlinearity to create a concern for the validity of our theories for engineering.

The most important reason for this nonlinearity is very well known though, but we probably never think about them. Hardly any text book [19, pp196-199] talks or provides any theory [20, pp167-179] for solving them. We call it saturation nonlinearity. Every engineering variable, like voltage, current, pressure, flow, etc. has some upper and lower bounds. They cannot go beyond that range. In terms of mathematical equation this situation can be described as

$$m_x \leq x \leq M_x,$$

where x is any physical engineering variable, m_x is the lower bound and M_x is the upper bound. Graphically the above equation can be represented by Figure 10. The figure shows that whenever the engineering variable x is within the range of [a,b] the output is linear and is equal to x. If x goes outside the boundary it gets clipped by M_x on the higher side and by m_x on the lower side. Note that m_x can be zero or negative also.

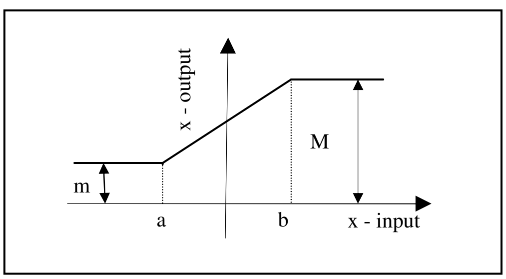


Figure. 10. Saturation Non-Linearity

Clearly Figure 10 shows severe nonlinearity and there is no escape from this in any engineering problems. We should not ignore its presence in engineering. These constraints are a kind of natural laws for our technology. Therefore we cannot treat our engineering problems using simple Linear Time Invariant (LTI) systems theory. Simply because there are no real LTI systems in engineering. If we do use such a theory then the performance of the system will be compromised.

It should be pointed out that it is almost impossible to keep the variables in the linear region. In most practical engineering problems there are hundreds of variables and hundreds of equations. It is not feasible to ensure that all variables will be within the linear range all the time. Their relationships are very complex. In addition there are many transients in all systems that can significantly alter the domain size of the variables. There is yet another important case where we have to demonstrate that the systems we build must behave normally when it goes to these limits. In addition, many systems have strict requirements that variables must go to this limit and stay there, like actuators in airplane wings. The wing flaps must reach their maximum angular positions and stay there for certain period of time.

In all engineering software, any conscientious programmer will always include the above nonlinearity test, usually called anti-windup, in their source code. And this code will automatically make the software, and hence our algorithm, nonlinear. A software engineer, barring few exceptions, does not know how mathematics works but knows how to make systems work. If you see any source code you will find many such patch work or kludge in the source code that are necessary to make the systems work. This necessity originates not only because of the lack of theoretical foundations of our algorithms but it will also be due to the RTOS and the interrupts of the background software which interferes with our theory.

All electronic hardware automatically includes such saturation nonlinearity in their systems. An automatic gain control mechanism, for example, actually is nothing but a nonlinear method of keeping the variables inside their linear regions. Because of this non-linearity any application of the LTI theory in engineering will violate the mathematical assumptions of the LTI theory. All transfer functions based approaches, like Laplace, or Fourier, are inappropriate for all engineering methods. They cannot work, because they violate the basic engineering assumptions. The transform approaches not only assume infinite time, they also assume linearity. If we use correct theory consistent with engineering models then we will definitely get much better results from our systems.

There is another very natural reason for using the non-linear theory in the design of our systems. Every engineer knows this one also but we want to mention it to strengthen our argument against the application of the LTI system theory. Most of the engineering requirements for today's technology are very stringent and we have experienced that our technology in most cases can support them to some extent. Because of these highly advanced and sophisticated requirements a simple linear model of engineering systems cannot achieve the objectives. We must make use of advanced nonlinear and dynamic or time varying adaptive system models.

A very well know and well established example that engineers have developed during the last thirty years is the Inertial Navigation System (INS). Today our requirement says that the first missile will make a hole in the building and the second missile must go through that hole. That is a very sophisticated and precise demand. The INS development shows that simple Newton's law of force equals to mass times acceleration cannot work. The equation has been extended to fill books of hundreds of pages. They derive starting with simple linear Newton's equation [21, p4] a complete and very highly nonlinear set of equations [22, pp73-77] to describe the motion of an object. These equations include, among many other things, Coriolis forces, earth's geodetic ellipsoidal models etc. Even after including all the nonliniearities that we know of we still had to integrate the INS with a Global Positioning System (GPS) to satisfy many of our requirements.

The simple Linear Time Invariant (LTI) system equation like the one given below:

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \qquad (38)$$

cannot solve most of our engineering problems, simply because the laws of nature are too complex and the demands from our technology are too precise. In engineering many nonlinear problems have been attempted using successive linearization methods with the application of theories of (38). These approaches do not work also, mainly because there are no theories with very general assumptions that establish their convergence, stability, and optimality. The above reason of nonlinearity is very well known to all of us.

We have added this section to highlight the need for the signal processing approach presented in this paper. We are proposing a software radio approach with finite time batch data processing, very high sampling rate, time domain theories, and global system level simultaneous interactions. We believe that this direction, has some theoretical foundations, and can be augmented to many nonlinear dynamic approaches.

The wavelets have become a very popular signal processing tools. It also has been used to extend the sampling theorem applications [23]. So we briefly talk about it in the next section.

10. Wavelet theory

The wavelet theory has many relations with the theories discussed in this paper. The Haar wavelet systems starts [24, p431] with the characteristic function of the [0,1] interval similar to the one defined by (14). These are also orthogonal functions as described in (15). The Shannon's scaling functions [24, p422] are the sinc functions $\sin \pi t/\pi t$ similar to (32). Wavelets are very useful signal processing tools for many image and voice related problems. However it is still at its developmental phase. It has not been demonstrated yet that it can be integrated, similar to Fourier or Laplace methods, into dynamic systems governed by differential equations.

The wavelet theory also gives a down sampling process [25, pp31-35]. Like Fourier theory the wavelet down sampling provides a lower resolution, as a consequence of the multi-resolution analysis, reconstruction of the original function. This is in contrast to the method presented in previous section, where the down sampled version still maintains the original resolution quality. The down sampling process is required for reducing the throughput requirements of the processor. Down sampling should not therefore reduce the quality of the signal.

Continuous wavelet transform [24, p366] of a function f(t) has been defined by the following integral:

$$W_\psi[f](a,b) = \int_{-\infty}^{\infty} f(t)\overline{\psi_{a,b}(t)}\, dt$$

Where the wavelet $\psi_{a,b}(t) \in L^2(R)$ is defined by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \qquad a, b \in R, a \neq 0$$

It is clear from the above definitions that the wavelet theory uses the infinite time assumption and therefore is not appropriate for signals of finite duration. It has the same application problem that the Fourier transform has. As mentioned before all practical problems are based on finite time assumptions. Since the scaling functions and the wavelets are used with translations on time axis, only very few and finite number of translations can be used over a finite duration interval.

It is also well known that the wavelet transform uses the linearity assumptions [24, p378]. Therefore it has the same problems discussed in Section 9 above and should not be used for most engineering applications.

Because many wavelets are orthogonal functions they will be very helpful in implementation of the function modulation systems described in Section 3. However it is not really known at this time how many band limited orthogonal wavelets can be constructed over a finite duration symbol time. Although wavelets are band limited but their bandwidth appears to be very high.

11. Conclusion and Future Work

We have given various proofs to show that k times, k>1, the Nyquist sample rate is necessary to improve the accuracy of recovering a function that is available only over finite time measurement window. We have shown that this k can be selected based on the required accuracy estimate ε.

The foundation of our derivations used the infinite dimensionality property of the function space. The concept essentially means that an infinite number of samples are necessary to precisely extract all the information from a function.

We have pointed out that many of our existing definitions and theories depend on the infinite time assumptions. We should systematically approach to eliminate this requirement from all our theories to make them realistic for our engineering problems.

12. Acknowledgement

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