Tensor-Based Recursive Least-Squares Algorithm with Low Computational Complexity

Ionuţ-Dorinel Fîciu, Cristian-Lucian Stanciu, Laura-Maria Dogariu, Camelia Elisei-Iliescu, and Constantin Paleologu
Department of Telecommunications, University Politehnica of Bucharest, Romania
Email: ionut_dorinel.ficiu@upb.ro, {cristian, ldogariu, pale}@comm.pub.ro, camelia.elisei@romatsa.ro

***Abstract*—Many system identification problems can be addressed based on tensor decomposition methods. In this framework, the conventional Recursive Least-Squares (RLS) algorithm requires a prohibitive amount of arithmetic resources and is sometimes prone to numerical stability issues. This paper presents a low-complexity RLS-based algorithm for multiple-input/single-output system identification, which results as a combination between the exponentially weighted RLS algorithm and the Dichotomous Coordinate Descent (DCD) iterations.**

***Index Terms*—adaptive filter; multilinear forms; recursive least-squares (RLS); dichotomous coordinate descent (DCD); tensor decomposition**

I. INTRODUCTION

The identification of multilinear forms (or linearly separable systems) can be efficiently exploited in the framework of different applications [1]. Such scenarios can appear in the framework of multichannel systems, e.g., with a large number of sensors and actuators. In these contexts, the basic approach relies on tensor decomposition and modeling techniques, since the multilinear forms can be modeled as rank-1 tensors. The main idea is to combine (i.e., "tensorize") the solutions of low-dimension problems, in order to efficiently solve a multidimensional system identification problem, which is usually characterized by a large parameter space.

For the system identification implementations, which can be decomposed using Multiple-Input/Single-Output (MISO) setups, several tensor-based models were recently proposed [1]. One such decomposition uses the Recursive Least-Squares (RLS) method based on Woodburry's identity to split the unknown system determination into multiple smaller adaptive systems. Despite the fact that the overall complexity is reduced, the algorithm previously introduced as tensor-based RLS (RLS-T) is still dependent on the square of each filter's length, and it is also prone to inherit the problems of classical least-squares solutions [2].

The combination with the Dichotomous Coordinate Descent (DCD) iterations has been established as a possible stable alternative with lower complexity traits, proportional to the filter's length [3]. Based on this idea, the current paper presents a tensorial decomposition for multilinear forms based on the RLS-DCD method. For the identification of an unknown MISO system based on its tensorial form, multiple RLS-DCD shorter filters are employed, which inherit the performance of the classical RLS versions and require lower arithmetic efforts.

In the following, Section II introduces the framework of multilinear forms, while Section III presents the proposed algorithm. Finally, several conclusions are summarized in Section IV.

II. MULTILINEAR FORMS

In the framework of a real-valued MISO system, the output signal (at discrete-time index n) is defined as

$$y(n) = \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} x_{l_1 l_2 \ldots l_N}(n) h_{1,l_1} h_{2,l_2} \cdots h_{N,l_N},$$

where $\mathbf{h}_i = [h_{i,1}\ h_{i,2}\ \cdots\ h_{i,L_i}]^T$ are N individual channels, each one of length $L_i,\ i = 1, 2, \ldots, N$, and the superscript ^{T} denotes the transpose operator. The input signals can be described in the tensorial form $\boldsymbol{\mathcal{X}}(n) \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_N}$, having the elements $(\boldsymbol{\mathcal{X}})_{l_1 l_2 \ldots l_N}(n) = x_{l_1 l_2 \ldots l_N}(n)$. Thus, the output signal becomes

$$y(n) = \boldsymbol{\mathcal{X}}(n) \times_1 \mathbf{h}_1^T \times_2 \mathbf{h}_2^T \times_3 \cdots \times_N \mathbf{h}_N^T, \qquad (1)$$

where $\times_i,\ i = 1, 2, \ldots, N$ denotes the mode-i product. As we can notice, $y(n)$ is a multilinear form, since it is a linear function of each $\mathbf{h}_i,\ i = 1, 2, \ldots, N$, when the other $N-1$ components are fixed.

Let us consider the rank-1 tensor $\boldsymbol{\mathcal{H}} \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_N}$, with the elements $(\boldsymbol{\mathcal{H}})_{l_1, l_2, \ldots, l_N} = h_{1,l_1} h_{2,l_2} \cdots h_{N,l_N}$, such that $\boldsymbol{\mathcal{H}} = \mathbf{h}_1 \circ \mathbf{h}_2 \circ \cdots \circ \mathbf{h}_N$, where $\circ$ denotes the outer product. In addition, we have $\text{vec}(\boldsymbol{\mathcal{H}}) = \mathbf{h}_N \otimes \mathbf{h}_{N-1} \otimes \cdots \otimes \mathbf{h}_1$, where $\text{vec}(\cdot)$ is the vectorization operation and $\otimes$ denotes the Kronecker product. Hence, we can rewrite (1) as $y(n) = \text{vec}^T(\boldsymbol{\mathcal{H}})\, \text{vec}[\boldsymbol{\mathcal{X}}(n)]$.

Furthermore, we can denote $\mathbf{x}(n) = \text{vec}[\boldsymbol{\mathcal{X}}(n)]$ and $\mathbf{g} = \text{vec}(\boldsymbol{\mathcal{H}})$. Here, $\mathbf{x}(n)$ is the input vector of length $L_1 L_2 \cdots L_N$ and $\mathbf{g}$ plays the role of a global impulse response (of the same length). Therefore, (1) finally becomes $y(n) = \mathbf{g}^T \mathbf{x}(n)$, while the reference signal results as

$$d(n) = \mathbf{g}^T \mathbf{x}(n) + w(n), \qquad (2)$$

where $w(n)$ is the measurement noise, which is uncorrelated with the input signals. The main goal is the identification of the global system $\mathbf{g}$. Equivalently, the identification problem can be formulated in terms of recursively estimating the individual components $\mathbf{h}_i,\ i = 1, 2, \ldots, N$.

III. RLS-Based Algorithms

The faster convergence rate is one of the main advantages of the RLS methods, with respect to the performances obtained by other families of algorithms. However, this convergence aspect comes together with a very high computational complexity. It was previously demonstrated that tensor-based algorithms could produce better results than the classical RLS approach, by splitting the long filter associated with the unknown system identification problem into multiple smaller filters, i.e., into multiple smaller system identification processes. This results in a significant decrease in the number of mathematical operations and improved convergence rates.

Let us consider the estimated impulse responses of the channels, $\widehat{\mathbf{h}}_i(n),\ i = 1, 2, \ldots, N$, and the estimated output, $\widehat{y}(n)$, such that the error signal results in N equivalent ways:

$$e(n) = d(n) - \widehat{y}(n) = d(n) - \widehat{\mathbf{h}}_i^T(n-1)\mathbf{x}_{\widehat{\mathbf{h}}_i}(n), \quad (3)$$

for $i = 1, 2, \ldots, N$, where

$$\mathbf{x}_{\widehat{\mathbf{h}}_i}(n) = \Big[\widehat{\mathbf{h}}_N(n-1) \otimes \widehat{\mathbf{h}}_{N-1}(n-1) \otimes \cdots \otimes \widehat{\mathbf{h}}_{i+1}(n-1)$$
$$\otimes \mathbf{I}_{L_i} \otimes \widehat{\mathbf{h}}_{i-1}(n-1) \otimes \cdots \otimes \widehat{\mathbf{h}}_2(n-1) \otimes \widehat{\mathbf{h}}_1(n-1)\Big]^T \mathbf{x}(n),$$

with $\mathbf{I}_{L_i}$ denoting the identity matrices of sizes $L_i \times L_i$.

At this point, we can apply the least-squares (LS) error criterion [2] in the context of (2) and (3). Thus, the cost functions can be formulated in N alternative ways, following the optimization procedure of the individual impulse responses. Furthermore, the minimization of these cost functions with respect to $\widehat{\mathbf{h}}_i(n),\ i = 1, 2, \ldots, N$ leads to the set of normal equations, which result in the updates of the individual filters of the RLS-T algorithm [1].

In this idea paper, we propose to use the combination between the DCD iterations and the RLS method, in the tensorial framework. The RLS-DCD algorithm [3] was employed in the past due to its low complexity arithmetic workloads and improved numerical stability. The advantages obtained by using the generalized tensorial model applied with low-complexity RLS algorithms can lead to high convergence/tracking speeds and acceptable computational requirements, an overall design that is suitable for efficient hardware applications.

The proposed tensor-based RLS-DCD (RLS-DCD-T) algorithm is summarized in Table I. For each of the corresponding channels, the RLS-DCD-T is designed with an overall complexity proportional to the length of the associated adaptive filter, in terms of additions and multiplications. No divisions are needed to perform the filter update process or to generate the output information. Consequently, the complexity of the Exponential Weighted RLS-DCD-T algorithm presented in Table I reflects its split functionality design. The overall computational effort is a sum of values proportional to the individual filter lengths, in terms of multiplications, respectively additions. Considering the fact that decompositions can be performed such that $L_i \ll L$, the proposed reduction in complexity represents a migration from a setup difficult to implement on hardware platforms to an attractive solution for multiple adaptive systems configurations.

Table I. Exponential weighted RLS-DCD-T algorithm (filter i).

Initialization :
$\widehat{\mathbf{h}}_i(0) = \mathbf{r}_i(0) = \mathbf{0}_{L_i \times 1},\ \mathbf{R}_i(0) = \xi_i \mathbf{I}_{L_i},\ \xi_i > 0,\ 0 < \lambda_i \le 1$
For $n = 1, 2, \ldots,$ number of iterations :
$\mathbf{R}_i^{(1)}(n) = \lambda_i \mathbf{R}_i^{(1)}(n-1) + \mathbf{x}_{\widehat{\mathbf{h}}_i}(n)\mathbf{x}_{\widehat{\mathbf{h}}_i}^{(1)}(n)$
$\quad\mathbf{R}_i^{(1)}(n)$ denotes the 1^{st} column of $\mathbf{R}_i^{-1}(n)$
$\quad\mathbf{x}_{\widehat{\mathbf{h}}_i}^{(1)}(n)$ denotes the 1^{st} element of $\mathbf{x}_{\widehat{\mathbf{h}}_i}(n)$
$e_{\widehat{\mathbf{h}}_i}(n) = d(n) - \widehat{\mathbf{h}}_i^T(n-1)\mathbf{x}_{\widehat{\mathbf{h}}_i}(n)$
$\mathbf{p}_{0,i}(n) = \lambda_i \mathbf{r}_i(n-1) + e_{\widehat{\mathbf{h}}_i}(n)\mathbf{x}_{\widehat{\mathbf{h}}_i}(n)$
$\mathbf{R}_i(n)\Delta\mathbf{h}_i(n) = \mathbf{p}_{0,i}(n) \xrightarrow{\text{DCD}} \Delta\widehat{\mathbf{h}}_i(n), \mathbf{r}_i(n)$
$\widehat{\mathbf{h}}_i(n) = \widehat{\mathbf{h}}_i(n-1) + \Delta\widehat{\mathbf{h}}_i(n)$

Using the conventional RLS family of algorithms [2] (i.e., the direct estimation of the global impulse response) could be very costly for large values of L, since the computational complexity order would be $\mathcal{O}(L^2)$. On the other hand, the computational complexity of the RLS-T algorithms is proportional to $\sum_{i=1}^{N} \mathcal{O}(L_i^2)$, which could be much more advantageous when $L_i \ll L$. In addition, since the RLS-T operates with shorter filters, improved performance is expected, as compared to the conventional RLS algorithm. The RLS-DCD-T brings an extra layer of efficiency by performing the same tasks with workloads of order $\sum_{i=1}^{N} \mathcal{O}(L_i)$. These observations are also supported by our preliminary experimental results, which will be reported in future works.

IV. Conclusion

This idea paper has introduced a low-complexity RLS-based adaptive algorithm for the identification of unknown systems based on tensorial decompositions. The resulting RLS-DCD-T algorithm benefits from the low computational requirements of the DCD iterations and could provide performance comparable with other established versions of tensorial based RLS methods. The reduction in complexity for the adaptive filter update process is important. Most importantly, the usage of the DCD iterations allows for the coefficient updates to be performed using only bit-shifts and additions.

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References

[1] L.-M. Dogariu et al., “Tensor-based adaptive filtering algorithms,” *Symmetry*, vol. 13, id 481 (27 pages), Mar. 2021.
[2] S. Haykin, *Adaptive Filter Theory*. Fourth Edition, Upper Saddle River, NJ, USA: Prentice-Hall, 2002.
[3] Y. V. Zakharov, G. P. White, and J. Liu, “Low-complexity RLS algorithms using dichotomous coordinate descent iterations,” *IEEE Trans. Signal Processing*, vol. 56, pp. 3150–3161, Jul. 2008.