Direct Adaptive Input Matrix Estimation Approaches for Linear Time-Invariant Dynamic Systems with Known Inputs

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***Abstract*—Equations Of Motion (EOM) can effectively describe physical dynamics within a prescribed set of assumptions and constraints. However, in many physical dynamic systems, performance can alternate and deteriorate with age, use, or alterations in operational points or environments. The culmination of these factors will be denoted as a health change later in the text. These physical changes can be characterized as alterations in the constitutive constants, which are denoted as mass, stiffness, and damping for a mechanical system, and internal interactions. If these health changes are not taken into account in the modeled EOM, discrepancies may emerge between the physical and model responses. The paper investigates two scenarios: 1) both the true plant and the input matrix experience a form of health change, and 2) only the input matrix experiences a form of health change. The control schemes depend on knowing the true system's input and output states. For case 1: Lyapunov stability proof guarantees internal and external state error convergence to zero asymptotically if the true system experiences health changes within the assumptions and constraints of the proposed control scheme. For case 2: the internal and external state errors converge asymptotically to a neighborhood of zero rather than to zero itself. This behavior results from the presence of a feedback filter in the adaptive input matrix estimation law, which restricts full convergence but enables faster state-error convergence compared to the case without a feedback filter.**

Keywords-Adaptive; Control; Estimation; Plant; Input Matrix.

I. INTRODUCTION

This article extends our original *Adaptive 2025* conference paper by considering feedback filters on the adaptive laws: adaptive state and adaptive input matrix estimation laws [1]. In earlier work, a feedback filter was successfully applied only to the adaptive state estimator law [2]. Initial attempts to apply feedback filters to both adaptive laws simultaneously, or to apply a feedback filter to one while omitting it from the other, did not yield promising simulation results. This is likely due to the imbalance in managing plant and input matrix uncertainty, while the feedback filter enables accelerated convergence relative to the non-feedback filtered case. Given the lack of favorable simulation outcomes, the final implementation applied the feedback filter solely to the adaptive input matrix estimator, while the adaptive state-estimation law was omitted. This article presents two proofs: 1) adaptive state and adaptive input matrix estimation, and 2) adaptive input matrix estimation using a feedback filter.

Before discussing the adaptive laws, it is import to understand their need. Equations Of Motion (EOM) can describe the dynamics of the true physical system within a set of assumptions and constraints. The EOM cannot guarantee anything beyond the prescribed conditions. With age, usage, changes in operational points, or environmental elements, the true physical system can undergo degradation. Failing to account for these degradations or health changes resulting from changes in internal interactions or constitutive constants—such as mass, stiffness, and damping in mechanical systems—can lead to an inaccurate depiction of the true dynamics. Equally important, but often overlooked is the potential decline of the system's actuator, which influences how inputs interact with the physical system. A substantial portion of control problems involves the regulation of output errors concerning a given input. Ignoring the health status changes in system dynamics or actuation can result in catastrophic failure if synthesized inputs does not adequately address these changes.

For traditional Luenberger or Kalman-like estimators to be practical, there has to be minimal uncertainty about the system [3], [4]. Unlike Luenberger Estimators, Kalman-like filters are renowned for their ability to account for noise and stochastic variations resulting from sensor or process disturbances, under the assumption that the noise follows a Gaussian distribution centered around zero. However, neither type of estimator is capable of accommodating changes in the health status of the system dynamics or the input matrix.

The sensitivity of Luenberger and Kalman-like estimators to minimal uncertainty regarding system dynamics motivates the development of robustness techniques to address model uncertainty [5], [6]. The control technique presented in this text can manage both plant and input matrix uncertainties. More importantly, it can also accommodate significant changes in system health, as defined in the derivation. This work builds upon our earlier findings, which indicated that only the true-physical plant experiences a health status change, causing changes in dynamics and constitutive constants [7]. In 2022, *Griffith* developed a closed-loop approach for input matrix estimation [8]. This paper explores the scenario in which the plant and the input matrix experience a change in health and is an extension of [1].

The implemented control architecture was designed for a general system and can be applied to any system that meets the assumptions and constraints outlined in the proof. The proof relies on two primary Lyapunov system stability criteria: Strict Positive Real (SPR) and Almost Strictly Dissipative (ASD). For a more formal definition and detailed explanation of SPR and ASD in the context of stability, please refer to [7], [9]. Moreover, since none of the estimated states are fed back to the true system, the estimator can operate without risking harm to the true system. Additionally, the proposed control scheme can be utilized offline and online. However, there can exist practical and numerical limits.

Following the introduction, this paper is divided into additional sections: III. Main Result - Theorem 1, IV. Main Result - Theorem 2, V. Illustrative Example - Theorem 1, VI. Illustrative Example - Theorem 2, and VII. Discussion. The beginning of Section III offers a summary of the derivation process for adaptive state and adaptive input matrix estimation, presenting one of the paper's theorem and control diagram. Sub-sections III-A and III-B provide the assumptions and constraints for both the true and reference systems while laying the foundation for updating the reference model. Sub-Section III-C defines the error states and their dynamics. In the error dynamics, residual terms exist; therefore, error states cannot be guaranteed to converge to zero. To address this issue, the error dynamic states are treated as energy-like terms. Then, an energy-like balance is constructed to remove residual terms, guaranteeing the error state to converge to zero globally as time approaches infinity. This process is detailed in Section III-D and Section III-E. A similar procedure is presented in Section IV for Theorem 2, adaptive input matrix estimation using a feedback filter. Following the derivations, Section V and Section VI present two Illustrative Examples detailing the implementation of the derived control schemes. These examples detail a scenario where the error state takes a relatively long and short time to converge. In particular, the interaction tuning terms $\{\gamma_u, \gamma_u\}$ were left unadjusted. These terms can affect the time at which the error state converges. Finally, the paper ends with two sections: VII. Discussion and VIII. Conclusion.

II. Nomenclature

A	=	True Plant
A_m	=	Model Plant
ASD	=	Almost Strictly Dissipative
B	=	Input Matrix
B_m	=	Input Matrix Model
$\in$	=	Belongs
C	=	Output Matrix
$(\cdot)^\dagger$	=	Conjugate Transpose
e_x	=	Internal State Error
$\hat{e}_y$	=	External State Error
$\hat{}$	=	Estimate
$\forall$	=	For All
L_*	=	Fixed Correction Matrix
γ	=	Interaction Tuning Term
ΔL	=	Variance Matrix
PR	=	Positive Real
SPR	=	Strictly Positive Real
σ	=	Set of Eigenvalues
$\ni$	=	Such that
Re	=	Real
$\exists$	=	There Exists
u	=	Input
x	=	Internal State
y	=	External (Output) State

III. Main Result - Theorem 1

Pertaining to the work being presented, the derived theorem and control laws, shown in Theorem 1 and Fig. 1, are catered to minimizing the internal state error (e_x) to zero between the true-physical system and reference model. This is achieved by accounting for discrepancies in the model plant (A_m) and input matrix (B_m), given a known input (u), output matrix (C), and external state (y). Uncertainty or variability in the model plant and input matrix means the convergence of the internal state error to zero cannot be guaranteed. As detailed in the derivation, to mitigate any variability, the error system is treated as an energy-like term. The aim is to dissipate all the energy of the error system, thereby ensuring the internal state error converges to zero as time approaches infinity, $e_x \xrightarrow[t\to\infty]{} 0$. To ensure error energy-like dissipation, the energy-like time rate of change for the error system is determined. Subsequently, residual energy-like time rate of change terms from any uncertainty are identified and countered. The remaining energy-like time rate of change term and the use of stability lemma, Barbalat-Lyapunov Lemma, ensures $e_x \xrightarrow[t\to\infty]{} 0$ asymptotically.

Theorem 1: Output Feedback on Reference Model for Adaptive Input Matrix, Plant, and State Estimation.
Consider the following state error system:

$$\begin{cases} \dot{e}_x = (A_m - KC)e_x + B_m\left(\Delta L_1 u + \Delta L_2 y\right) \\ \hat{e}_y = Ce_x = C(\hat{x} - x) \\ L_1 = \Delta L_1 + L_{1*} \\ L_2 = \Delta L_2 + L_{2*} \\ \dot{L}_1 = \Delta \dot{L}_1 = -\hat{e}_y u^\dagger \gamma_u \\ \dot{L}_2 = \Delta \dot{L}_2 = -\hat{e}_y y^\dagger \gamma_y \end{cases}, \quad (1)$$

where e_x is the estimated internal state error, $\hat{e}_y$ is the external estimated state error, $\{L_{1*}, L_{2*}\}$ are fixed-correction matrices, $\{\Delta L_1, \Delta L_2\}$ are the variability-uncertainty terms, K is a fixed gain, and $\{\gamma_u, \gamma_y\} > 0$ are the interaction tuning terms. Given:

1) The triples of (A, B, C) and (A_m, B_m, C) are ASD and SPR respectively.
2) A model plant (A_m) must exist.
3) A model input matrix (B_m) must exist.
4) Output matrix (C) is known.
5) Allow $B \in \text{Sp}\{B_m L_{1*}\} \ni B \equiv B_m L_{1*}$.
6) Allow $A \in \text{Sp}\{A_m, B_m L_{2*} C\} \ni A = A_m + B_m L_{2*} C$.
7) The set of eigenvalues (σ) of the true and reference plant are stable (i.e. $\text{Re}(\sigma(A)) < 0$ & $\text{Re}(\sigma(A_m)) < 0$).

If conditions are met, then $\{e_x, \hat{e}_y\} \xrightarrow[t\to\infty]{} 0$ asymptotically. $\{\Delta L_1, \Delta L_2\}$ are guaranteed to be bounded; however, no guarantee of $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$. If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the dynamics of the true system or some energy equivalence have been numerically captured.

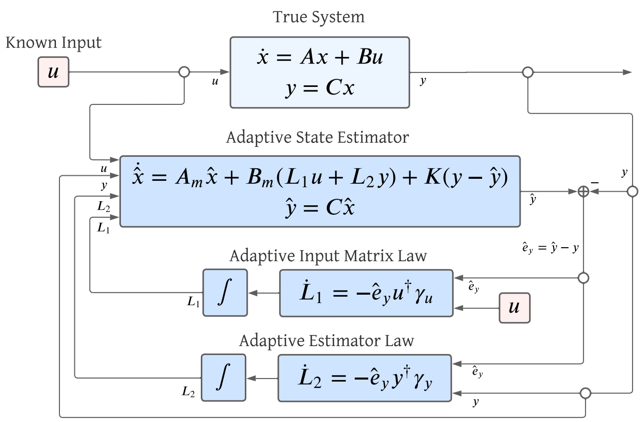

Figure 1. Control diagram for adaptive plant, input matrix, and state estimation given a known input (u), output matrix (C), and external state (y).

A. Defining True System Dynamics

Assume the dynamics of the true-physical system is linear time-invariant and therefore can be expressed in state-space form such that:

$$\text{True System}\begin{cases}\dot{x} = Ax + Bu\\ y = Cx.\end{cases} \tag{2}$$

Both the true system's plant (A), assumed to be stable (i.e. $\text{Re}(\sigma\{A\}) < 0$), and the input matrix (B) experience a health change caused by age or use, altering the constitutive constants and system dynamics. Output matrix (C) and external (output) state (y) are known. The input (u) can be any bounded-continuous waveform the user provides, possibly a known disturbance.

B. Overview of Updating the Reference Model

Subsequent sections will derive a control scheme and laws to minimize the error between the true and reference systems, Eq. (2) and Eq. (3), respectively. Note that both true and model systems match in dimension size, but can differ in constitutive constant values and internal interactions. Allow the reference model to have the following state space characteristics:

$$\text{Reference Model}\begin{cases}\dot{x}_m = A_m x_m + B_m u\\ y_m = Cx_m.\end{cases} \tag{3}$$

To update the input matrix model (B_m), assume that B_m can be corrected via a input matrix fixed correction term (L_{1*}) such that:

$$B \equiv B_m L_{1*}. \tag{4}$$

The true plant is assumed to be decomposed into two components: an initial plant model (A_m) and plant matrix correction term $(B_m L_{2*} C)$ such that:

$$A \equiv A_m + B_m L_{2*} C. \tag{5}$$

Both Eq. (4) and Eq. (5) assumed decompositions are structured such that they can be modified via an estimator. In the estimator, the initial input matrix and plant are updated via their respective correction term $\{L_1, L_2\}$:

$$L(t) = \Delta L + L_* \xrightarrow[t\to\infty]{} L(t) = L_*, \tag{6}$$

where ΔL is the variability-uncertainty term. If both variability term converges to zero, $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the input matrix and true plant (or energy equivalent) have been numerically captured. For the control scheme to apply, the true and reference systems must be ASD and SPR, respectively.

C. Estimated State Error

Given that the true plant (A) and input matrix (B) experiences a health change caused by age or use, and the internal state (x) is often blended into a linear combination or missing from the external state (y), an estimator can be created using the reference model:

$$\text{Estimator}\begin{cases}\dot{\hat{x}} = A_m\hat{x} + B_m(L_1 u + L_2 y)\\ \hat{y} = C\hat{x}.\end{cases} \tag{7}$$

To minimize the error between the true and estimated systems, consider the following error state equations:

$$\begin{cases}e_x = \hat{x} - x\\ \hat{e}_y = \hat{y} - y = Ce_x.\end{cases} \tag{8}$$

To capture the internal state of the true system, the internal state error must converge to zero as time approaches infinity. To investigate the internal state error dynamics, take the time derivative of the internal state error and substitute Eq. (2) and Eq. (7):

$$\begin{aligned}\dot{e}_x &= \dot{\hat{x}} - \dot{x}\\ &= A_m\hat{x} + B_m(L_1 u + L_2 y) - (Ax + Bu).\end{aligned} \tag{9}$$

From Eq. (9), consider the difference between input matrices:

$$\begin{aligned}B_m(\underbrace{\Delta L_1 + L_{1*}}_{=L_1})u - \underbrace{B_m L_{1*}}_{=B} u &= B_m \underbrace{\Delta L_1 u}_{=w_u}\\ &= B_m w_u.\end{aligned} \tag{10}$$

Again, using Eq. (9) as a reference, consider the difference between the model and true plants, where $A \equiv A_m + B_m L_{2*} C$:

$$\begin{aligned}A_m\hat{x} + B_m(\underbrace{\Delta L_2 + L_{2*}}_{=L_2})y - Ax =& A_m e_x + B_m \underbrace{\Delta L_2 y}_{=w_y}\\ =& A_m e_x + B_m w_y.\end{aligned} \tag{11}$$

Therefore, the error system can be written as:

$$\begin{cases}\dot{e}_x = A_m e_x + B_m(w_u + w_y)\\ \hat{e}_y = Ce_x.\end{cases} \tag{12}$$

Additionally, the estimator can be extended to use a fixed gain (K):

$$\begin{cases}\dot{\hat{x}} = A_m\hat{x} + B_m(L_1 u + L_2 y) + K(y - \hat{y})\\ \hat{y} = C\hat{x}.\end{cases} \tag{13}$$

Resulting in the following error equation:

$$\begin{cases} \dot{e}_x = (\underbrace{A_m - KC}_{=A_c})e_x + B_m(w_u + w_y) \\ \hat{e}_y = Ce_x. \end{cases} \tag{14}$$

To use Eq. (14), find a fixed gain $(K) \ni \text{Re}(\sigma\{A_m - KC\}) < 0$.

Regardless of the estimator selected, the internal state error (e_x) can not be guaranteed to converge such that $e_x \xrightarrow[t\to\infty]{} 0$ due to the residual terms $\{w_u, w_y\}$ existing in the error equation. To adequately address these residual components, additional considerations are needed.

D. Lyapunov Stability for the Estimated State Error

Lyapunov stability analysis represents dynamic systems in terms of energy-like functions to describe the convergence of a particular or a set of states. For this case study, Lyapunov stability is used to guarantee the convergence of internal state error $(e_x) \ni e_x \xrightarrow[t\to\infty]{} 0$.

Given the state error equation as described in Eq. (12), consider the following energy-like Lyapunov equation with assumed real scalars:

$$V_e = \frac{1}{2}e_x^\dagger P_x e_x; P_x > 0, \tag{15}$$

where the $(\cdot)^\dagger$ is the conjugate transpose and where $P_x > 0$ represents a matrix P_x that is symmetric $(P_x = P_x^\dagger)$ and positive-definite $(\text{Re}(\sigma\{P_x\}) > 0)$.

To determine the energy-like time rate of change of V_e, take the time derivative of V_e and substitute Eq. (12) for the error dynamics:

$$\begin{aligned} 2\dot{V}_e =& \dot{e}_x^\dagger P_x e_x + e_x^\dagger P_x \dot{e}_x \\ =& (A_m e_x + B_m(w_u + w_y))^\dagger P_x e \\ &+ e_x^\dagger P_x(A_m e_x + B_m(w_u + w_y)) \\ =& e_x^\dagger(A_m^\dagger P_x + A_m P_x)e_x + 2\underbrace{e_x^\dagger P_x B_m(w_u + w_y)}_{=(B_m(w_u+w_y))^\dagger P_x e_x}. \end{aligned} \tag{16}$$

Modifying the SPR stability condition for the reference model:

$$\begin{cases} A_m^\dagger P_x + P_x A_m = -Q_x \\ P_x B_m = C^\dagger \end{cases} ; Q_x > 0. \tag{17}$$

From here, the SPR condition can be applied to Eq. (16), resulting in:

$$\begin{aligned} 2\dot{V}_e &= -e_x^\dagger Q_x e_x + 2\underbrace{e_x^\dagger C^\dagger}_{=\hat{e}_y^\dagger}(w_u + w_y) \\ &= -e_x^\dagger Q_x e_x + 2\hat{e}_y^\dagger w_u + 2\hat{e}_y^\dagger w_y \\ &= -e_x^\dagger Q_x e_x + 2\underbrace{(\hat{e}_y, w_u)}_{=(w_u,\hat{e}_y)} + 2\underbrace{(\hat{e}_y, w_y)}_{=(w_y,\hat{e}_y)}. \end{aligned} \tag{18}$$

By removing the residual terms $\{(\hat{e}_y.w_u), (\hat{e}_y, w_y)\}$ in Eq. (18), results in $\dot{V}_e \leq 0$.

To counter the residual terms, consider the following energy-like functions:

$$V_u + V_y = \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger) + \frac{1}{2}\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta L_2^\dagger), \tag{19}$$

where $\{\gamma_u, \gamma_y\} > 0$. The energy-like time rate of change for $V_u + V_y$ follows:

$$\dot{V}_u + \dot{V}_y = \underbrace{\text{tr}(\Delta \dot{L}_1 \gamma_u^{-1} \Delta L_1^\dagger)}_{=\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta \dot{L}_1^\dagger)} + \underbrace{\text{tr}(\Delta \dot{L}_2 \gamma_y^{-1} \Delta L_2^\dagger)}_{=\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta \dot{L}_2^\dagger)}. \tag{20}$$

A control law for the input matrix and plant variance time rate of change $\{\Delta \dot{L}_1, \Delta \dot{L}_2\}$ can be defined as the following:

$$\begin{cases} \Delta \dot{L}_1 = -\hat{e}_y u^\dagger \gamma_u \\ \Delta \dot{L}_2 = -\hat{e}_y y^\dagger \gamma_y. \end{cases} \tag{21}$$

Substituting Eq. (21) into Eq. (20):

$$\begin{aligned} \dot{V}_u + \dot{V}_y =& \text{tr}(\underbrace{-\hat{e}_y u^\dagger \gamma_u}_{\Delta \dot{L}_1} \gamma_u^{-1} \Delta L_1^\dagger) \\ &+ \text{tr}(\underbrace{-\hat{e}_y y^\dagger \gamma_y}_{\Delta \dot{L}_2} \gamma_y^{-1} \Delta L_2^\dagger) \\ =& -\text{tr}(\hat{e}_y \underbrace{u^\dagger \Delta L_1^\dagger}_{=w_u^\dagger}) - \text{tr}(\hat{e}_y \underbrace{y^\dagger \Delta L_2^\dagger}_{=w_y^\dagger}) \\ =& -\text{tr}(\hat{e}_y w_u^\dagger) - \text{tr}(\hat{e}_y w_y^\dagger) \\ =& -\text{tr}(w_u^\dagger \hat{e}_y) - \text{tr}(w_y^\dagger \hat{e}_y) \\ =& -w_u^\dagger \hat{e}_y - w_y^\dagger \hat{e}_y \\ =& -(w_u, \hat{e}_y) - (w_y, \hat{e}_y). \end{aligned} \tag{22}$$

For notation purposes, allow the following:

$$\begin{cases} V_{euy} = V_e + V_u + V_y \\ \dot{V}_{euy} = \dot{V}_e + \dot{V}_u + \dot{V}_y. \end{cases} \tag{23}$$

From here, the estimate state error closed-loop energy-like function can be written as:

$$\begin{aligned} V_{euy} =& \frac{1}{2}e_x^\dagger P_x e_x + \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ &+ \frac{1}{2}\text{tr}(\Delta L_2 \gamma_y^{-1} \Delta L_2^\dagger). \end{aligned} \tag{24}$$

Therefore, the estimated state error closed-loop energy-like time rate of change can be written as:

$$\begin{aligned} \dot{V}_{euy} =& -\frac{1}{2}e_x^\dagger Q_x e_x + (w_u, \hat{e}_y) + (w_y, \hat{e}_y) \\ &- (w_u, \hat{e}_y) - (w_y, \hat{e}_y) \\ =& -\frac{1}{2}e_x^\dagger Q_x e_x \leq 0. \end{aligned} \tag{25}$$

Having $\dot{V}_{euy} \leq 0$ means that $\{e_x, \Delta L_1, \Delta L_2\}$ are guaranteed to be bounded. Due to $\dot{V}_{euy}$ negative-semi-definite nature, no additional information can be said about the error internal state (e_x) converging $\ni e_x \xrightarrow[t\to\infty]{} 0$.

E. Applying Barbalat-Lyapunov Lemma on $\dot{V}_{euy}$

To guarantee $e_x \xrightarrow[t\to\infty]{} 0$, consider Barbalat-Lyapunov Lemma - Given:

1) V is lower bounded.
2) $\dot{V}$ is negative-semi-definite.
3) $\dot{V}$ is uniformly continuous in time.

If all conditions are met, then $\dot{V} \xrightarrow[t\to\infty]{} 0$ according to [10].

The first two conditions of the Barbalat-Lyapunov Lemma are satisfied with Eq. (24) and Eq. (25). The third condition, $\dot{V}_{euy}$ being uniformly continuous in time, can be satisfied by showing that $\ddot{V}_{euy}$ is bounded [10].

To prove $\ddot{V}_{euy}$ is bounded, consider W_{euy}:

$$W_{euy} \geq -2\dot{V}_{euy} \geq 0. \tag{26}$$

Taking the time derivative of W_{euy} results in the following:

$$\begin{aligned}\dot{W}_{euy} =& 2e_x^\dagger Q_x \dot{e}_x \\ =& 2e_x^\dagger Q_x (A_m e_x + B_m(w_u + w_y)) \\ =& 2e_x^\dagger Q_x (A_m e_x + B_m(\Delta L_1 u + \Delta L_2 y)).\end{aligned} \tag{27}$$

From Eq. (25), $\{e_x, \Delta L_1, \Delta L_2\}$ are bounded. Input (u) can be any bounded-continuous waveform. Following, the true plant is assumed stable (i.e., $\mathrm{Re}(\sigma\{A\}) < 0$); therefore, a bounded input will result in a bounded output (y) [11]. Combining all bounded results yields: $\dot{W}_{euy}$ is indeed bounded. Making $\ddot{V}_{euy}$ bounded.

Given that all the conditions of Barbalat-Lyapunov are satisfied, $\dot{V}_{euy}$ evolution in time can be expressed as:

$$\dot{V}_{euy} \xrightarrow[t\to\infty]{} 0. \tag{28}$$

Therefore, proves $e_x \xrightarrow[t\to\infty]{} 0$ is asymptotically guaranteed. However, regardless of Barbalat-Lyapunov being satisfied, Lyapunov stability results only guarantees $\{\Delta L_1, \Delta L_2\}$ to be bounded. If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$ numerically, the true input matrix and plant or an energy equivalence have been captured. Additionally, without loss of generality, the derived Lyapunov stability proof can be modified for the error system using fixed gain, Eq. (12).

Altogether, assuming the reference (A_m, B_m, C) and true (A, B, C) systems are SPR and ASD respectfully, such that the decomposition of the true input matrix (B) and plant (A) can be written as $B \equiv B_m L_{1*}$ and $A \equiv A_m + B_m L_{2*} C$. Then adaptive laws (Eq. (21)) and diagram (Fig. 1) can be formulated such that the internal state error is guaranteed to converge to zero asymptotically. Lyapunov stability proof only guarantees that $\{\Delta L_1, \Delta L_2\}$ will be bounded. However, if $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, then the true input matrix and plant or energy equivalent have been numerically captured.

IV. Main Result - Theorem 2

An initial attempt was made to incorporate feedback filters into Theorem 1, similar to those as described in [2]. However, numerical results were not fruitful. See Section VII for additional details. This section further extends *Griffith* 2022 findings by incorporating the feedback filter into the input matrix estimation law [8]. The addition of the feedback filter to the input matrix estimation law enables faster convergence of the internal state error (e_x) compared to the non-feedback filter case numerically. The tradeoff of using the feedback filter is that the internal state (e_x) can only be guaranteed to converge to a neighborhood about zero $(e_x \xrightarrow[t\to\infty]{} R_*)$, versus the internal error converging to zero $(e_x \xrightarrow[t\to\infty]{} 0)$ in the non-feedback filter case.

A key difference between Theorem 1 and 2 is knowledge of the true system's plant (A). To apply Theorem 2, the true system plant (A) must be known, while the input matrix model (B_m) must remain. The introduction of the feedback term in the input matrix estimator law disables the use of the Barbalat-Lyapunov Lemma and increases the complexity of the analysis. Much of the analysis consists of bounding terms using a combination of algebraic manipulation, Sylvester's Inequality, and Cauchy-Schwarz inequality. Applying the feedback filter to the adaptive input matrix estimator is summarized in Theorem 2 and Fig. 2.

Theorem 2: Output Feedback on Reference Model for Adaptive Input Matrix using a Feedback Filter and State Estimation. Consider the following state error system:

$$\begin{cases}\dot{e}_x = (A - KC)e_x + B_m \Delta L_1 u \\ \hat{e}_y = Ce_x = C(\hat{x} - x) \\ L_1 = \Delta L_1 + L_{1*} \\ \dot{L}_1 = \Delta\dot{L}_1 = -\hat{e}_y u^\dagger \gamma_u - \alpha L_1\end{cases}, \tag{29}$$

where e_x is the estimated internal state error, $\hat{e}_y$ is the external estimated state error, L_{1*} is fixed-correction matrix, ΔL_1 is the variability-uncertainty terms, K is a fixed gain, and $\gamma_u > 0$ is the interaction tuning term. Given:

1) The triples of (A, B, C) and (A, B_m, C) are both SPR.
2) A true plant (A) must exist.
3) A model input matrix (B_m) must exist.
4) Output matrix (C) is known.
5) Allow $B \in \mathrm{Sp}\{B_m L_{1*}\} \ni B \equiv B_m L_{1*}$.
6) The set of eigenvalues (σ) of the true plant are stable (i.e. $\mathrm{Re}(\sigma(A)) < 0$).

If conditions are met, then the internal error state converges to a neighborhood about zero $(e_x \xrightarrow[t\to\infty]{} R_*)$ asymptotically. ΔL_1 is guaranteed to be bounded; however, no guarantee of $\Delta L_1 \xrightarrow[t\to\infty]{} 0$. If $\Delta L_1 \xrightarrow[t\to\infty]{} 0$, then the true system input matrix or some energy equivalence has been numerically captured.

A. Defining the True and Reference System

The true system dynamics will follow all of the assumptions and constraints outlined in Section III-A. In contrast, allow the reference model to have the following state space characteristics:

$$\text{Reference Model}\begin{cases}\dot{x}_m = Ax_m + B_m u \\ y_m = Cx_m,\end{cases} \tag{30}$$

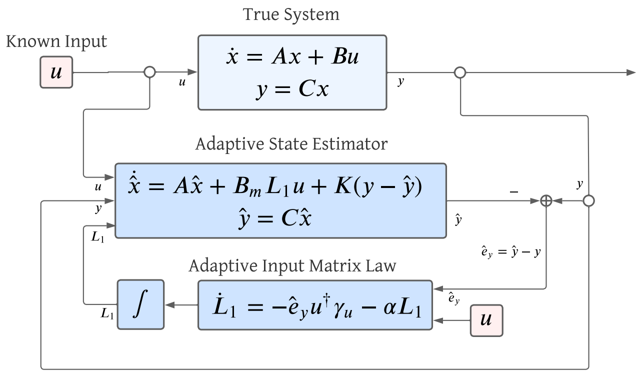

Figure 2. Control diagram for adaptive input matrix with a feedback filter and state estimation given a known input (u), output matrix (C), and external state (y).

where the true plant (A) is assumed to be known. The input matrix model (B_m) can be corrected using a fixed correction term as outlined in Eq. (4) and Eq. (6). For the control scheme to apply, the true and reference systems must be SPR.

B. Estimated State Error

Assuming that the input matrix (B) only experiences a form of health change and the true internal states (x) are often blended into some weighted linear combination of the external state (y), an estimator can be formed using the reference model:

$$\text{Estimator} \begin{cases} \dot{\hat{x}} = A\hat{x} + B_m L_1 u \\ \hat{y} = C\hat{x}. \end{cases} \tag{31}$$

Following a similar protocol as described in Section III-C, error states are defined (Eq. (8)) and the error system dynamics follow:

$$\begin{cases} \dot{e}_x = Ae_x + B_m \underbrace{(\Delta L_1 u)}_{=w_u} \\ \hat{e}_y = Ce_x. \end{cases} \tag{32}$$

The estimator defined in Eq. (31) can be extended to use a fixed gain (K):

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B_m L_1 u + K(y - \hat{y}) \\ \hat{y} = C\hat{x}. \end{cases} \tag{33}$$

Resulting in the following error equation:

$$\begin{cases} \dot{e}_x = (\underbrace{A - KC}_{=A_c})e_x + B_m w_u \\ \hat{e}_y = Ce_x. \end{cases} \tag{34}$$

Again, regardless of the estimator selected, the estimated states cannot be guaranteed to converge to the true states due to a residual term (w_u) existing in the error equation. To accommodate the residual term, additional constraints are needed.

C. Lyapunov Stability for the Estimated States

The initial Lyapunov Stability analysis will follow a similar protocol as outlined in Section III-D. Using the same Lyapunov function as shown in Eq. (15), the energy-like time rate of change of V_e follows:

$$\begin{aligned} 2\dot{V}_e =& \dot{e}_x^\dagger P_x e_x + e_x^\dagger P_x \dot{e}_x \\ =& (Ae_x + B_m w_u)^\dagger P_x e + e_x^\dagger P_x (Ae_x + B_m w_u) \\ =& e_x^\dagger (A^\dagger P_x + AP_x) e_x + 2 \underbrace{e_x^\dagger P_x B_m w_u}_{=(B_m w_u)^\dagger P_x e_x}. \end{aligned} \tag{35}$$

Modifying the SPR stability condition for the reference model:

$$\begin{cases} A^\dagger P_x + P_x A = -Q_x \\ P_x B_m = C^\dagger \end{cases} ; Q_x > 0. \tag{36}$$

Now, the SPR condition can be applied onto Eq. (35):

$$\begin{aligned} 2\dot{V}_e &= -e_x^\dagger Q_x e_x + 2 \underbrace{e_x^\dagger C^\dagger}_{=\hat{e}_y^\dagger} w_u \\ &= -e_x^\dagger Q_x e_x + 2\hat{e}_y^\dagger w_u \\ &= -e_x^\dagger Q_x e_x + 2 \underbrace{(\hat{e}_y, w_u)}_{=(w_u, \hat{e}_y)}. \end{aligned} \tag{37}$$

By removing the residual term $((w_u, \hat{e}_y))$ in Eq. 37, results in $\dot{V}_e \leq 0$.

To counter the residual term, consider the following energy-like function:

$$V_u = \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger), \tag{38}$$

where $\gamma_u > 0$. The energy-like time rate of change for V_u follows:

$$\dot{V}_u = \underbrace{\text{tr}(\Delta \dot{L}_1 \gamma_u^{-1} \Delta L_1^\dagger)}_{=\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta \dot{L}_1^\dagger)}. \tag{39}$$

A control law for the input matrix variance time rate of change $(\Delta \dot{L}_1)$ can be defined as:

$$\Delta \dot{L}_1 = -\hat{e}_y u^\dagger \gamma_u - \alpha L_1, \tag{40}$$

where αL_1 acts as a feedback filter. Additional constraints will be added for α later in the derivation, but for now, allow α to be a scalar with a value greater than zero $(\alpha > 0)$.

Substituting Eq. (40) into Eq. (39):

$$\begin{aligned} \dot{V}_u &= \text{tr}(\underbrace{(-\hat{e}_y u^\dagger \gamma_u - \alpha L_1)}_{\Delta \dot{L}_1} \gamma_u^{-1} \Delta L_1^\dagger) \\ &= \text{tr}(-\hat{e}_y u^\dagger \gamma_u \gamma_u^{-1} \Delta L_1^\dagger - \alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ &= -\text{tr}(\hat{e}_y \underbrace{u^\dagger \Delta L_1^\dagger}_{=w_u^\dagger}) - \text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ &= -\text{tr}(\underbrace{\hat{e}_y w_u^\dagger}_{=w_u^\dagger \hat{e}_y}) - \text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ &= -(w_u, \hat{e}_y) - \text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger). \end{aligned} \tag{41}$$

Therefore, the closed-loop energy-like function can be written as:

$$V_{eu} = V_e + V_u = \frac{1}{2}e_x^\dagger P_x e_x + \frac{1}{2}\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger). \quad (42)$$

While the closed-loop energy-like time rate of change follows:

$$\begin{aligned}\dot{V}_{eu} = \dot{V}_e + \dot{V}_u =& -\frac{1}{2}e_x^\dagger Q_x e_x + (w_u, \hat{e}_y) - (w_u, \hat{e}_y) \\ & - \text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger) \\ =& -\frac{1}{2}e_x^\dagger Q_x e_x - \text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger).\end{aligned} \quad (43)$$

Due to a residual term existing Eq. (43) ($-\text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger)$) Barabalat-Lyapunov methods, as implemented in Section III-E, cannot be utilized. Therefore, $e_x \xrightarrow[t\to\infty]{} 0$ cannot be guaranteed. However, an upper limit or radius of convergence (R_*) can be determined for the internal state error (e_x).

D. Determining the Radius of Convergence for the Internal Error

The derivation to determine the radius of convergence (R_*) for the internal error (e_x) follows a similar approach as described in the *Fuentes* 2025 [2]. The approach requires bounding each component of the right-hand side of Eq. (43), then utilizing algebraic manipulation to find the limit-supremum of the internal state error as time approaches infinity.

1) Bounding $\frac{1}{2}e_x^\dagger Q_x e_x$: Applying Sylvester's Inequality to Eq. (15) results in:

$$\frac{\lambda_{\min}(P_x)}{2}\|e_x\|^2 \le V_e = \frac{1}{2}e_x^\dagger P_x e_x \le \frac{\lambda_{\max}(P_x)}{2}\|e_x\|^2, \quad (44)$$

where $\{\lambda_{\min}(P_x), \lambda_{\max}(P_x)\}$ are the minimum and maximum eigenvalues of matrix P_x. Following, $\|e_x\|$ is the norm of the internal error. The upper bound of V_e is representative of the right-hand side of Eq. (44). Therefore, the upper bound of $\|e_x\|^2$ can be expressed as:

$$\frac{2V_e}{\lambda_{\max}(P_x)} \le \|e_x\|^2. \quad (45)$$

Applying Sylvester's Inequality to $-\frac{1}{2}e_x^\dagger Q_x e_x$, the upper bound can be determined as:

$$-\frac{1}{2}e_x^\dagger Q_x e_x \le -\frac{\lambda_{\min}(Q_x)}{2}\|e_x\|^2. \quad (46)$$

Substituting Eq. (45) into Eq. (46):

$$\begin{aligned}-\frac{1}{2}e_x^\dagger Q_x e_x &\le -\frac{\lambda_{\min}(Q_x)}{2}\underbrace{\frac{2V_e}{\lambda_{\max}(P_x)}}_{=\|e_x\|^2} \\ &\le -\frac{\lambda_{\min}(Q_x)}{\lambda_{\max}(P_x)}V_e = -2\alpha V_e.\end{aligned} \quad (47)$$

Due to both $\{P_x, Q_x\} > 0$, means both have sets of positive eigenvalues and therefore require $2\alpha \ge \frac{\lambda_{\min}(Q_x)}{\lambda_{\max}(P_x)} > 0$ and be of scalar value.

2) Bounding $-tr(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger)$: In order to bound $-\text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger)$ from Eq. (43), recall that $L_1 = \Delta L_1 + L_{1*}$. Therefore, the residual term can be updated as:

$$\begin{aligned}\text{tr}(\alpha L_1 \gamma_u^{-1} \Delta L_1^\dagger) =& \alpha \text{tr}(\underbrace{(\Delta L_1 + L_{1*})}_{=L_1}\gamma_u^{-1}\Delta L_1^\dagger) \\ =& \alpha \underbrace{\text{tr}(\Delta L_1 \gamma_u^{-1} \Delta L_1^\dagger)}_{=2V_u} \\ &+ \alpha \text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger) \\ =& 2\alpha V_u + \alpha \text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger).\end{aligned} \quad (48)$$

Substituting both Eq. (47) and Eq. (48) into Eq. (43) results in:

$$\dot{V}_{eu} = \underbrace{-2\alpha V_e - 2\alpha V_u}_{=-2\alpha V_{eu}} - \alpha \text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger). \quad (49)$$

Therefore, Eq. (49) can alternatively be written as:

$$\dot{V}_{eu} + 2\alpha V_{eu} = -\alpha \text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger). \quad (50)$$

The right-hand side of Eq. (50) can be upper bounded by taking the absolute value of terms:

$$\dot{V}_{eu} + 2\alpha V_{eu} \le \alpha|\text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)|. \quad (51)$$

3) Upper Bounding $|tr(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)|$: Applying the Cauchy-Schwarz inequality on $|\text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)|$:

$$\begin{aligned}&|\text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)| \le \\ \text{tr}(L_{1*}\gamma_u^{-1}L_{1*}^\dagger)^{\frac{1}{2}}&\underbrace{\text{tr}(\Delta L_1 \gamma_u^{-1}\Delta L_1^\dagger)^{\frac{1}{2}}}_{=(2V_u)^{\frac{1}{2}} \le (2V_{eu})^{\frac{1}{2}}}.\end{aligned} \quad (52)$$

Using the trace properties and Cauchy-Schwarz inequality, $\text{tr}(L_{1*}\gamma_u^{-1}L_{1*}^\dagger)^{\frac{1}{2}}$ can be alternatively written as:

$$\begin{aligned}\text{tr}(L_{1*}\gamma_u^{-1}L_{1*}^\dagger)^{\frac{1}{2}} &\le |\text{tr}(L_{1*}^\dagger L_{1*}\gamma_u^{-1})^{\frac{1}{2}}| \\ &\le \text{tr}(L_{1*}^\dagger L_{1*})^{\frac{1}{2}}\text{tr}(\gamma_u^{-1})^{\frac{1}{2}}.\end{aligned} \quad (53)$$

For notation purposes, allow the following assumptions:

$$M_k > 0 \ni \text{tr}(L_{1*}^\dagger L_{1*})^{\frac{1}{2}} \le M_k \quad (54)$$

and

$$\text{tr}(\gamma_u^{-1}) \le \left(\frac{1}{\alpha M_k}\right)^2. \quad (55)$$

Substituting Eq. (54) and Eq. (55) into Eq. (53) results in:

$$\begin{aligned}\text{tr}(L_{1*}\gamma_u^{-1}L_{1*}^\dagger)^{\frac{1}{2}} &\le \text{tr}(L_{1*}^\dagger L_{1*})^{\frac{1}{2}}\text{tr}(\gamma_u^{-1})^{\frac{1}{2}} \\ &\le M_k\left(\frac{1}{\alpha M_k}\right) \\ &\le \frac{1}{\alpha}.\end{aligned} \quad (56)$$

Thus, substituting Eq. (56) into Eq. (52) results in:

$$\begin{aligned}|\text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)| \le& \underbrace{\text{tr}(L_{1*}\gamma_u^{-1}L_{1*}^\dagger)^{\frac{1}{2}}}_{=\frac{1}{\alpha}}(2V_{eu})^{\frac{1}{2}} \\ \le& \frac{1}{\alpha}(2V_{eu})^{\frac{1}{2}}.\end{aligned} \quad (57)$$

4) Combing Bounds to Determine Radius of Convergence (R_*): Combining the results in Eq. (57) into Eq. (51) results in:

$$\dot{V}_{eu} + 2\alpha V_{eu} \le \alpha \underbrace{|\text{tr}(L_{1*}\gamma_u^{-1}\Delta L_1^\dagger)|}_{=\frac{1}{a}(2V_{eu})^{\frac{1}{2}}} \le (2V_{eu})^{\frac{1}{2}}. \tag{58}$$

Thus, Eq. (58) can be rewritten as:

$$\frac{\dot{V}_{eu} + 2\alpha V_{eu}}{V_{eu}^{\frac{1}{2}}} \le \sqrt{2}. \tag{59}$$

Consider the following expression:

$$\begin{aligned}\frac{d}{dt}(2e^{\alpha t}V_{eu}^{\frac{1}{2}}) &= 2(\alpha e^{\alpha t}V_{eu}^{\frac{1}{2}} + e^{\alpha t}\frac{1}{2}V_{eu}^{-\frac{1}{2}}\dot{V}_{eu}) \\ &= \frac{2\alpha e^{\alpha t}V_{eu} + e^{\alpha t}\dot{V}_{eu}}{V_{eu}^{\frac{1}{2}}} \\ &= \frac{\dot{V}_{eu} + 2\alpha V_{eu}}{V_{eu}^{\frac{1}{2}}}e^{\alpha t}.\end{aligned} \tag{60}$$

Therefore, Eq. (59) can be alternatively expressed as:

$$\frac{d}{dt}(2e^{\alpha t}V_{eu}^{\frac{1}{2}}) \le \sqrt{2}e^{\alpha t}. \tag{61}$$

Integrating and evaluating both sides of Eq. (61) from 0 to τ results in:

$$2e^{\alpha\tau}V_{eu}(\tau)^{\frac{1}{2}} - 2e^{\alpha 0}V_{eu}(0)^{\frac{1}{2}} \le \frac{\sqrt{2}}{\alpha}\left(e^{\alpha\tau} - e^{\alpha 0}\right). \tag{62}$$

Solving for $V_{eu}(\tau)^{\frac{1}{2}}$:

$$V_{eu}(\tau)^{\frac{1}{2}} \le e^{-\alpha\tau}V_{eu}(0) + \frac{\sqrt{2}}{2\alpha}(1 - e^{-\alpha\tau}). \tag{63}$$

From Eq. (63), given that $\alpha > 0$, then it can be shown that $V_{eu}(\tau)^{\frac{1}{2}}$ is bounded for all $\tau \ge 0$ and therefore bounding $V_{eu}(\tau)$ for all $\tau \ge 0$. By $V_{eu}(\tau)$ being bounded for all $\tau \ge 0$, this also implies that both $\{e_x, \Delta L_1\}$ remain bounded as they compose V_{eu}.

$V_{eu}^{\frac{1}{2}}$ can be further lower bounded by considering the following inequality:

$$\frac{\lambda_{\min}(P_x)^{\frac{1}{2}}}{\sqrt{2}}\|e_x\| \le V_e^{\frac{1}{2}} + V_u^{\frac{1}{2}} \le V_{eu}(\tau)^{\frac{1}{2}}. \tag{64}$$

Solving for $\|e_x\|$ in Eq. (64) results in:

$$\begin{aligned}\|e_x\| \le &\frac{\sqrt{2}}{\lambda_{\min}(P_x)^{\frac{1}{2}}}e^{-a\tau}V_{eu}(0) \\ &+ \frac{1}{\alpha\lambda_{\min}(P_x)^{\frac{1}{2}}}(1 - e^{-\alpha\tau}).\end{aligned} \tag{65}$$

Taking the limit and supremum (upper bound) of $\|e_x\|$ in Eq. (65) results in the radius of confidence (R_*) as the adaptive input matrix estimator with feedback term is used:

$$\limsup_{\tau\to\infty}\|e_x\| = \frac{1}{\alpha\lambda_{\min}(P_x)^{\frac{1}{2}}} \equiv R_*. \tag{66}$$

Recall $\|e_x\|$ being bounded is based on the assumption of both Eq. (54) and Eq. (55). Equation (55) can notably be upper bounded by applying Sylvester's Inequality, altering the radius of confidence by a scaling factor.

Following, V_u can be bounded by applying the properties of the trace and the Cauchy–Schwarz inequality:

$$V_u = \frac{1}{2}\text{tr}(\Delta L_1^\dagger\Delta L_1\gamma_u^{-1}) \le \frac{1}{2}\underbrace{\text{tr}(\Delta L_1^\dagger\Delta L_1)}_{=\|\Delta L_1\|_F^2}\text{tr}(\gamma_u^{-1}), \tag{67}$$

where $\|\Delta L_1\|_F$ is the Frobenius norm of ΔL_1. Using the assumption of Eq. (55), $V_u^{\frac{1}{2}}$ can be expressed as:

$$V_u^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|\Delta L_1\|_F\left(\frac{1}{\alpha M_k}\right). \tag{68}$$

Using a similar procedure for deriving the limit and supremum of $\|e_x\|$, this can be done to determine the limit and supremum of $\|\Delta L_1\|_F$:

$$\frac{1}{\sqrt{2}}\|\Delta L_1\|_F\left(\frac{1}{\alpha M_k}\right) \le V_e^{\frac{1}{2}} + V_u^{\frac{1}{2}} \le V_{eu}(\tau)^{\frac{1}{2}}. \tag{69}$$

Resulting in the following radius of confidence:

$$\limsup_{\tau\to\infty}\|\Delta L_1\| = M_k \equiv R_{**}. \tag{70}$$

E. Summarizing Results

Assuming that the true (A, B, C) and reference (A, B_m, C) systems are SPR and the true input matrix satisfies $B \equiv B_m L_{1*}$, then the adaptive input matrix estimation law using the feedback filter (Eq. (40)) and diagram (Fig. 2) guarantees $\|e_x\| \xrightarrow[t\to\infty]{} R_*$ asymptotically. Following, Lyapunov Stability Analysis only guarantees that ΔL_1 will remain bounded. Finally, the Lyapunov stability proof can also be applied to the error system under the use of fixed gains in Eq. (34).

V. Illustrative Example - Theorem 1

The following is an illustrative example of applying Theorem 1 and the control diagram (Fig. 1) on a general case study. Numerical values for (A_m, B_m, C) and (A, B, C) are derived and modified from [12].

A. State Space Representations for Reference and True Systems

Allow the reference model as defined in Eq. (3) have the following properties:

$$\begin{aligned}A_m &= \begin{bmatrix}-7 & 2 & 4\\ -2 & -1 & 2\\ -2 & 2 & -1\end{bmatrix}; \\ B_m &= \begin{bmatrix}0\\ .7\\ 2\end{bmatrix}; C = \begin{bmatrix}0.5 & 0 & 1\end{bmatrix}; x(0) = 0.\end{aligned} \tag{71}$$

To apply the control scheme as defined in Theorem 1 and show in Fig. 1, allow the true system as defined by Eq. (2) have the following properties:

1) $B \in \text{Sp}\{B_m L_{1*}\} \ni B \equiv BL_{1*}$.

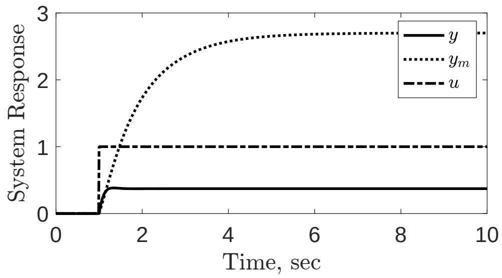

Figure 3. True (y) and reference model (y_m) output response given a unit step input (u).

2) $A \in \mathrm{Sp}\{A_m, B_m L_{2*} C\} \ni A \equiv A_m + BL_{2*}C$.
Assume the health change for the input matrix and the plant can be described by $\{L_{1*}, L_{2*}\} \ni L_{1*} = 2$ and $L_{2*} = -5$. Therefore, the true system can be defined by the following:

$$\begin{aligned} A \equiv& A_m + B_m L_{2*} C = \begin{bmatrix} -7 & 2 & 4 \\ -3.75 & -1 & -1.5 \\ -7 & 2 & -11 \end{bmatrix}; \\ B \equiv& B_m L_{1*} = \begin{bmatrix} 0 \\ 1.4 \\ 4 \end{bmatrix}; C = \begin{bmatrix} 0.5 & 0 & 1 \end{bmatrix}; x(0) = 0. \end{aligned} \tag{72}$$

Recall that the constitutive constants of the true plant (A) and input matrix (B) are unknown. However, an initial estimate of the plant (A_m) and input matrix (B_m) exists.

When both the reference and true systems, as defined in Eq. (71) and Eq. (72), are given a unit step input, as shown in Fig. 3, the differences in rise times and output response become evident. These differences can be further explained by examining the eigenvalues of the reference and true plants:

$$\begin{aligned} \sigma(A_m) &= \{-1, -3, -5\} \\ \sigma(A) &\approx \{-2.28, -8.36 \pm i5.05\}. \end{aligned} \tag{73}$$

B. Defining the Known Input (u)

To implement the control scheme, a bounded and continuous input must be used. In practice, this input can be a known disturbance. For this example, allow the input to be defined as:

$$u = 2 + \sin(2t). \tag{74}$$

C. Adaptive State and Input Matrix Estimation

In this section, the presented control scheme, detailed in Fig. 1, is implemented with two cases: with and without the use of a fixed gain (K) term.

1) Adaptive Control Scheme without the use of Fixed Gain $(K = 0)$: The control scheme detailed in Fig. 1 is implemented without using the fixed gain term $(K = 0)$ and $\{\gamma_u, \gamma_y\} = I$. As derived in the proof, Fig. 4 demonstrates the convergence of the internal state, where $e_x \xrightarrow[t\to\infty]{} 0$. Given that the internal state error converges to zero, equivalently, the external state error converges $\ni \hat{e}_y \xrightarrow[t\to\infty]{} 0$. Meaning that the estimated output $(\hat{y})$ converges to the true output (y).

Although the proof only guarantees that the adaptive variance will be bounded, numerically $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$. For this case study, the true input matrix and plant have been numerically captured, Fig. 5 and Fig. 6.

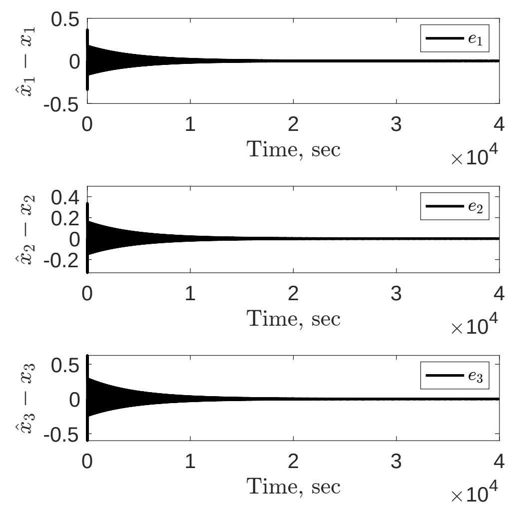

Figure 4. Internal state error converging to zero without the use of the fixed gain $(K = 0)$.

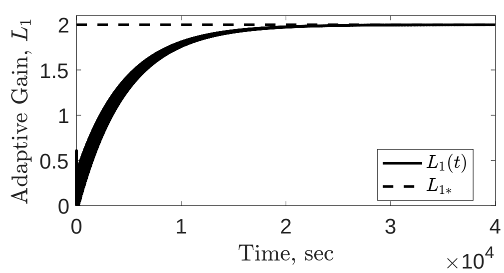

Figure 5. Input Matrix adaptive term $L_1(t)$ converging to L_{1*} without the use of the fixed gain $(K = 0)$.

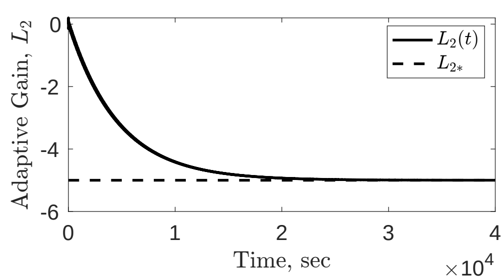

Figure 6. Plant correction adaptive term $L_2(t)$ converging to L_{2*} without the use of the fixed gain $(K = 0)$.

2) Adaptive Control Scheme with the use of Fixed Gain $(K \neq 0)$: The control scheme detailed in Fig. 1 is implemented using the fixed gain term $(K \neq 0)$ and $\{\gamma_u, \gamma_y\} = 1$. The fixed gain term K was derived using a Linear Quadratic Regulator (LQR) where $Q = I_3$ and $R = 1$. Similarly to the result of Section V-C1, $e_x \xrightarrow[t\to\infty]{} 0$, shown in Fig. 7. Again, since the internal state error converges to zero, the external error will converge to zero for the true and estimator systems. Moreover, as $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$, the true input matrix and plant are numerically captured in Fig. 8 and Fig. 9.

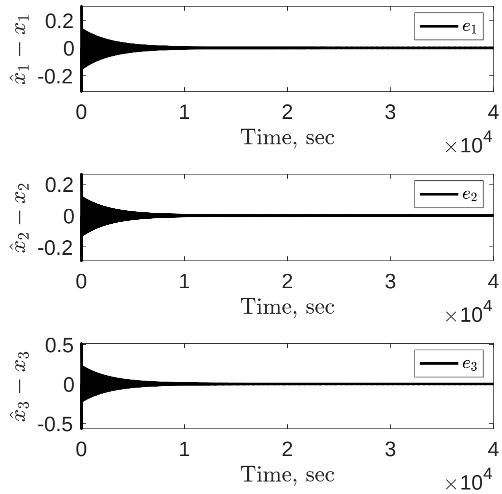

Figure 7. Internal state error converging to zero with the use of the fixed gain ($K \neq 0$).

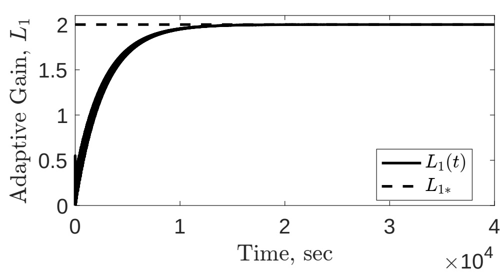

Figure 8. Input Matrix adaptive term $L_1(t)$ converging to L_{1*} with the use of the fixed gain ($K \neq 0$).

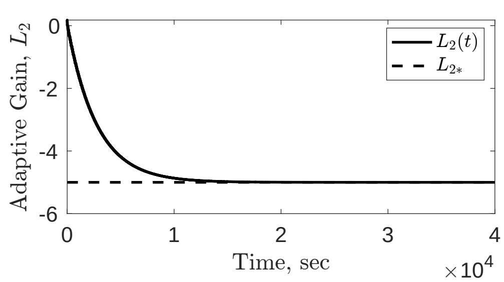

Figure 9. Plant correction adaptive term $L_2(t)$ converging to L_{2*} with the use of the fixed gain ($K \neq 0$).

There can be a benefit of using a fixed gain term in the estimator, as the term can affect the time in which internal states and adaptive terms converge, compare Fig. 5 and Fig.8. More crucially, both adaptive tuning terms $\{\gamma_u, \gamma_y\}$ can be adjusted to amplify or dampen the effects of the adaptive controller, directly impacting the convergence of the error state. For this particular example, setting $\gamma_u = 1.3$ and $\gamma_y = 1.85$ reduces the time in which $e_x \xrightarrow[t\to\infty]{} 0$ and $L \xrightarrow[t\to\infty]{} L_*$ by order of magnitude faster than the depicted figures in this text. However, there are numerical limits for the tuning terms $\{\gamma_u, \gamma_y\}$. Making the adaptive controller too sensitive to changes may lead to divergent artifacts.

3) Kalman-Bucy Filter-Kalman Filter in Continuous Time: The previous two sections, Sections V-C1 and V-C2, illustrate an example that implements an adaptive estimator both with and without a fixed gain (K) term. When the adaptive laws are removed, and only the fixed gain estimator is used, the reference model is no longer updated within the estimator. This is evident in Fig. 10 and Fig. 11, where it can be seen that both the internal and external errors reach a non-zero steady-state value. The value of the fixed gain term (K) is derived using LQR, where $Q = I_3$ and $R = 1$. Although the observed time window for the error dynamics response is limited, the system response does not change as time approaches infinity.

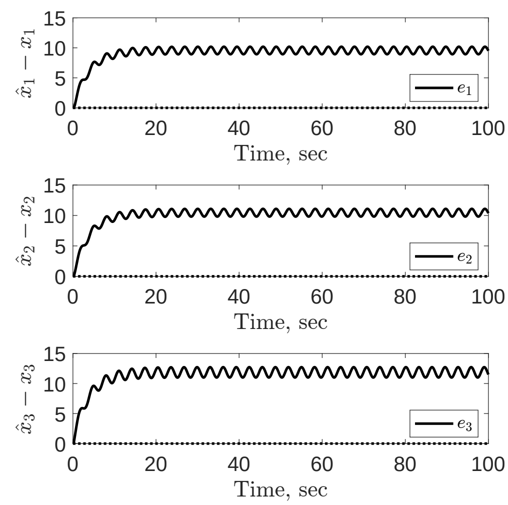

Figure 10. Internal state error (e_x) not converging in the absence of the adaptive estimator. Only the fixed gain (K) is used in the estimator.

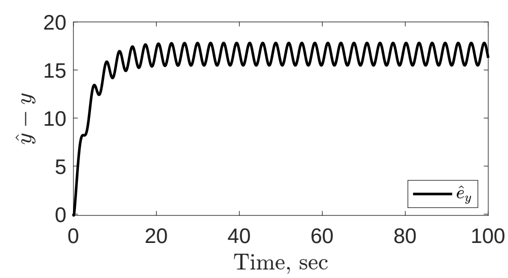

Figure 11. External state error ($\hat{e}_y$) not converging in the absence of the adaptive estimator.

VI. Illustrative Example - Theorem 2

The following is an illustrative example of applying Theorem 2 and the control diagram (Fig. 2) on a general case study. Numerical values for (A, B_m, C), (A, B, C), and input (u) are the same from Section V.

A. Input Matrix Estimation with the Feedback Filter ($\alpha > 0$ *and* $K = 0$)

In this section, the control scheme outlined in Fig. 2 is implemented with only one case study, without the use of the fixed gain ($K = 0$) term. The reason for this is that Section V already considers the case where the adaptive laws are coupled with and without fixed gain terms.

As derived in the proof, Fig. 12 shows that the internal state error converges to a neighborhood of zero when the adaptive input matrix law is coupled with the feedback term ($\|e_x\| \xrightarrow[t\to\infty]{} R_*$). As the internal-state error settles within this neighborhood, the external-state error correspondingly follows. In Fig. 13, we see that L_1 approaches L_{1*}, but does not fully converge, consistent with the theoretical result, which guarantees only that the adaptive variance (ΔL_1) remains bounded. Although, there exist variance in L_1, the updated input matrix ($B_m L_1$) more closely aligns with the true input matrix ($B \equiv B_m L_{1*}$) than the nominal input matrix model (B_m). Values used on the adaptive input matrix estimation law follows: $\alpha = .05$ and $\gamma_u = I$.

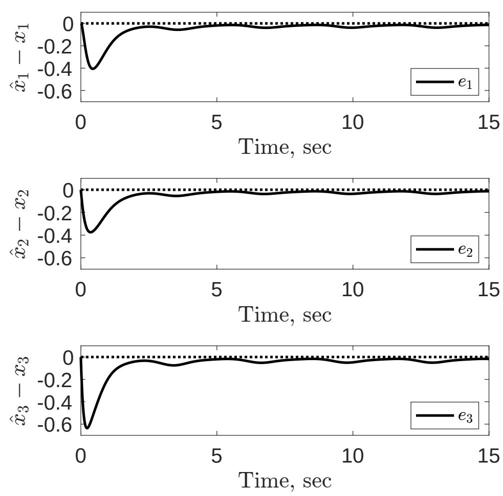

Figure 12. Internal state error converging to the neighborhood about zero without the use of the fixed gain ($K = 0$) and using the feedback filter on the adaptive input matrix law.

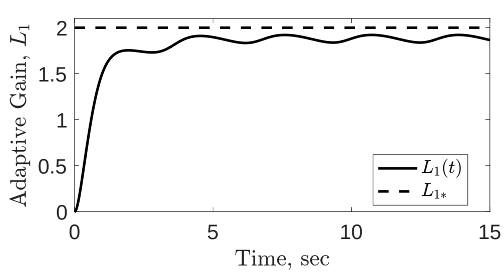

Figure 13. Input Matrix adaptive term $L_1(t)$ approaching L_{1*} without the use of the fixed gain ($K = 0$) and using the feedback filter on the adaptive input matrix law.

B. Input Matrix Estimation without the Feedback Filter ($\alpha = 0$ *and* $K = 0$)

The control scheme detailed in Fig. 2 is implemented without using a fixed gain term ($K = 0$) and without the feedback filter ($\alpha = 0$). After truncation, proof follows the results presented in *Griffith* 2022 [8]. The adaptive input matrix estimation law uses $\gamma_u = I$. In the absence of the feedback filter, $\|e_x\| \xrightarrow[t\to\infty]{} 0$, as demonstrated in Fig. 14, consistent with theoretical results. As the internal state error converges to zero, the external state error will follow. Lastly, as $\Delta L_1 \xrightarrow[t\to\infty]{} 0$, the true input matrix is captured, as shown in Fig. 15. Note that theoretical results only guarantee ΔL_1 to remain bounded, whereas the numerical results exhibit full convergence.

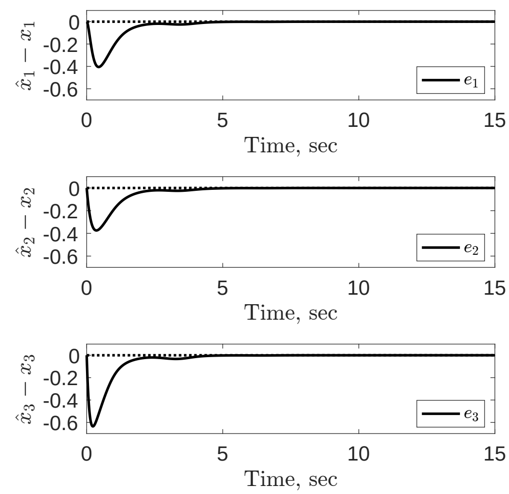

Figure 14. Internal state error converging to zero without the use of the fixed gain ($K = 0$) and not using the feedback filter on the adaptive input matrix law.

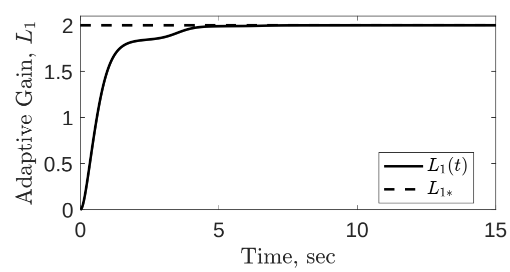

Figure 15. Input Matrix adaptive term $L_1(t)$ converging to L_{1*} without the use of the fixed gain ($K = 0$) and not using the feedback filter on the adaptive input matrix law.

C. Kalman-Bucy Filter-Kalman Filter in Continuous Time

The previous two sections, Sections VI-A and VI-B, presented examples of implementing the adaptive input-matrix estimator both with and without a feedback term, while not using a fixed gain term ($K = 0$). When the adaptive laws are removed, and only the fixed gain term is retained, the input matrix model is no longer updated within the estimator. This lack of adaptation results in larger internal and external state errors compared to the adaptive cases, as shown in Fig. 16 and Fig. 17. The value for the fixed gain term is obtained using LQR, where $Q = I_3$ and $R = 1$.

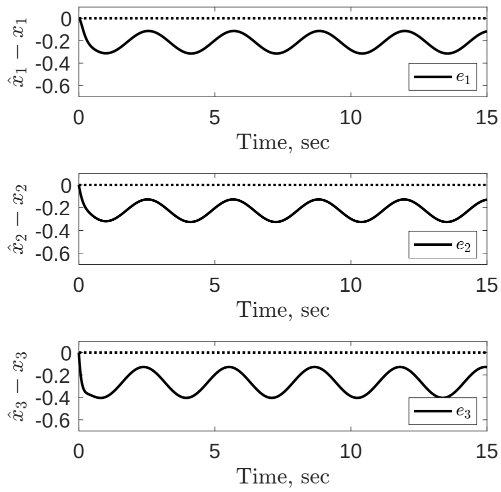

Figure 16. Internal state error (e_x) not converging to zero in the absence of the adaptive input matrix estimator. Only fixed gain (K) used in the estimator.

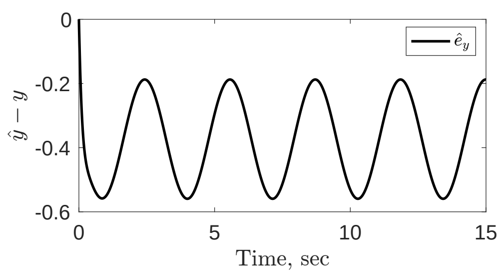

Figure 17. External state error $(\hat{e}_y)$ not converging to zero in the absence of the adaptive input matrix estimator.

VII. Discussion

The theorem presented in this text pertains to globally stable LTI systems. By utilizing local linearization, the theorems can be applied to both linear and non-linear systems that exhibit stable behavior within certain neighborhoods. Local linearization around these neighborhoods is a common practice in engineering, particularly because many systems incorporate non-linear elements.

Knowing the output matrix (C) is a common assumption in control design. As the output matrix denotes which linear combination of the internal state is being measured, typically denoted as a sensor. These sensors can degrade and experience health changes as well, but this variable is outside of the specified presenting theorems.

There exist theoretical hard constraints such that the true plant and input matrices must follow a specific decomposition; however, these constraints are required for the proof. Future work could explore methods to relax and bypass some of these constraints, as well as consider the influence of stochastic processes.

The theorems presented do not require feedback from the true system, allowing the estimation technique to be implemented without jeopardizing the true system's stability. Consequently, these estimation schemes can run alongside various input methods used to regulate the system's output. This parallel implementation enables continuous monitoring of changes in the true system's dynamics. If a significant health change occurs, identified through decomposition, technicians and operators can be alerted to inspect and replace affected areas of the system.

Due to the coupled nature of the adaptive laws, adaptive gains depend on one another to change. When combined with asymptotic convergence requirements, the error states can take orders of magnitude of time to converge, as demonstrated in the illustrative example shown in Section V. Tuning the adaptive gains and introducing a fixed gain can reduce the time in which the error state converges to zero by orders of magnitude; however, the amount of time for the error state to converge is still a pressing issue. An attempt had been made to introduce a feedback filter on the adaptive laws as described in *Fuentes's 2025* robust study, as successful implementation can further reduce the convergence time of the error states [2]. However, numerical results were not promising. This motivated the implementation of the feedback filter only on the adaptive input-matrix estimation law. Comparing the illustrative examples in Section V and Section VI reveals a clear difference in the time scales required for the error states and adaptive gains to reach steady state. This further highlights the challenge of incorporating two adaptive mechanisms simultaneously to manage system uncertainty.

Future work will focus on decoupling the adaptive laws by introducing an additional output matrix for the input matrix, which will serve as an encoder for the input and input matrix response. Nevertheless, this approach presents new challenges, particularly concerning the observability conditions for the input matrix. Considering the numerical results, the presented Theorem 1 is likely the best fit for offline use or slow dynamic systems such as structures, whereas Theorem 2 is better suited for faster dynamic systems such as robotics.

VIII. Conclusion

The true system dynamics and input matrix can be influenced by a health status change, resulting in potential changes in constitutive constants and internal interactions. If these changes are not considered in the modeled EOM, discrepancies will emerge between the system model and the true system response. To address the impact of the health status change of the true system's plant and input matrix, a set of coupled adaptive laws were derived. These laws ensure that the error states between the model and the true system converge to zero; however, their effectiveness depends on a specific decomposition. Due to their coupled nature and asymmetric convergence characteristics, the error states may require a significant amount of time to converge. Consequently, these laws may not be practical for fast dynamics, such as in robotics, when applied in real-time. Instead, they are more suited for offline applications or for relatively slow dynamic systems, like structures. In contrast, when using only a single adaptive law in the estimator, convergence time can differ by several orders of magnitude, making the approach more suitable for online applications. Future work will focus on relaxing some of the decomposition constraints and exploring methods to decouple the adaptive laws.

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