# Global Stability of Positive Different Fractional Orders Nonlinear Feedback Systems

Tadeusz Kaczorek
Faculty of Electrical Engineering
Bialystok University of Technology
Bialystok, Poland
e-mail: kaczorek@ee.pw.edu.pl

Lukasz Sajewski
Faculty of Electrical Engineering
Bialystok University of Technology
Bialystok, Poland
e-mail: l.sajewski@pb.edu.pl

*Abstract*—**The global stability of continuous-time fractional orders nonlinear feedback systems with positive linear parts is investigated. New sufficient conditions for the global stability of these class of positive nonlinear systems are established. The effectiveness of these new stability conditions is demonstrated on a simple example.**

***Keywords-global stability; fractional order; positive; nonlinear; feedback system.***

## I. INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1][4][9]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models, and so on. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine. An overview of state of the art in positive systems theory is given in the monographs [1][4][9][13][18].

Mathematical fundamentals of the fractional calculus are given in the monographs [13][18][23][24]. Positive fractional linear systems have been investigated in [3][5][7][10]-[14][17][21][24][25][26]. Positive linear systems with different fractional orders have been addressed in [10][11][28]. Descriptor positive systems have been analyzed in [2][28]. Linear positive electrical circuits with state feedback have been addressed in [2][18]. The superstabilization of positive linear electrical circuits by state feedback has been analyzed in [16] and the stability of nonlinear systems in [17][18]. The global stability of nonlinear systems with negative feedback and not necessary asymptotically stable positive linear parts has been investigated in [6][8]. The global stability of nonlinear standard and fractional positive feedback systems has been considered in [15].

In this paper, the global stability of nonlinear fractional orders feedback systems with positive linear parts will be addressed.

The paper is organized as follows. In Section 2, the basic definitions and theorems concerning the positive different fractional orders linear systems are recalled. New sufficient conditions for the global stability feedback nonlinear systems with positive linear parts are established in Section 3. Concluding remarks are given in Section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $M_n$ - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $I_n$ - the $n\times n$ identity matrix.

## II. PRELIMINARIES

Consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t)\,, \tag{1a}$$

$$y(t) = Cx(t)\,, \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors, $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$. In this paper, the following Caputo definition of the fractional derivative of $\alpha$ order will be used [13][18][23][24]

$${}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1, \tag{2}$$

where $\dot f(\tau) = \frac{df(\tau)}{d\tau}$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $\mathrm{Re}(x)>0$ is the Euler gamma function.

**Definition 1.** [13][18] The fractional system (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x(0)\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

**Theorem 1.** [13] [18] The fractional system (1) is positive if and only if

$$A\in M_n\,,\; B\in\Re_+^{n\times m}\,,\; C\in\Re_+^{p\times n}\,. \tag{3}$$

**Definition 2.** The fractional positive linear system (1) is called asymptotically stable (and the matrix *A* Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x(0) \in \Re_+^n . \tag{4}$$

The positive fractional system (1) is asymptotically stable if and only if the real parts of all eigenvalues $s_k$ of the matrix *A* are negative, i.e. $\mathrm{Re}\, s_k < 0$ for $k = 1,...,n$ [13] [18].

**Theorem 2.** The positive fractional system (7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{5}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1,...,n$ such that

$$A\lambda < 0 \text{ or } \lambda^T A < 0 . \tag{6}$$

The transfer matrix of the system (1) is given by

$$T(s^\alpha) = C[I_n s^\alpha - A]^{-1} B . \tag{7}$$

Now, consider the fractional linear system with two different fractional order

$$\begin{bmatrix} \dfrac{d^\alpha x_1(t)}{dt^\alpha} \\ \dfrac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) , \tag{8a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} , \tag{8b}$$

where $0 < \alpha, \beta < 1$, $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are the state vectors, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{p \times n_i}$; *i*, *j* = 1,2; $u(t) \in \Re^m$ is the input vector and $y(t) \in \Re^p$ is the output vector. Initial conditions for (8) have the form

$$x_1(0) = x_{10} ,\ x_2(0) = x_{20} \text{ and } x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} . \tag{9}$$

**Remark 1.** The state equation (8) of fractional continuous-time linear systems with two different fractional orders has a similar structure as the 2D Roeesser type models.

**Definition 3.** The fractional system (8) is called positive if $x_1(t) \in \Re_+^{n_1}$ and $x_2(t) \in \Re_+^{n_2}$, $t \ge 0$ for any initial conditions $x_{10} \in \Re_+^{n_1}$, $x_{20} \in \Re_+^{n_2}$ and all input vectors $u \in \Re_+^m$, $t \ge 0$.

**Theorem 3.** The fractional system (8) for $0 < \alpha < 1;\ 0 < \beta < 1$ is positive if and only if

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_N ,\ \overline{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{N \times m} ,$$

$$C = [C_1 \quad C_2] \in \Re_+^{p \times n}\ (N = n_1 + n_2) . \tag{10}$$

**Theorem 4.** The positive fractional system (8) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - \overline{A}] = s^n + \overline{a}_{n-1}s^{n-1} + ... + \overline{a}_1 s + \overline{a}_0 \tag{11}$$

are positive, i.e. $\overline{a}_i > 0$ for $i = 0,1,...,n-1$.

2) There exists strictly positive vector $\lambda = [\lambda_1 \ \cdots \ \lambda_n]$, $\lambda_k > 0$, $k = 1,...,n$ such that

$$\overline{A}\lambda < 0 \text{ or } \lambda^T \overline{A} < 0 . \tag{12}$$

**Theorem 5.** The solution of the equation (8a) for $0 < \alpha < 1;\ 0 < \beta < 1$ with initial conditions (9) has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \Phi_0(t) x_0 + \int_0^t M(t-\tau) u(\tau) d\tau , \tag{13}$$

where

$$\begin{aligned} M(t) &= \Phi_1(t) B_{10} + \Phi_2(t) B_{01} \\ &= \begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^1(t) \\ \Phi_{21}^1(t) & \Phi_{22}^1(t) \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \Phi_{11}^2(t) & \Phi_{12}^2(t) \\ \Phi_{21}^2(t) & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} \Phi_{11}^1(t) B_1 + \Phi_{12}^2(t) B_2 \\ \Phi_{21}^1(t) B_1 + \Phi_{22}^2(t) B_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^2(t) \\ \Phi_{21}^1(t) & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \tag{14a}$$

and

$$\Phi_0(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha + l\beta}}{\Gamma(k\alpha + l\beta + 1)} , \tag{14b}$$

$$\Phi_1(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha + l\beta - 1}}{\Gamma[(k+1)\alpha + l\beta]} , \tag{14c}$$

$$\Phi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]}, \tag{14d}$$

$$T_{kl} = \begin{cases} I_n & \text{for } k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & \text{for } k + l > 1 \end{cases} \tag{14e}$$

The proof is given in [11].
Note that, if $\alpha = \beta$, then from (13) we have

$$\Phi_0\big|_{\alpha=\beta}(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{15}$$

The transfer matrix of the system (8) is given by

$$T(s^\alpha, s^\beta) = \overline{C}\left[\begin{bmatrix} I_{n_1}s^\alpha & 0 \\ 0 & I_{n_2}s^\beta \end{bmatrix} - \overline{A}\right]^{-1} \overline{B}. \tag{16}$$

## III. FRACTIONAL DIFFERENT ORDERS NONLINEAR FEEDBACK SYSTEMS WITH POSITIVE LINEAR PARTS

Consider the nonlinear feedback system shown in Figure 1, which consists of the positive linear part, the nonlinear element with characteristic $u = f(e)$ and the positive scalar feedback. The positive linear part is described by the equations

$$\begin{bmatrix} \dfrac{d^\alpha x_1(t)}{dt^\alpha} \\ \dfrac{d^\beta x_2(t)}{dt^\beta} \end{bmatrix} = \overline{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overline{B}u(t),$$
$$y(t) = \overline{C}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{17}$$

where $0 < \alpha, \beta < 1$, $x_1 = x_1(t) \in \Re^{n_1}$ and $x_2 = x_2(t) \in \Re_+^{n_2}$ are the state vectors, $u = u(t) \in \Re$ is the input vector, $y = y(t) \in \Re$ is the input vector, matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ for $p = m = 1$ are defined by (10).

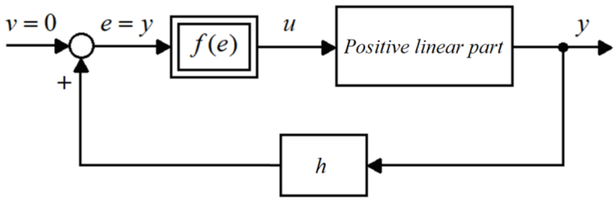


Figure 1. The nonlinear feedback system.

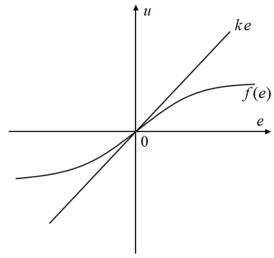


Figure 2. Characteristic of the nonlinear element.

The characteristic of the nonlinear element is shown in Figure 2 and it satisfies the condition

$$0 < f(e) < ke, \quad 0 < k < \infty. \tag{18}$$

It is assumed that the positive linear part is asymptotically stable (the matrix $\overline{A} \in M_n$ is Hurwitz).

**Definition 4.** The nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in \Re_+^n$.

The following theorem gives sufficient conditions for the global stability of the positive nonlinear system.

**Theorem 6.** The nonlinear system consisting of the positive linear part, the nonlinear element satisfying the condition (18) and the positive scalar feedback *h* is globally stable if the matrix

$$\overline{A} + kh\overline{B}\overline{C} \in M_n \tag{19}$$

is asymptotically stable (Hurwitz matrix).
Matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ are given by (10).

**Proof.** The proof will be accomplished by the use of the Lyapunov method [19][20]. As the Lyapunov function $\overline{V}(x)$, we choose

$$\overline{V}(x) = V_1(x) + V_2(x) = \lambda_1^T x_1 + \lambda_2^T x_2 \geq 0 \text{ for}$$
$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Re_+^n, \quad \overline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \Re_+^n, \tag{20}$$

where $\overline{\lambda}$ are strictly positive vectors with all positive components.
Using (20) and (17), we obtain

$$\frac{d^\alpha V_1(x)}{dt^\alpha}+\frac{d^\beta V_2(x)}{dt^\beta}=[\lambda_1^T \quad \lambda_2^T]\begin{bmatrix}\frac{d^\alpha x_1}{dt^\alpha}\\ \frac{d^\beta x_2}{dt^\beta}\end{bmatrix} \tag{21}$$
$$=\overline{\lambda}^T(\overline{A}\overline{x}+\overline{B}u)\le\overline{\lambda}^T(\overline{A}x+kh\overline{B}\overline{C})\overline{x}$$

since $u=f(e)\le ke=kh\overline{C}\overline{x}$.

From (21), it follows that $\frac{d^\alpha V_1(x)}{dt^\alpha}+\frac{d^\beta V_2(x)}{dt^\beta}<0$ if the matrix (19) is Hurwitz and the nonlinear system is globally stable. □

**Example 1.** Consider the nonlinear system with the positive linear part with the matrices

$$A=\begin{bmatrix}-3 & 0.5 & 0.2 & 0.1\\ 1 & -2 & 0.2 & 0.3\\ 0.2 & 0.3 & -5 & 0.4\\ 0.3 & 0.4 & 0.5 & -4\end{bmatrix},\ B=\begin{bmatrix}0.5\\0.2\\0.6\\0.4\end{bmatrix},\ C=[0.2 \quad 0.4 \quad 0.5 \quad 0.3],$$
$$h=0.5,\ \ \alpha=0.4,\ \ \beta=0.6,\ \ n_1=n_2=2\,, \tag{22}$$

the nonlinear element satisfying the condition (18) and the positive feedback with gain $h$. Find $k$ satisfying (19) for which the nonlinear system is globally stable for $h=0.5$.
Using (14) and (17) for $h=0.5$, we obtain

$$\hat{A}=\overline{A}+kh\overline{B}\overline{C}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}+kh\begin{bmatrix}B_1\\B_2\end{bmatrix}[C_1 \quad C_2]$$
$$=\begin{bmatrix}-3+0.05k & 0.5+0.1k & 0.2+0.125k & 0.1+0.075k\\ 1+0.02k & -2+0.04k & 0.2+0.05k & 0.3+0.03k\\ 0.2+0.06k & 0.3+0.12k & -5+0.15k & 0.4+0.09k\\ 0.3+0.04k & 0.4+0.08k & 0.5+0.1k & -4+0.06k\end{bmatrix}. \tag{23}$$

The characteristic polynomial of the matrix (23) has the form

$$\det(I_4 s-\hat{A})=s^4+(14-0.3k)s^3+(70.05-3.31k)s^2$$
$$+(146.39-11.99k)s+(104.64-14.28k) \tag{24}$$

and its coefficients are positive, which implies that the nonlinear system with (22) is globally stable for $k<7.33$.

**Remark 1.** The determinant of the matrix (23) has the form

$$\det(\hat{A})=104.64-14.28k \tag{25}$$

and it is equal to zero for $k=7.33$.

## IV. CONCLUSIONS

The global stability of continuous-time different fractional orders nonlinear feedback systems with positive linear parts and positive scalar feedback has been investigated. New sufficient conditions for the global stability of this class of positive nonlinear systems have been established (Theorem 6). The effectiveness of these new stability conditions has been demonstrated on simple a example of positive nonlinear different orders system. The considerations can be extended to discrete-time standard fractional different orders nonlinear systems with positive linear parts and scalar feedback. An open problem is an extension of the considerations to nonlinear different orders fractional systems with interval matrices of their positive linear parts.

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