Interstitial b-SHAP-Owen Amalgam for the Enhancement of Artificial Intelligence System-Centric Sequential Decision-Making

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***Abstract*—The leveraging of an Interstitial b-SHAP-Owen Amalgam (IbSOA) is the focus of this paper; interestingly, the amalgam can potentially lend towards the enhancement of Artificial Intelligence (AI) System (AIS)-Centric multi-stage Sequential Decision-Making (DM) (SDM) amidst Ambiguity and Uncertainty (A/U) by leveraging certain previously peer-reviewed preset modules — Lower Ambiguity, Higher Uncertainty (LAHU) and Higher Ambiguity, Lower Uncertainty (HALU) Module (LHM) — and certain subordinate modules — LHM's Isomorphic Paradigm (IsoP) Comparator Similarity Measure Module (ICSM2) and LHM's Metaheuristic Algorithm Module (MAM) — as well as a new module in the form of a Conversational AI Robustness (CAIR) Accelerant (CAIR-A) Module (as a proxy case study example); CAIR-A addresses the challenge of Robust Dialogue Management (RDM) with the objective of sufficiently supporting a Conversational AI Agent (CAA) so as to be able to maintain consistency, coherency, and validity throughout an ongoing dialogue. To achieve this, the involved Reasoning Mechanisms (RMs) — Monotonic Reasoning (MR) and Non-Monotonic Reasoning (NMR) — need to be well counterpoised; the apropos harmonizing of MR/NMR can help ensure the RDM's plasticity for enhanced conversational coherence. In turn, the successful counterpoising of MR/NMR is predicated upon insights into the Monotonic/Non-Monotonic Transition Zones (MNTZs); after all, shifts from a monotonic to a non-monotonic paradigm and vice versa in the MNTZs can occur at a higher than anticipated rate. IbSOA-related insights into the myriads of interplays among local, glocal, and global facets can better contextualize the behavior within the MNTZs and better support the CAA's SDM amidst A/U. As previously noted, the CAA is simply a proxy application, and the implications of the referenced counterpoising have broader implications for AIS SDM.**

Keywords-artificial intelligence systems; machine learning; LAHU; HALU; LHM; IsoP; Monotonic Reasoning; Non-Monotonic Reasoning; MNTZ; Interstitial b-SHAP-Owen Amalgam (IbSOA); Sequential Decision-Making (SDM).

I. INTRODUCTION

Conversational AI Agents (CAA) endeavor to emulate natural human conversation through, among other modes, text and/or voice (CAA capability has also been extended to include holographic form, such as offered by Holoconnects, Ravatar, Proto, and others). Other companies focus on certain realism aspects, and offerings include those by Hume (https://www.hume.ai/?tab=evi), ElevenLabs (https://elevenlabs.io/conversational-ai), etc. Conventional CAA leverages an amalgam of constituent technologies, such as Automatic Speech Recognition (ASR), Natural Language Processing (NLP), Natural Language Understanding (NLU), Natural Language Generation (NLG), Large Language Models (LLMs), and Machine Learning (ML), among others, so as to engage in meaningful dialogue and provide robust "human-like responses." In contrast to prototypical chatbots, which are underpinned by "rule-based" prescribed scripts, CAA can engage in unscripted dialogues, learn/tune from each and every engagement, as well as leverage external sources/systems. In terms of realism, CAA should be able to perform in real-time without any awkward delays. This paper builds upon [1], which was submitted to Future Computing 2025 on March 1, 2025 (now published), as well as touches upon certain elements of [2], which was submitted to AI-based Systems and Services (AISyS) 2025 on June 10, 2025 (accepted; in press), and fundamentally, this paper, which was substantially completed in April 2025 (following the journal invitation), underscores the notion that CAA responses need to remain consistent, coherent, and valid. This is non-trivial to achieve, and a number of technical challenges exist within this ecosystem. Among other challenges, first, the CAA must contend with the SDM challenge; in essence, the series of decisions that the CAA makes will have a cumulative effect as to how robust the overall conversation (which may ensue across multiple interactions) will be. Second, the CAA needs to undertake SDM amidst varying degrees of A/U, whose definitions are provided in Table I below.

TABLE I. AMBIGUITY (A) VERSUS UNCERTAINTY (U)

DM Notions	*Definitions*
Ambiguity (A)	Ambiguity is prevalent when the involved circumstance(s) and/or the involved *information* can be construed in varying ways (including contradictory).
Uncertainty (U)	Uncertainty is prevalent when the outcome of the involved circumstance(s) is unknown/unpredictable, and there exists a lack of knowledge (wherein *information* pertains to recitals of fact and/or descriptors while *knowledge* pertains to the comprehension and operationalization of that *information*).

Interestingly, the issue of SDM extends far beyond the CAA case study, as will be delineated in Section IA.

A. The SDM Challenge

Autonomous Systems (AS), such as in the form of Autonomous Artificial Intelligence Systems (AIS), may include Autonomous Underwater Vehicles (AUVs), Unmanned Surface Vehicles (USVs), Unmanned Aerial Vehicles (UAVs), unmanned spacecraft, etc. A more commonly recognized manifestation might be in the form of self-driving cars. Some of the involved companies are quite well-known in the marketplace: Amazon's Zoox, General Motors' Cruise, Tesla, and Waymo, with the latter two, in particular, asserting their desire to achieve the Society of Automotive Engineers' (SAE) sixth level of driving-related automation – "5 (full automation)," which is contrasted to "0 (no automation)." This type of sixth level autonomous AI-centric vehicle, as well as certain "near-to-this-level" paradigms — as pertains to the other referenced drones — are necessarily reliant upon robust Sequential Decision-Making (DM) (SDM), which is often referred to as multi-stage DM since the SDM challenge itself resides in the fact that multiple DM points — over an elongated temporal span (i.e., a longer time horizon) — are interrelated. Each decision made at a DM point potentially constitutes a DM Inflection Point (DMIP) and impacts the ensuing prospective possible courses of action and their outcomes, such as described by Wanke, for which an abridged version is shown in Figure 1 below [3].

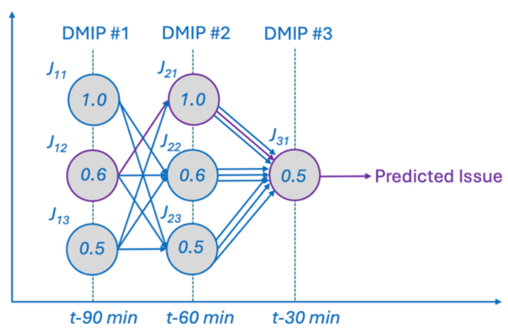


Figure 1. Various DM points comprising an exemplar SDM problem

Utilizing Wanke's presented case, 1.0 equates to "no resolution actions will be taken," 0.5 equates to "resolve the anticipated issue," 0.6 equates to a "partial resolution," J_{ij} equates to "resolution cost distribution at decision point *i* of option j," and the mean cost for the indicated path in purple of 0.6-1.0-0.5 is expressed in (1) [3].

$$E(J)=J_{12}+E(J_{21}|J_{12})+E(J_{31}|J_{21}|J_{12}) \qquad (1)$$

It should be clear from Figure 1 and (1) that SDM stands in stark contrast to single-stage DM, wherein downstream consequences are not necessarily considered. After all, a core challenge of SDM is to optimize the cumulative outcome over the various multi-stage DM points. Practically speaking, Real-World Scenarios (RWS) typically involve SDM amidst ever-evolving environs; along this vein, RWS SDM often occurs amidst varying degrees of A/U, whose proposed handling, among other approaches, is discussed in the following Section IB.

B. Tackling SDM Amidst A/U

At the core, RWS AIS SDM A/U (AIS-SDM-A/U) tends to involve several steps/DM points/DMIPs amidst a complex and dynamic ecosystem. Apart from the transportation-related venues previously described in Section IA, AIS-SDM-A/U has also become more prevalent in the energy sector (e.g., for the positing of energy demand over a certain period of time and optimizing the involved power grid to handle the associated consumption), the financial sector (e.g., for the determination of the allocation and re-allocation of financial assets to optimize Return on Investment or ROI), the healthcare sector with regards to patient monitoring, diagnosis, and treatment planning (e.g., for the selection of a series of treatments to attain, ultimately, a more optimal outcome), etc. Apart from these referenced sectors, AIS-SDM-A/U is becoming prevalent in other sectors as well. Yet, at its core, SDM is a formidable area to tackle with the well-known Bellman notion of the "curse of dimensionality" (wherein contending with an increasing number of features can segue to an exponential increase in computational cost and/or a prospective marked decline in performance), and *the question of how to handle A/U remains an ongoing challenge* [4]; along this vein, features, dimensions, and criteria should be differentiated. Features typically equate to what the involved entity "is" or "does" (i.e., property/characteristic, functionality), dimensions relate to how the entity can be delineated or classified/categorized, and criteria include the standards and/or benchmarks utilized to assess the entity. Given these distinctions and the associated backdrop, certain *preset* approaches, such as that of a Lower Ambiguity, Higher Uncertainty (LAHU) and Higher Ambiguity, Lower Uncertainty (HALU) Module (LHM), have been advanced in an effort to tackle the AIS-SDM-A/U challenge. The functions of the LAHU and HALU LHM components are described in Table II.

TABLE II. LAHU/HALU MODULE (LHM)

LHM Components	***Definitions***
Lower Ambiguity, Higher Uncertainty (LAHU)	"Under a Compressed Decision Cycles (CDC) 'paradigm or tight time constraints,' the LHM 'accepts higher uncertainty (i.e., sparse data) given the condition of lower ambiguity' (i.e., Lower Ambiguity, Higher Uncertainty or LAHU)," "and this roughly translates to the consideration that an isomorphic scenario has manifested previously within the available historical data" [5], [6], and those Digital Object Identifiers (DOIs) of Table IV.
Higher Ambiguity, Lower Uncertainty (HALU)	Under a paradigm of Uncompressed Decision Cycles (UDC), meaning, "if there exists a condition of Higher Ambiguity" conjoined with Lower Uncertainty (i.e., HALU), "wherein the isomorphic scenario is nonexistent within the historical data, there will be a proactive seeking of more data 'to lower uncertainty' so as to move towards a more acceptable state" [5], [6], and those DOIs of Table IV.

With regards to the LHM's repertoire of experience, this can include the various data categories of Table III: *Non-Operational Data*, *Situational Awareness Data*, and *Operational Data,* among others. These comprise, among other facets, the "repertoire of experience" (e.g., the encountering of similar scenarios) of the LHM.

TABLE III. LHM DATA CATEGORIES

Data Categories	***Definitions***
Non-Operational Data (NOD)	"historical and forensic data that has been ingested, apriori" or near-contemporaneously, "to serve as a baseline" for contextualizing the *Operational Data*" [first DOI of Table IV].
Situational Awareness Data (SAD)	environs data that is "contextualized and integrated with *Operational Data*" for the purpose of appraising *Operational Data* "prior to the exigency circumstance;" "in this way…lessons can be learned and leveraged without necessarily needing the immediate performance required of *Operational Data*" [first DOI of Table IV].
Operational Data (OD)	"this data is, when contextualized by *Situational Awareness Data* and *Non-Operational Data*, can be quite indicative and better lend to the immediate performance expected of quasi-real-time data for decision-making" [first DOI of Table IV].

To recap Tables II and III, and as discussed in the first DOI of Table IV, under a LAHU paradigm, given a sufficient repertoire of experience, the tolerance for uncertainty is raised, thereby lowering "the need to turn *Non-Operational Data* and *Situational Awareness Data* into immediate performance data" [first DOI of Table IV]. Suffice it to say, for future circumstances, wherein immediate DM is necessary, the now formulated set of *Deep Belief Heuristics* may be utilized. The practical utility of the LHM (i.e., its enablement) is delineated in [5], [6], and those DOIs of Table IV below.

TABLE IV. EXEMPLAR DIGITAL OBJECT IDENTIFIERS (DOIS) PERTAINING TO THE NOTIONS OF LAHU/HALU

Facet	***DOI***
LAHU; HALU	• 10.1109/IEMCON.2019.8936241 • 10.1109/IAICT62357.2024.10617473 • 10.1109/GEM61861.2024.10585580 • http://dx.doi.org/10.2139/ssrn.5183492 • http://dx.doi.org/10.2139/ssrn.4984663 (DOIs are generally not assigned to patents, such as [5] and [6]).

As described by the last DOI of Table IV, central to the LHM are various supporting modules, such as that of an Isomorphic Paradigm (IsoP) Comparator Similarity Measure (ICSM) Module (ICSM2) as well as a Metaheuristic Algorithm Module (MAM), among other modules. These are briefly described as follows, and further clarification will be provided in Section II.

1) LHM's ICSM2

The ICSM's decided actions may be predicated upon whether the compared paradigm actually needs to undergo a computationally more extensive IsoP examination. For example, in some cases, such as for unordered sets, the ordering of the edges (and their weights) may not necessarily be relevant, as only the nodes (and their values) (hereinafter, "N+V") need to be compared; in other cases, such as for ordered sets, the edges and the sequencing of the nodes is of significance. Exemplar ICSM2 considerations are presented, in somewhat logical order, within Table V below.

TABLE V. EXEMPLAR CONSIDERATIONS PRIOR TO AND INCLUDING THAT OF ISOMORPHIC COMPARISON

ICSM2 Considerations	***Definition***
Unordered Set (UnS)	Set of disparate constituents, wherein the order of the constituents is not relevant. By way of example, {1, 2, 3, 4, 5} equates to {5, 3, 1, 4, 2}.
Equal Sets (EqS)	A pair of sets S and S' is equal if and only if (iff) each constituent of S is also a constituent of S'; moreover, the order of the constituents is not relevant. By way of example, if S = [1, 2, 3, 8, 9, 10] and S' = {9, 3, 1, 2, 10, 8}, then S=S'.
Equivalent Sets (EquivS)	A pair of sets S and S' is considered equivalent if the number of constituents in S and S' is the same (i.e., same cardinality). By way of example, if S = {1, 3, 5, 7, 9} and S' = {2, 4, 6, 8, 10}, then S and S' are considered to be equivalent.
Ordered Set (OrS)	Set of disparate constituents, wherein the order of the constituents is relevant, and the constituents can be ordered and compared via operators, such as <. By way of example, an ordered set might be {1, 2, 3, 5, 6, 8, 9, 10}, whereas an unordered set might be {6, 5, 1, 2, 3, 10, 9, 8}.
Partially Ordered Set (POSET)	Set of disparate constituents, wherein the constituents might or might not be able to be ordered and compared, since operators such as <= can yield different variations. By way of example, Calcworkshop (https://calcworkshop.com/relations/partial-order/) provides some examples, which we extrapolate upon in the way of {a < b < c < d <= e <= f }, {a <= b <= c < d < e <= f < g}, and {a < b < c <= d <= e < f}, which are shown diagrammatically in Table VI.
Unordered Sets with Isomorphism (UnS-Iso)	For a set of disparate constituents, wherein the constituents are unordered, if there is a one-to-one relationship (i.e., bijection), then the unordered sets are likely isomorphic. By way of example, if S={1, 2, 3, 4, 5}, S'={a, b, c, d, e}, and 1<->a, 2<->b, 3<->c, 4<->d, and 5<->e (wherein each constituent in S relates to a unique constituent in S'), then S and S' are considered to be isomorphic.
POSETs with Isomorphism (POSET-Iso)	For a set of disparate constituents, wherein the constituents are considered to be within a POSET, if there is a bijection, then the POSETs are likely isomorphic. By way of example, if S={S_1, S_2, S_3, S_4, S_5}, S'={S'$_2$, S'$_4$, S'$_5$, S'$_3$, S'$_1$}, and S_1<->S'$_2$, S_2<->S'$_5$, S_3<->S'$_1$, S_4<->S'$_4$, and S_5<->S'$_3$, (wherein each constituent in S relates to a unique constituent in S'), then S and S' are considered to be isomorphic. To demonstrate this, online tools are available, such as https://graphonline.top/en/?graph=xPLjwOkrglDRgYeS,

	among others.
Isomorphism Variants (IV)	Extrapolating upon the POSETs with isomorphism, there are also permutations that are actually isomorphism variants (e.g., automorphism, which is a particular type of isomorphism that has a symmetrical structure), which Lemons nicely casts and for which examples are shown in Table VII as IV#1 through #3 [7].

TABLE VI. EXEMPLARS OF ISOMORPHIC PARTIALLY-ORDERED SETS (POSETS)

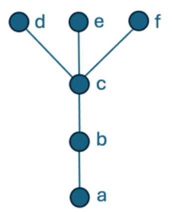
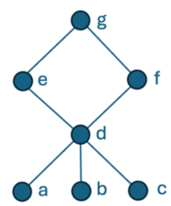
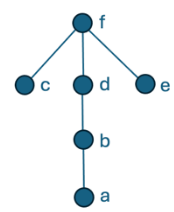
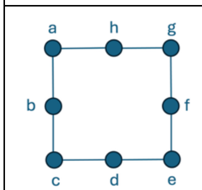
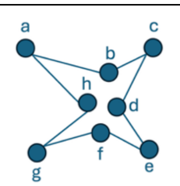
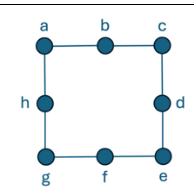
POSET #1	***POSET #2***	***POSET #3***
		
Maximal: d, e, f	Maximal: g	Maximal: f
Greatest: none	Greatest: g	Greatest: f
Minimal: a	Minimal: a, b, c	Minimal: a
Least: a	Least: none	Least: a

TABLE VII. EXEMPLARS OF ISOMORPHIC VARIANTS (IVS)

IV #1	***IV #2***	***IV #3 (automorphism)***
		

The LHM ICSM2's considered pathways for undertaking the requisite pre- and isomorphic/non-isomorphic comparisons, such as alluded to by Table V, are underpinned by the LHM's MAM. After all, depending upon the time available (as was depicted in Figure 1), certain pathways may likely be more computationally tractable. The notion of Criteria Weights (CW), such as that of an Adaptive Criteria Weighting System (ACWS), is crucial in this regard, and exemplar prior experimentation is listed in Table VIII; it should be remembered that CW/ACWS is pivotal for Multi-Criteria Decision-Making (MCDM) and that the notion of the CW/ACWS dovetails with the Interstitial bespoke (b) Shapley Additive Explanations (SHAP)-Owen Amalgam or Interstitial b-SHAP-Owen Amalgam (IbSOA), which designates an importance value to each feature; after all, CW/ACWS systematically assesses and ranks alternatives or features based upon their relative importance (as determined by the CW). IbSOA will be further addressed in Section IIID.

TABLE VIII. EXEMPLAR DOIS PERTAINING TO CRITERIA WEIGHTING EXPERIMENTATION

Facet	***DOI***
CW, ACWS	• 10.1109/ICSGTEIS60500.2023.10424230 • 10.1109/ICPEA56918.2023.10093212 • 10.1109/AIIoT61789.2024.10579033 • http://dx.doi.org/10.2139/ssrn.4984663 • http://dx.doi.org/10.2139/ssrn.5183492 • 10.1109/AIIoT65859.2025.11105315

2) LHM's MAM

Proceeding from the ICSM2 discussion to that of the MAM, the MAM's decided actions, particularly amidst the temporal limitations of CDC, may also be predicated upon what type of Decision-Making Problem (DMP) is encountered. In the case of SDM, each DM point may be handled by whether the DMP is any of, among others, the following: (1) Programmed DMP, (2) Analytical DMP, and (3) Non-Programmed DMP. The definitions for these are shown in Table IX below.

TABLE IX. TYPES OF DECISION-MAKING PROBLEMS (DMPS)

Type of DMP	***Definition***
Programmed DMP (PDMP)	Prototypical Structured Problems (PSP), as pertains to DMP, with well-established Standard Operating Procedures (SOPs) and approach vectors.
Analytical DMP (ADMP)	Mid-range Semi-Structured Problems (MRSSP), as pertains to DMP, that require more in-depth analysis and a more comprehensive approach; it may also require decomposing the DMP into smaller, more manageable (i.e., computationally tractable) DMPs.
Non-Programmed DMP (NPDMP)	More Complex Unstructured Problems (CUP), as pertains to DMP, that may require a bespoke approach.

As the MAM, by way of example, considers each DMP, the insights provided by the IbSOA will be invaluable, as the likely behavior within the MNTZ may be pivotal in ensuring that a prudent pathway, within the given time constraints, is chosen. This will be further expounded upon in Sections IIC and IIID. In the context of the MAM, as noted in the 6th DOI of Table VIII, according to Hjeij, in the journal paper, "A brief history of heuristics: how did research on heuristics evolve," "the use of a heuristic is 'inevitable where no method to find an optimal solution exists or is known,' particularly when 'the problem and/or the optimality criterion is ill-defined'." To add to this, Bobadilla-Suarez, in the journal paper, "Fast or Frugal, but Not Both: Decision Heuristics Under Time Pressure," notes "that heuristics do not form a uniform class and that "more frugal heuristics will not necessarily be faster to implement than less frugal ones. Similarly, less frugal strategies can be fast given the right stimuli [i.e., accelerants]" [6th DOI of Table VIII]. Drake as well as Fisher & Thompson add to what Bobadilla-Suarez asserted; as delineated in the journal paper, "A Novel Cooperative Multi-State Hyper-Heuristic for Combination Optimization Problems," "'mixing and combining different Low-Level Heuristics [LLH] produced better quality solutions than if they were applied separately' and 'showed that individual heuristics may be particularly effective at certain stages…but perform poorly at others,' as each individual heuristic may involve particular methods (as Zayat and Watrobski well noted) [6th DOI of Table VIII]. In essence, 6th DOI of Table VIII features the case study, wherein it "might be prudent to avoid utilizing AHP, SAW, and [Preference Ranking Organization Method for Enrichment Evaluation] PROMETHEE when handling negative values" [6th DOI of Table VIII].

The presets of ICSM2 and MAM (two of the key components of LHM) have now been touched upon. Prior to the unpacking of the proposed third component of LHM, CAIR-A, some background information will be provided. For the reader's convenience, a listing of the acronyms utilized thus far as well as for the sections that follow is being provided in Table X below.

TABLE X. LISTING OF ACRONYMS UTILIZED

Acronym	*Full Form*
A	Ambiguity
ACWS	Adaptive Criteria Weighting System
ADMP	Analytical DMP
AI	Artificial Intelligence
AICDS	Artificial Intelligence Control and Decision System
AIS	Artificial Intelligence System
AS	Autonomous System
ADMB	Automatic Differentiation Model Builder
ASR	Automatic Speech Recognition
A/U	Ambiguity/Uncertainty
AUV	Autonomous Underwater Vehicle
b-SHAP	Bespoke Shapley Additive Explanation
b-SHAP-Owen	Bespoke Shapley Additive Explanation Owen
c-SHAP	classical Shapley Additive Explanation
CAA	Conversational Artificial Intelligence Agent
CAIR	Conversational Artificial Intelligence Robustness
CAIR-A	Conversational Artificial Intelligence Robustness Accelerant
CAP	Component Assignment Problem
CBR	Case-Based Reasoning
CDC	Compressed Decision Cycle
CDS	Control and Decision System
CPRLD	**C**onstriction Factor-**P**article Swarm Optimization-**R**obust Convex Relaxation-**L**ong Short-Term Memory-**D**eep Convolutional Neural Network
CF	Constriction Factor
CRITIC	CRiteria Importance through Intercriteria Correlation
CWS	Criteria Weighting System
CWT	Continuous Wavelet Transforms
CVA	Conversational Virtual Agent
CW	Criteria Weight
DCGAN	Deep Learning Generative Adversarial Network
DCNN	Deep Convolutional Neural Network
DL	Deep Learning
DM	Decision-Making; Decision Maker
DMP	Decision-Making Problem
DMIP	Decision-Making Inflection Point
DOI	Digital Object Identifier
E	Execution
ELECTRE	ÉLimination Et Choix Traduisant la REalité
EM	Estimation-based Method
EUT	Expected Utility Theory
FCSO	Finite-Change Shapley-Owen
FN	Fuzzy Number
FS	Fuzzy Set
GAN	Generative Adversarial Network
GBR	Graph-Based Reasoning
G*f*FN	Generalized *f-sided* Fuzzy Number
GFN	Generalized Fuzzy Number
GFS	Generalized Fuzzy Set
G*Hp*FN	Generalized *Heptagonal* Fuzzy Number
G*Hx*FN	Generalized *Hexagonal* Fuzzy Number
GI	Graph Isomorphism
GL*f*FN	Generalized Linear *f-sided* Fuzzy Number
GLIV*Pe*FN)	Generalized Linear Interval-Valued *Pentagonal* Fuzzy Number
Glocal	a portmanteau "global" and "local"
GLIV*Pe*FN	Generalized Linear Interval-Valued *Pentagonal* Fuzzy Number
GL*Pe*FN	Generalized Linear *Pentagonal* Fuzzy Number
GN	Grey Number
G*N*FN	Generalized *N-sided* Fuzzy Number
GNL*f*FN	Generalized Non-Linear *f-sided* Fuzzy Number
GNLIV*Pe*FN	Generalized Non-Linear Interval-Valued *Pentagonal* Fuzzy Number
GNL*Pe*FN	Generalized Non-Linear *Pentagonal* Fuzzy Number
G*Oc*FN	Generalized *Octagonal* Fuzzy Number
G*Pe*FN	Generalized *Pentagonal* Fuzzy Number
GPL	General Public License
GPU	Graphics Processing Unit
GS	Grey Set
GSO	Generic Shapley-Owen
G*Tp*FN	Generalized *Trapezoidal* Fuzzy Number
G*Tr*FN	Generalized *Triangular* Fuzzy Number
HALU	Higher Ambiguity, Lower Uncertainty
HH	Hyper-Heuristic
I	Interpretability
IA	Independence Axiom
IBE	Inference to the Best Explanation
ICSM	Isomorphic Paradigm Comparator Similarity Measure
ICSM2	Isomorphic Paradigm Comparator Similarity Measure Module
IFF	If and Only If
IFN	Intuitionistic Fuzzy Number
IFS	Intuitionistic Fuzzy Set
IM	Intelligent-based Method
IPOPT	Interior Point OPTimizer
IsoP	Isomorphic Paradigm
IUC	Inherent Uncertainty Construct
IVFN	Interval-Valued Fuzzy Number
IVFS	Interval-Valued Fuzzy Set
IVIFS	Interval-Valued Intuitionistic Fuzzy Set
LAHU	Lower Ambiguity, Higher Uncertainty
LLH	Lower-Level Heuristic
LHM	**L**ower Ambiguity, Higher Uncertainty/**H**igher Ambiguity, Lower Uncertainty **M**odule
LLM	Large Language Model
LOE	Line of Effort
LSTM	Long Short-Term Memory
MADM	Multi-Attribute Decision-Making
MADM/MODM	Multi-Attribute Decision-Making/Multi-

SM/OM	Objective Decision-Making Subjective Method/ Objective Method
MAM	Metaheuristic Algorithm Module
MCDM	Multi-Criteria Decision-Making
M_i	Imprecision Membership
MILP	Mixed Integer Linear Programming
MINLP	Mixed Integer Non-Linear Programming
MIP	Mixed Integer Programming
ML	Machine Learning
ML2	Machine Learning on Machine Learning
M_m	Median Membership
MMSO	**M**ulti-Attribute Decision-Making/**M**ulti-Objective Decision-Making **S**ubjective Method/ **O**bjective Method
MNTZ	Monotonic/Non-Monotonic Transition Zone
MODM	Multi-Objective Decision-Making
MR	Monotonic Reasoning
NDZ	Non-Detection Zone
NLG	Natural Language Generation
NLP	Natural Language Processing
NLU	Natural Language Understanding
NMR	Non-Monotonic Reasoning
NOD	Non-Operational Data
NPDMP	Non-Programmed DMP
OD	Operational Data
OM	Objective Method
OSNS	Optimal Shapley-Nondominated Solution
OSONS	Optimal Shapley-Owen-Nondominated Solution
OUF	Over/Under Frequency
OUV	Over/Under Voltage
P-RM	Primary Reasoning Mechanism
PDMP	Programmed DMP
PJD	Phase Jump Detection
POSET	Partially Ordered Set
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluation
PSO	Particle Swarm Optimization
r-IsoP	Relaxed Isomorphic Paradigm
RMS	Relationship/Membership Stream
RR	Rank Reversal
RCAP or dP/dt	Rate of Change of Active Power
RCR	Robust Convex Relaxation
RDM	Robust Dialogue Management
RM	Reasoning Mechanism
ROCOF of df/dt	Rate of Change of Frequency
ROCPAD	Rate of Change of Phase Angle Difference
ROYG	Red-Orange-Yellow-Green
ROI	Return on Investment
RTIA or R-III	Reverse Triple I Algorithm
RWS	Real-World Scenario
S	Sensitivity
S-RM	Secondary Reasoning Mechanism
SAD	Situational Awareness Data
SAE	Society of Automotive Engineers
SCSO	"Squared Cohorts" Shapley-Owen
SDM	Sequential Decision-Making
SDP	Semi-Definite Programming
SQP	Sequential Quadratic Programming
SHAP	Shapley Additive Explanation
SNOPT	Sparse Nonlinear OPTimizer
SIS	Subsethood Inference Subsethood
SM	Subjective Method
SOP	Standard Operating Procedure (SOP)
STEA	System Transparency Explainability & Accountability
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution

TPU	Tensor Processing Unit
TIA	Triple I Algorithm
T1FS	Type 1 Fuzzy Set
T2FS	Type 2 Fuzzy Set
U	Uncertainty; [performance under] Uncertainty
UAV	Unmanned Aerial Vehicle
UDC	Uncompressed Decision Cycle
USV	Unmanned Surface Vehicle
V	Validity
VAP	Value-added Proposition
VBSO	Variance-Based Shapley-Owen
VIKOR	VlseKriterijumska Optimizacija I Kompromisno Resenje
vTHD	Voltage Imbalance and Total Harmonic Detection
VVS	Voltage Vector Shift
WM	Wavelet Transform-based Method

With these acronyms and their full forms in hand, it seems prudent to provide a quasi-dependencies equation so as to facilitate the parsing of this paper. The first equation will take the form presented in (1a), and the standard convention of a dashed arrow (wherein the dependent element is to the left of the tail, and the element that it is dependent upon is to the right of the arrowhead) is utilized.

AIS⇢**S**DM⇢**L**HM⇢ICSM2+MAM (1a)

The extended version for constituents supporting the module for LHM⇢ICSM+MAMF, for the purposes of this paper, becomes (1b).

LHM⇢**I**CSM2+**M**AM+**C**AIR-A (1b)

As it is understood that the LHM instantiation now includes CAIR-A as well, such as is reflected in (1b), (1) will simply be referred to as ASL-IMC. Proceeding on, the second equation will take the form, as presented in (2), using the same convention.

RDM ⇢**M**R/NMR⇢**M**NTZ⇢**I**bSOA (2)

For convenience, (2) will be referred to as RMMI. The follow-on third equation will then take the form, as is presented in (3), using the same convention.

ASL-IMC⇢RMMI (3)

For this paper, (3) will be the overarching operative quasi-dependencies equation, and (3) will be unpacked within this paper in the following fashion. Section I had provided an overview of the SDM challenge and provided some exemplar *presets* (e.g., LHM, ICSM2, MAM) as well as introduced a new module (e.g. CAIR-A) to be utilized for the experimentation herein; Section I also addressed (1a). The remainder of this paper is organized as follows. Section II presents the backdrop/background for an AIS approach towards DMP, such as SDM, which is predicated upon a robust harmonizing/counterpoising of MR and NMR; in turn, a successful MR/NMR counterpoising is dependent upon robust insights into the MNTZs. The MNTZs, in turn, are informed by IbSOA. Then, the previously referenced bespoke mechanism in the form of the referenced CAIR-A Module, which can assist with the ongoing CAA challenge of Robust Dialogue Management (RDM), is touched upon. Section III lays out some theoretical foundations and picks back up on the discussion regarding IbSOA and its derived insights, which can be instrumental at the MNTZs by gleaning the interplays among local, glocal (a portmanteau "global" and "local"), and global, particularly when conjoined with a robust MADM/MODM SM/OM (MMSO) mechanism; the section then proceeds with presenting some precursor experimentation as well as an interim discussion on IbSOA. Section IV provides some further thoughts as well as concluding remarks, and proposed future work closes the paper.

II. Background

The AIS approach towards DMP, such as SDM, may involve a variety of Reasoning Mechanisms (RMs). In the context of the referenced LHM, Analogical Reasoning, Abductive Reasoning, Inductive Reasoning, Probabilistic Reasoning, and Temporal Reasoning, among others, may be utilized. These primary RMs are described in Table XI.

TABLE XI. Types of primary RMs (P-RMs)

Type of Primary RM	*Definition*
Analogical Reasoning (AnaR)	Firt put it quite nicely and simply by saying that analogical reasoning is a "type of thinking that relies upon an analogy" [8]. Gentner affirms and deems it to be "the ability to perceive and use relational similarity between two situations or events" [9]. In a similar fashion, Thibaut refers to it as a "process in which a base domain and a target domain are compared in order to find relational correspondences" [10].
Probabilistic Reasoning (ProbR)	Nandi opines that probabilistic reasoning "is a framework used to make inferences and decisions under uncertainty" [11]. Nafar concurs, cites Oaksford and Chater, and notes that "probabilistic reasoning often aligns with Bayesian Rationalism," and within an AI context, Nafar refers to it as a "mapping [of] uncertainty…to a Bayesian probabilistic framework" [12].
Temporal Reasoning (TempR)	Leeuwenberg posits that temporal reasoning is "the process of combining different temporal cues into a coherent temporal view" [13]. Xiong unpacks this further and asserts that some of the types of temporal reasoning include "sequencing, duration, frequency, simultaneity, temporal relation, comparative analysis and facts extraction," which can be construed roughly as determining chronological order, how long an event lasted, how much time elapsed between the start of one event and the start of another, when a particular event occurred, whether events happened at the same time, and contrasting multiple events, respectively [14].
Deductive Reasoning (DedR)	Grote-Garcia put it quite well by noting that "deductive reasoning is the process of using general premises to draw specific conclusions" [15]. Along this vein, Taylor & Francis recaps this by stating that deductive reasoning "involves moving from the general to the specific, and it is used to draw conclusions based on known facts or assumptions" [16].
Inductive Reasoning (IndR)	The University of Illinois Springfield puts it succinctly: "inductive reasoning is the ability to combine pieces of information that may seem unrelated to form general rules or relationships" [17]. In essence, inductive reasoning is a technique of deriving conclusions by progressing from the specific to the general.
Abductive Reasoning (AbdR)	Sandoval-Hernandez asserts that "abductive reasoning involves starting with surprising or puzzling observations and inferring the most likely explanations" [18]. Thagard views it in a slightly more subdued fashion, wherein "explanatory hypotheses are formed and evaluated" [19]. Belzen follows Thagard and asserts that "abductive reasoning describes the theory-based attempt of explaining a phenomenon by a cause" [20].

In turn, both Analogical Reasoning and Abductive Reasoning may leverage Case-Based Reasoning (CBR), while the former may also utilize Graph-Based Reasoning (GBR). These are described in Table XII below.

TABLE XII. Types of secondary RMs (S-RMs)

Type of RM	*Definition*
Case-Based Reasoning (CBR)	Yan describes CBR as being "based on the cognitive assumption that similar problems have similar solutions" [21]. Taylor & Francis captures this as CBR being "a problem-solving approach that involves using past successful solutions to similar problems to solve new problems" [22].
Graph-Based Reasoning (GBR)	GBR approaches have been utilized for the purposes of, among others, that described in [23]. In essence, it is a technique that buttresses reasoning capabilities by characterizing problems as graphs. Cao adds an interesting addendum to this notion by putting forth the "Reasoning Graph Verifier" (a.k.a., "GraphReason") to "analyze and verify the solutions generated by LLMs," which Cao asserts "enhances the reasoning abilities of LLMs" and "also outperforms existing verifier methods in terms of improving these models' reasoning performance" [24];

Interestingly, GBR is often applied by AIS for Fault Detection (discerning anomalies and positing causes/root causes), Diagnosis, and NLP/NLU (for positing the context and intended meaning), such as described by the papers associated with the DOIs of Table XIII.

TABLE XIII. EXEMPLAR DOIS PERTAINING TO THE NOTIONS OF DETECTION, DIAGNOSIS, NLP/NLU

Facet	*DOI*
Detection	• 10.1109/CMD48350.2020.9287173 • 10.1109/CMD48350.2020.9287173 • 10.1109/CMD48350.2020.9287281 • 10.1109/CMD48350.2020.9287262 • 10.1109/CMD48350.2020.9287299
Diagnosis	• 10.1109/ICOIACT46704.2019.8938444
NLP/NLU	• https://ssrn.com/abstract=3789767

A. AIS for Various Sectors and a Proxy Application

AIS approaches, particularly for AI Control and Decision Systems (CDS) (AICDS), as pertains to DMP/SDM, remain an active research area not only for the sector touched upon in Section IA, but also for the sectors of, among others, healthcare, finance, and energy. These sectors tend to involve elongated temporal spans for evaluation metrics, and definitive results may be challenging to come by. While the referenced AUVs, USVs, UAVs, and self-driving cars can indeed be somewhat comprehensively evaluated, the benchmarking thereof can be cost prohibitive and may require elongated temporal spans (e.g., testing them under various weather conditions, such as detection capability against the white background of snow). Accordingly, the analysis of Abductive Reasoning and benchmarking of NLU as well as NLG can, potentially, be more cost effective and economize on time when using a proxy application, such as Conversational AI (wherein AI is used to mimic human-like conversational dialogue). For a CAA (a.k.a., Conversational Virtual Agent or CVA), beyond the essential first step of ASR, NLU and NLG are formidable downstream steps. It is imperative that the AIS be able to discern the conversational context as well as the overarching intent of the user (i.e., NLU) and respond accordingly (i.e., NLG). Presuming that NLU and meaningful NLG can be satisfactorily achieved, RDM becomes central. In essence, RDM involves the CAA being able to carry on a coherent conversation, wherein validity is, ideally, sustained both within the single interaction as well as over the course of several interactions (a.k.a., multi-turn conversations). Even within a single interaction, the conversation may be quite nuanced with a variety of inflection points. These inflections may also occur over the course of the multi-turn conversation. Maintaining consistency, coherency, and validity throughout these multi-turn conversations is central for meaningful/productive dialogues (i.e., RDM). Significantly, consistent validity inspires trust in the CAA. As the time-dependent priorities of the user and information being consumed by the CAA may dynamically change, the varying context could contradict prior information. Therefore, it is critical that the CAA has the capability for the graceful handling of contradictions with due care so as to help ensure conversational fluidity and maintain ongoing confidence and trust in the CAA. This capability is rooted in the successful counterpoising of the CAA's Monotonic Reasoning (MR) and Non-Monotonic Reasoning (NMR) and the handling of MR/NMR and the MNTZs (a.k.a., CAIR-A). MR/NMR (and the MNTZs) will be further unpacked in the following Sections IIB through IIC.

B. MR and NMR Counterpoising

Against the backdrop of AI and RMs, MR and NMR are two of the major pillars for AI-centric logic/RMs. These are described in Table XIV below.

TABLE XIV. TYPES OF AI-CENTRIC LOGIC/RMS

Type of AI-centric logic/RMs	**Definition**
Monotonic Reasoning (MR)	MR-centric responses will remain consistent throughout time despite whatever new information might arrive.
Non-Monotonic Reasoning (NMR)	NMR-centric responses allow for modification and/or retraction of prior assertions.

As noted in Section IA, "as the priorities of the user and arriving information may alter the context and/or contradict prior information," the harmonizing and counterpoising of the MR and NMR becomes critical for maintaining logical consistency and interconnectedness among the multi-turn conversations (i.e., coherence); if the constituent elements of the multi-turn conversation are indeed logically related, then the overall dialogue should be relatively free of contradictions. This constitutes "conversational coherence." There are a variety of evaluation tools in this regard. For example, the "ConversationCoherence Evaluator" is purported to be "a tool designed to check the coherence of conversations by an AI," as "it evaluates whether each response in a conversation logically follows from the previous messages, ensuring that the AI maintains context and relevance throughout the interaction" [25]. However, "conversational coherence" is, actually, quite difficult to maintain in RWS, where information is often sparse/incomplete and/or ambiguous/uncertain. Depending upon "what" and "when" the information is made available (the issues of provenance/pedigree regarding the "who" and the "where" will not be discussed here; rather, they will be discussed in future work), a specific RM may be used, such as shown in Table XV.

TABLE XV. TYPES OF PRIMARY REASONING MECHANISMS (P-RMS)

Information Available: "What"/"When"		***Type of RM Utilized***	***Conclusion***
• Facts • Accepted Truths • Rules • Scientific Laws • Mathematical Theorems • Established Principles • Specific	Used in an iterative fashion	Deductive Reasoning	Guaranteed to be true, if the premises and argument are valid.

Information • Logical Connections			
Starts with the same set as Deductive Reasoning (if available), but also involves Probabilistic approaches	Typically front-loaded, but can also "unfold in a bottom-up" fashion	Inductive Reasoning	Likely to be true, but it could be false despite the observations being accurate.
Starts with the same set as Deductive Reasoning (if available), but also involves hypotheses, assessments, and best-fit approximations	Typically front-loaded, but as it is trend-sensitive as well as aberration-sensitive, it can also "unfold in a bottom-up" fashion	Abductive Reasoning	Plausible best guess approximation or a posit as to the optimal explanation; along this particular vein, Harman is well-known for his research involving "Inference to the Best Explanation" (IBE) [26].

With regards to the various RMs presented in Table XV, first, deductive reasoning will be addressed. Ferguson proclaims that "classic deductive logic entails that once a conclusion is sustained by a valid argument, the argument can never be invalidated, no matter how many new premises are added. This derived property of deductive reasoning is known as *monotonicity*" [27]. Bundy and Wallen restates this as "the *monotonicity* of deductive logic," wherein "the addition of new axioms to a set of axioms can never decrease the set of theorems or facts" [28]. Fuhrmann summarizes the aforementioned with the notion that "deductive inference, at least according to the canons of classical logic, is *monotonic*; if a conclusion is reached on the basis of a certain set of premises, then that conclusion still holds if more premises are added" [29].

Next, turning to abductive reasoning, Hentenryck asserts that "as a form of reasoning appropriate for handling incomplete information, abduction is also closely related to non-monotonic reasoning" [30]. Paul affirms this by clearly stating: "abduction is a form of non-monotonic reasoning" [31]. Lagerkvist follows on by summarizing as follows: "one of the best-known examples of non-monotonic reasoning is *abductive* reasoning" [32].

Proceeding along to inductive reasoning, Leidinger cites Hans with regards to testing "nonmonotonic reasoning among other inductive reasoning tasks" [33]. What is particularly interesting is that Hans notes that in his testing, Large Multimodal Models (LMMs)/LLMs (e.g., GPT-3.5, GPT-4) were not able to capture "human behavior on the non-monotonicity phenomena," and actually took "the opposite" position [34]. Along this vein, although GPT-4 was much more successful in capturing human behavior than GPT-3.5, Hans highlighted the "notable exception" in "its failure to capture the phenomenon of premise non-monotonicity" with regards to inductive reasoning (a.k.a., property induction) [34]. *Hence, the handling of non-monotonic reasoning remains an ongoing challenge that may be impacting the performance of the newer LMMs/LLMs being released*. Generally speaking, Kazemi notes that LLMs have difficulty contending with contradictory information (thereby segueing into the challenges of contending with the non-monotonic realm). [35].

In brief, deductive reasoning is often construed as a form of MR (wherein the addition of relevant information buttresses conclusions reached based upon recitals of fact and evidentiary material) while abductive reasoning is often considered to be a form of NMR (wherein new information can potentially reverse [i.e., cause to retract] prior inferences reached through evidentiary material and reasoning). Hence, inductive reasoning is typically taken to be non-monotonic, but the literature has also noted cases where it is weakly monotonic; for example, Janke delineates how NMR "is inherently required in several approaches to inductive inference" [36]. Overall, insights into MR/NMR behavior (within the MNTZ) and an apropos harmonizing/counterpoising of MR/NMR can provide the requisite flexibility for RDM and the desired ensuing CAA conversational coherence.

C. *Insights into the MNTZ*

As noted at the end of Section IIB, the successful counterpoising of MR/NMR necessitates meaningful insights into the MNTZ. Yet, as discussed in [2], the discerning/comprehending of the behavior at the MNTZ can be quite challenging. In addition, [2] noted that "maintaining coherence and monotonicity is non-trivial, as the involved AIS might discern connections (particularly those that are non-monotonic) within the evolving dataset. In the context of CAIR at-large (CAIR-A, specifically), non-monotonic aspects can arise as incoming information can re-contextualize and/or contradict matters. Yet, enforcing a strict monotonic paradigm can segue to an unnatural rigidity and/or incorrect/irrelevant responses by the" CAA. Hence, the counterpoising of MR/NMR is non-trivial.

As further noted in [2], "the shift of the involved variables from a monotonic to a non-monotonic paradigm can be quite unexpected and occur more frequently than anticipated/desired." For AIS-SDM-A/U, the spawning of "non-monotonic, non-polynomial, and even non-continuous functions" does indeed occur more frequently than expected or desired [37]. Restated, as pertains to the MNTZ, there exists a proclivity for "spawning to the NP-Hard side," and this is similar to the situation, "wherein the transformation of non-convex Mixed Integer Non-Linear Programming (MINLP) to convex problems often spawn further non-convex MINLP problems" [2]. With regards to

Monotonic/Non-Monotonic and Linear/Non-linear (wherein Non-Monotonic can be Continuous or Discontinuous and Non-Linear can be Polynomial or Non-Polynomial), Figure 2 conveys some of the prospective pathways to convex form (in green) as well as the pathways that remain nonconvex (in red) [2]; MIP equates to Mixed Integer Programming, and MILP equates to Mixed Integer Linear Programming.

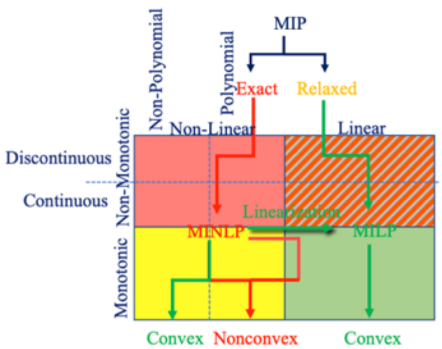


Figure 2. "Non-convex to convex transformation pathways (e.g., non-convex non-monotonic, non-polynomial, non-continuous MINLPs to convex form)" [2].

For the convex form, a myriad of Semi-Definite Programming (SDP) solvers can be leveraged to handle the computations in polynomial time ("presuming further spawning does not occur," since "NP-hard-related spawn can potentially congest matters with an indefinite impasse") [2]. There are a variety of mechanisms that can be useful in this regard, and some of these facets are discussed in Section IID.

D. Formulating a Bespoke Mechanism for the MNTZ

As noted in Sections IIA and IIB, the type of reasoning and prospective RM combinatorial (Analogical, Probabilistic, Temporal, Deductive, Inductive, Abductive, etc.) can indeed impact the MR/NMR amalgam and the likely behavior within the MNTZ. For example, as noted in the prologue to Section II, various forms of primary RM (and various secondary RMs, which they may leverage) are utilized in the context of the referenced LHM. Taking the examples of Tables XI and XII, Table XVI is presented, which delineates the referenced RMs and their MR/NMR categorization.

TABLE XVI. EXAMPLE RMS AND THEIR MR AND/OR NMR CATEGORIZATION

RM	***MR/NMR Categorization***
Analogical Reasoning	*Non-Monotonic*; Kerber notes that "two modi of analogical reasoning" "rely on different forms of relevance knowledge that cause non-monotonicity" [38].
Probabilistic Reasoning	*Monotonic* and *Non-Monotonic*; Liu notes that "it has been found that the ability to handle incomplete information or to perform nonmonotonic reasoning does not exist in some probabilistic reasoning mechanisms" (i.e., these mechanisms can only contend with Monotonic), while Grosof defines "an approach to non-monotonic probabilistic reasoning" [39][40].
Temporal Reasoning	*Monotonic* and *Non-Monotonic*; Baral asserts that temporal reasoning is monotonic much of the time, but amidst new incoming information, the understanding of past, present, and future can alter, and this necessitates "that goals be changed non-monotonically" [41].
Deductive Reasoning	*Monotonic*; As noted earlier in Section IIB, Ferguson notes that Deductive Reasoning is monotonic [27]. However, it should be noted that Hunter makes the caveat that deductive argumentation is non-monotonic [42].
Inductive Reasoning	*Weakly Monotonic* and *Non-Monotonic*; as noted earlier in Section IIB, Jantke notes that NMR "is inherently required in several approaches to inductive inference" [36]. Jantke builds upon this by asserting, "monotonic and non-monotonic reasoning is introduced into inductive inference," discusses a "weakly monotonic inductive inference algorithm," and notes that "consistency and monotonicity can hardly be achieved simultaneously" [37]. The general idea is that a monotonic approach can be used to formulate hypotheses in an incremental fashion, but during this formulation, revisions/retractions may be necessary, and this segues into the area of non-monotonicity. Jantke summarizes with the statement "in the area of inductive inference of total recursive functions monotonicity can rarely be guaranteed" [43].
Abductive Reasoning	*Non-Monotonic*; as noted earlier in Section IIB, Hentenryck and Paul have noted that abductive reasoning is in the realm of non-monotonic reasoning [30][31]. Pereira affirms this [44].
Case-Based Reasoning (CBR)	*Cautiously Monotonic* and *Non-Monotonic*; the premise of CBR is that prior cases are examined to determine a best-fix approximation (i.e., ascertaining the most similar cases) to resolve a current situation. However, new cases may reverse the determinations of prior cases, and new information may result in revision/retraction to an earlier conclusion. However, it should be noted that Paulino-Passos makes the caveat by defining a variation, which is cautiously monotonic [45].
Graph-Based Reasoning (GBR)	*Monotonic* and *Non-Monotonic*; when the addition of edges reinforces the conclusion, the paradigm is monotonic, but when the subtraction/negation of edges or addition of edges that obviate prior paths occurs, it is significant to note that Bochman asserts that GBR-related representations and nonmonotonic inheritance/nonmonotonic reasoning are intricately connected [46].

When Table XVI is sorted by the prevalence of monotonicity to non-monotonicity, the results are as depicted in Table XVII with Red-Orange-Yellow-Green (ROYG) color coding, wherein Monotonic is indicated by green and Non-Monotonic is indicated by red; weakly monotonic is indicated by orange, and cautiously monotonic is indicated by yellow.

TABLE XVII. RM-CENTRIC SORTING OF MR AND NMR BY PREVALENCE

RM	***MR/NMR Categorization***	
Deductive Reasoning (DedR)	*Monotonic*	
Probabilistic Reasoning (ProbR)	*Monotonic*	*Non-Monotonic*
Temporal Reasoning (TempR)	*Monotonic*	*Non-Monotonic*
Graph-Based Reasoning (GBR)	*Monotonic*	*Non-Monotonic*
Case-Based Reasoning (CBR)	*Cautiously Monotonic*	*Non-Monotonic*
Inductive Reasoning (IndR)	*Weakly Monotonic*	*Non-Monotonic*
Analogical Reasoning (AnaR)	*Non-Monotonic*	
Abductive Reasoning (AbdR)	*Non-Monotonic*	

Depending upon the amalgam of RMs and ensuing RM pathways initially selected, such as by the LHM-at-large, the related DM points/DMIPs will likely impact the downstream behavior within the MNTZ in terms of a preponderance of monotonic or non-monotonic behavior. Furthermore, as previously noted in the Abstract, "shifts from a monotonic to a non-monotonic paradigm and vice versa in the MNTZs can occur at a higher than anticipated rate." This is also previously in Section IIC noted: "the shift of the involved variables from a monotonic to a non-monotonic paradigm can be quite unexpected and occur more frequently than anticipated/desired" [2]. Accordingly, given the propensity to migrate towards non-monotonic, the ability to well-handle non-monotonic and somewhat mitigate against the shift from a monotonic to a non-monotonic paradigm seems crucial and prudent. Again, the involved counterpoising is key, as the simpler strategy of reliance (readily segueing into overreliance) upon a non-monotonic paradigm can readily lead to complicated models that are prone to overfitting and, quite possible, consume a not insignificant portion of the computational resources available.

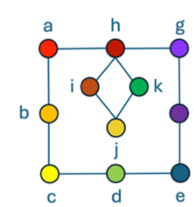
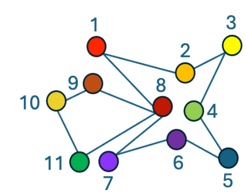
While it seems that focusing on certain RMs (e.g., deductive) might seem computationally more tractable, it would preclude more RWS RM approaches (e.g., ProbR, TempR, and GBR). Yet, these approaches can segue to non-monotonic unexpectedly. Moreover, the much-desired CBR and IndR approaches are at much higher risk for segueing to non-monotonic (as they are already cautiously monotonic and weakly monotonic, respectively). Finally, some of the desired "more sophisticated approaches" (e.g., AnaR, AbdR) squarely reside within the non-monotonic realm, as is evidenced by Table XVII.

Within the realm of analogical reasoning, isomorphism exhibits some promise for being computationally tractable, and certain approaches leverage potentially more accelerated pathways. The amalgam of ICSM2, MAM, and CAIR-A is one such triumvirate approach.

1) ICSM2

As noted in Section IB, the LHM is buttressed by, among other modules, an ICSM2, which determines (should it be needed) antecedent occurrences of IsoPs. In graph theory, if there exists a one-to-one correspondence between the vertice set of S and S', then S and S' are isomorphic. This is shown in Table XVIII, and should the reader desire, this can be affirmed via a variety of tools, such as the one available at https://graphonline.top/en/?graph=xPLjwOkrglDRgYeS. Rather than the graphs, adjacency matrices can also be utilized to determine isomorphism. This can be affirmed via a variety of tools, such as the one available at https://graphonline.top/en/create_graph_by_matrix.

TABLE XVIII. EXEMPLAR ISOMORPHISM BETWEEN S AND S'

Isomorphism between S and S'	***Graph S***	***Graph S'***
f(a) = 1 *f(b)* =2 *f(c)* =3 *f(d)* =4 *f(e)* =5 *f(f)* =6 *f(g)* =7 *f(h)* =8 *f(i)* =9		

Prior work on "graph theory isomorphisms" or Graph Isomorphism (GI) included aberration detection as pertains to the smart grid. Exemplar publications on this thematic are shown in Table XIX.

TABLE XIX. EXEMPLAR DIGITAL OBJECT IDENTIFIERS (DOI) PERTAINING TO GRAPH ISOMORPHISM (GI)

Facet	***DOI***
GI	• 10.1109/IEMCON.2019.8936241 • 10.1109/IAICT62357.2024.10617473 • 10.1109/GEM61861.2024.10585580 • http://dx.doi.org/10.2139/ssrn.5183492 • http://dx.doi.org/10.2139/ssrn.4984663

For that particular Line of Effort (LOE), various aberration detection techniques were utilized, and these are shown in Table XX below.

TABLE XX. EXEMPLAR DOIS PERTAINING TO ABERRATION DETECTION TECHNIQUES

Short-Form Acronym for various Aberration Detection Techniques	***Long-Form Name***
OUV	Over/Under Voltage
OUF	Over/Under Frequency
PJD	Phase Jump Detection
RCAP or dP/dt	Rate of Change of Active Power
ROCOF of df/dt	Rate of Change of Frequency
VVS	Voltage Vector Shift
vTHD	Voltage Imbalance and Total Harmonic Detection (THD)
ROCPAD	Rate of Change of Phase Angle Difference
IM	Intelligent-based Method
EM	Estimation-based Method
WM	Wavelet Transform-based Method

The techniques presented in Table XX were then sorted by Non-Detection Zone (NDZ) and Speed, wherein green indicates either a good NDZ or speed, and red indicates either a poor NDZ or speed, such as shown in Table XXI, row 2. This is then recast in radar chart form, which is shown in Table XXI, row 4. For this particular comparison, the main point to convey, for the presented example in Table XXI, is that the structure itself holds no meaning, as only the values associated with each technique has any significance. Hence, no IsoP testing/comparison needs to be performed (with the associated savings of computational resources).

TABLE XXI. EXAMPLE WHEREIN THE STRUCTURAL TOPOLOGY IS IRRELEVANT; NO ISOP IS NEEDED

Techniques sorted by NDZ

Sorted by NDZ

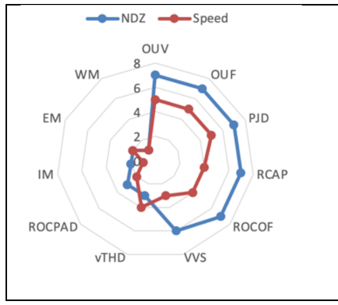
Technique	NDZ	Speed
OUV	7	5
OUF	7	5
PJD	7	5
RCAP	7	4
ROCOF	7	4
VVS	6	3
vTHD	3	4
ROCPAD	3	2
IM	2	1
EM	2	2
WM	1	1

Techniques sorted by Speed

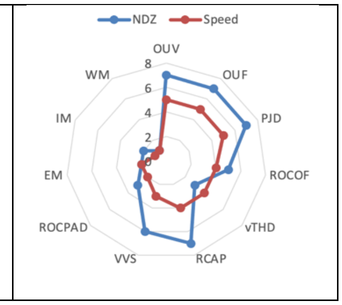
Sorted by Speed

Technique	NDZ	Speed
OUV	7	5
OUF	7	5
PJD	7	5
ROCOF	5	4
vTHD	3	4
RCAP	7	4
VVS	6	3
ROCPAD	3	2
EM	2	2
IM	2	1
WM	1	1

Radar Chart of Techniques characterized by NDZ and Speed (sorted by NDZ)



Radar Chart of Techniques characterized by NDZ and Speed (sorted by speed)



For many ecosystems, wherein addressing computational forays with graph-based approaches can be invaluable, “non-graph-based approaches” can be of comparable value-added proposition (VAP). By way of example, the robust treatment of various figures of speech (e.g., simile, metaphor) and argument (e.g., analogy) (a.k.a., collectively, elements of figurative language) can be invaluable, such as reviewed in Table XXII.

TABLE XXII. EXEMPLAR ELEMENTS OF FIGURATIVE LANGUAGE

Exemplar Element	***Description***
Simile	is a comparison of two disparate entities, via words, such as “like” or “as.”
Metaphor	is a direct comparison and asserts that two disparate entities are the same, via words, such as “is,” “was,” etc. (wherein the words “like” or “as” are not utilized).
Analogy	creates a comparison of how a seemingly disparate entity is akin to, relates to, or is similar to another disparate entity for the purpose of explaining/demonstrating.
Allegory	embodies a more complex/symbolic comparison and leverages a narrative to convey an abstract notion/concept; Wearing points out that “allegories make sense when they’re interpreted literally while (most) metaphors do not” [47]. Holme asserts that “arguably, an allegory frames all the events of a story inside an extended metaphor” [48]. More simplistically, Burton claims that “an allegory is a complete narrative that seems to be about one thing but is actually about another” [49].

Hofstadter has argued that analogy is “the core of cognition,” and Holyoak seems to affirm [50][51]. Hofstadter further states, “without concepts there can be no thought, and without analogies there can be no concepts” [52]. Given the significance of analogies, it seems prudent to review analogies at various levels of intricacy. Wijesiriwardene asserts that analogies can be viewed “at four distinct levels of complexity: lexical, syntactic, semantic, and pragmatic,” and he further notes that “as the analogies become more complex, they require increasingly extensive, diverse knowledge beyond the textual content, unlikely to be found in the lexical co-occurrence statistics that power LLMs” [53].

A discussion on Large Concept Models (LCMs), to assist in this regard, can be found within those DOIs shown table XXIII below.

TABLE XXIII. EXEMPLAR DOIS/TITLES OF PAPERS PERTAINING TO LARGE CONCEPT MODELS (LCMS)

Facet	***DOI***
LCM	• 10.1109/IAICT65714.2025.11100570
	• A Prospective Monotonic/Non-Monotonic Transition Zone Impediment for Concept Model-Centric Artificial Intelligence Systems

Wijesiriwardene depicts the increasing levels of analogy complexity, and a version is shown in Figure 3 below.

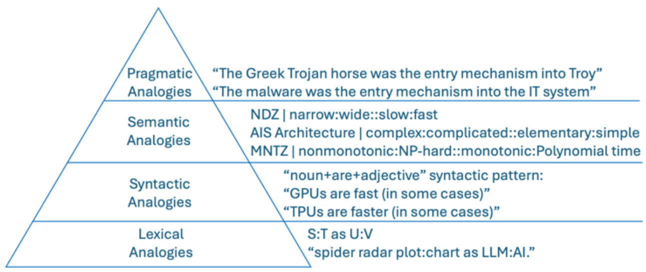


Figure 3. "Increasing Levels of Analogy Complexity" [53].

These increasing levels of analogical complexity are further delineated in Table XXIV.

TABLE XXIV. EXEMPLAR ANALOGY TYPES

Analogy Types	***Descriptors***
Lexical Analogies	Lexical analogies center upon the affinity and shared relationships between concepts despite the concepts being quite distinct and disparate. An example would be S:T as U:V or "spider radar plot is to chart as LLM is to AI."
Syntactic Analogies	Syntactic analogies undertake reasoning by comparison so as to discern structural affinity and relationships between syntactic structures. An example would be that of the "noun+are+adjective" syntactic pattern, which can be seen in the following: "Graphics Processing Units (GPUs) are fast (in some cases)" and "Tensor Processing Units (TPUs) are faster (in some cases)."
Semantic Analogies	Semantic analogies center upon the affinity and shared relationships between entities (while the underlying structure may not be the same). Examples include antonymy (e.g., narrow:wide::slow:fast), synonymy (e.g., complex:complicated::elementary:simple), cause & effect (nonmonotonic:NP-hard::monotonic:Polynomial time), etc.
Pragmatic Analogies	Pragmatic analogies are often predicated upon figurative language, wherein literal translations may not convey the implied meaning intended. A classic example would be, "Reading this paper was a piece of cake;" in essence, the intended meaning was that the paper was easy to read rather than the literal translation that the paper was an edible slice of cake. Wijesiriwardene's example of the Trojan horse theme and comparing it to malware is particularly appropriate for modern times.

Wijesiriwardene explains that "pragmatic analogies are the most complex [of the analogy types], spanning several sentences (often a paragraph) that elaborate on both the source and target domains, contain multiple concept or entities related by diverse relationships, contain abstractions (modeled as subgraphs), and require us to map concepts/entities, relationships and subgraphs between source and target contextualized by external knowledge and a purpose" [53]. To further explain this, if S and S' are isomorphic, then they are analogous in their underlying linkages/relationships and structure. However, while the various forms of analogies exhibit similarities, there may not be a seamless structural match. It then follows that while the gamut of isomorphic cases can be considered analogies, not all analogies are isomorphic. This particular determination is central. If IsoP can be avoided, then the expenditure (of an unknown level of computational resource expenditure) can be bypassed. However, if an IsoP comparison is indeed required, then the ICSM2 operationalizes the IsoP comparison via the derivation of a specific amalgam, which is comprised of two key facets, such as shown in Figure 4 below.

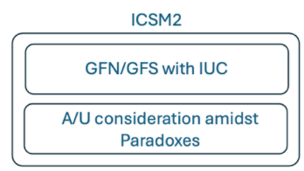


Figure 4. ICSM2's Derivation of a specific amalgam: (1) a pertinent GFN/GFS and Inherent Uncertainty Construct (IUC), and (2) A/U Consideration Amidst Paradoxes

Figure 4 needs some unpacking. First, starting with the more discrete facet, Generalized Fuzzy Numbers (FN) (GFN) can be quite useful. The type of shape of the involved GFN is typically captured by the "*f-sided*" notation within the long form of "Generalized *f-sided* Fuzzy Number" as well as by "*f*" notation in the short-form of the "G*f*FN" acronym. The G*f*FN membership function can extend into a more sophisticated shape beyond the more common and simplistic 3-sided (i.e., triangular) or 4-sided (i.e., trapezoidal, which has a minimum of one pair of parallel sides) prototypical shapes. Examples of the involved notation for these more sophisticated shapes are shown in Table XXV below.

TABLE XXV. EXEMPLAR GENERALIZED FUZZY NUMBERS (GFNS) WITH SHAPE TYPES

Short-Form Acronym	***Long-Form Name***
G***Tr***FN	Generalized ***Triangular*** Fuzzy Number

G***Tp***FN	Generalized ***Trapezoidal*** Fuzzy Number
G***Pe***FN	Generalized ***Pentagonal*** Fuzzy Number
G***Hx***FN	Generalized ***Hexagonal*** Fuzzy Number
G***Hp***FN	Generalized ***Heptagonal*** Fuzzy Number
G***Oc***FN	Generalized ***Octagonal*** Fuzzy Number

In some cases, within the literature, the *Tr, Tp, Pe, Hx, Hp, Oc*, etc. are supplanted by "*N*," and the utilized acronym is that of G*N*FN. Regardless, the *"N-sided"/"f-sided"* nature allows the membership function to be defined by a greater multiple of linear and/or non-linear functions for better capturing/representing the U of A/U than that by the simpler [prototypical] shapes (given their limitations with regard to their constituent number of linear and/or non-linear functions able to be represented). However, a further distinction is also made. A G*f*FN can not only have a varying number of *f* sides, but also the sides comprising the boundary of the G*f*FN might not necessarily be linear (i.e., the sides can be non-linear, such as in the case of a Gaussian curve, etc.). Accordingly, the notation can be of the form GL*f*FN for a Generalized Linear *f-sided* FN and GNL*f*FN for a Generalized Non-Linear *f-sided* FN. Taking the example case of 5-sides, GL*Pe*FN denotes the involved "Generalized Linear *Pentagonal* FN," wherein the sides are all linear; in contrast, GNL*Pe*FN denotes Generalized Non-Linear *Pentagonal* FN, wherein the side(s) might be non-linear. Chakraborty depicts a GL*Pe*FN in [54], and a variation is shown in Figure 5.

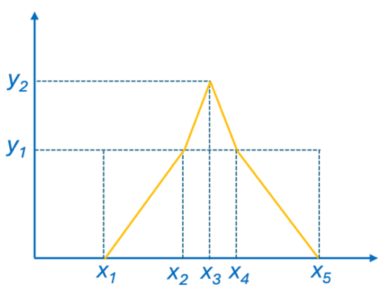


Figure 5. An exemplar GLPeFN

Chakraborty also provides an example of a GNL*Pe*FN, and a version is reproduced below in Figure 6 [54].

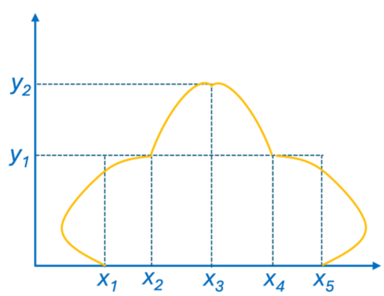


Figure 6. An exemplar GNL*Pe*FN

In some cases, the GL*Pe*FN approximation can suffice, rather than computing the GNL*Pe*FN, and Velu & Ramalingam provide an example of this, for which a variation is shown below in Figure 7 [55].

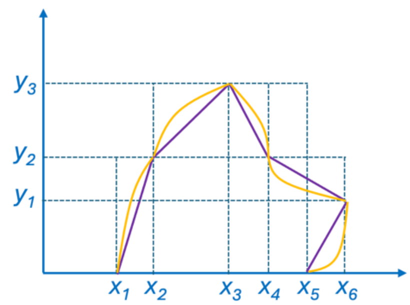


Figure 7. An exemplar GNL*Pe*FN

Before plowing forward, some background information is necessary; an Interval-Valued Fuzzy Number (IVFN) is a fuzzy number, wherein the degree of membership is denoted by an interval (as a range of prospective membership values) instead of a single value. Along this vein, Chakraborty further presents a Generalized Linear Interval-Valued *Pentagonal* Fuzzy Number (GLIV*Pe*FN), and a variation is shown in Figure 8 (with symmetry) along with a *Hexagonal* rendition (GNLIV*Hx*FN) in Figure 9 (with asymmetry) [54].

Figure 8. An exemplar GLIV*Pe*FN (with symmetry)

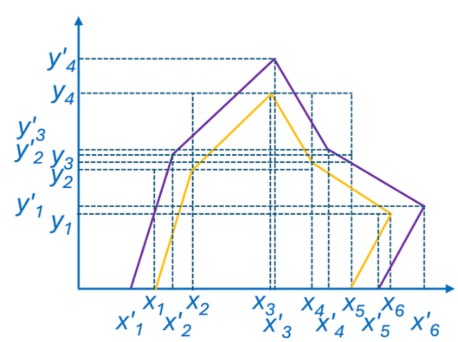


Figure 9. An exemplar GLIV*Hx*FN (with asymmetry)

Chakraborty further presents a Generalized Non-Linear Interval-Valued *Pentagonal* Fuzzy Number (GNLIV*Pe*FN), but to be consistent with Figure 9, a Hexagonal rendition (GNLIV*Hx*FN) is cast below in Figure 10 (with asymmetry).

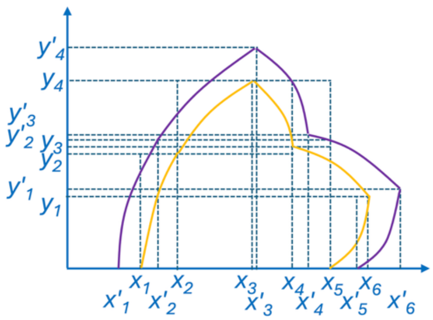


Figure 10. An exemplar GNLIV*Hx*FN (with asymmetry)

Again, as previously shown in Figure 7, for some cases, best-fit approximations (or even more rudimentary approximations) may suffice. This raises the issue of which particular approach vector might be prudent (depending upon the need). Prior to unpacking this, Intuitionistic Fuzzy Numbers (IFNs) will be explained. In essence, IFN embody both membership and non-membership degrees, thereby providing a more nuanced delineation of uncertainty. Mert depicts a Generalized Non-Linear P [Intuitionistic Logic] *Pentagonal* Fuzzy Number (with asymmetry), but to be consistent with Figures 9 and 10, a hexagonal version is cast in Figure 11 [55].

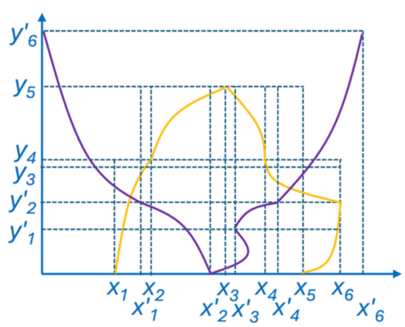


Figure 11. An exemplar GNLIL*Pe*FN (with asymmetry)

As can be gleaned, IFNs and IVFNs depict U in varying ways. For some cases, IFNs are able to be expressed in IVFN form (e.g., when membership as well as non-membership values are expressed as single-point intervals), such as in the case wherein (0.9, 0.1) segues to ([0.9, 0.9], [0.1, 0.1]). In contrast, the reverse is not necessarily true, such as in the case, wherein ([0.1, 0.9], [0.3, 0.7]) might not necessarily translate to a "single, representative IFN." For certain basic operations (e.g., "and", "or"), equivalence can be demonstrated. However, this is not necessarily the case for more complicated operations.

Proceeding from the various GFNs to the broader category of Generalized Fuzzy Sets (GFSs) can be quite useful as well. By way of context, the classical FS involves a membership function that assigns a degree of membership (e.g., between 0 and 1) to each element within the involved FS. However, this does not necessarily reflect the *hesitation* for scenarios with incomplete/uncertain information. In contrast, GFS encompass a range of extensions to the classical Fuzzy Set (FS), such as that of Intuitionistic FS (IFS), Interval-Valued FS (IVFS), etc. Along this vein, IFR equates to Intuitionistic Fuzzy Reasoning and IVFR equates to Interval-Valued Fuzzy Reasoning. In terms of isomorphism, IFSs and IVFSs have been shown to be formally equivalent [56][57]. In essence, they share equivalent mathematical structures in spite of their differences in representing U. Luo had expressed this diagrammatically, for which a version is reproduced in Figure 12 [57]. With regards to Figure 12, TIA equates to the Triple I Algorithm (i.e., "a fuzzy reasoning algorithm," which posits an output given a specified input), RTIA equates to the Reverse Triple I (a.k.a., R-III) Algorithm (i.e., "a fuzzy reasoning algorithm," which inverts the TIA so as to ascertain an apropos input given a specified desired output), SIS equates to Subsethood Inference Subsethood (i.e., the degree of containment of a FS within another FS), *b* equates to bijection, and *i* equates to isomorphic [57].

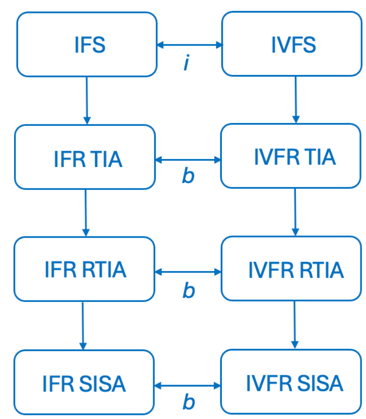


Figure 12. The IFS and IVFS Isomorphism and various Bijections [57]

The significance of this resides in the paradigm that, generally, IFS tend to be quicker to compute than IVFS (particularly in the area of basic comparisons/operations). Moving beyond IVFS, the Interval-Valued Intuitionistic Fuzzy Set (IVIFS) yields even "more precise results" [58]. After all, while an IVFS depicts the uncertainty of membership via an interval, an IVIFS leverages both an interval-centric degree of membership as well as an interval-centric degree of non-membership. Thus, IVIFS is even more nuanced than IVFS. Fortunately, it is a generalization of IFS, which is isomorphic to IVFS, as Bustince well depicted, and a version is reproduced in Figure 13 [118]. Along the vein of Figure 13, the Type 2 Fuzzy Set (T2FS) had been discussed in, among other papers, the second DOI of Table IV, which discussed the notion that the "Type-2 Fuzzy Set (T2FS)…can accommodate the uncertainty of membership fluidity, whereas the Type-1 Fuzzy Set (T1FS) only accommodates membership invariableness" [second DOI of Table IV][first

DOI of Table VIII][second DOI of Table VIII]. Along with IVFS, IFS, IVIFS, there are also the notions of Vague Set (VS) and Grey Set (GS).

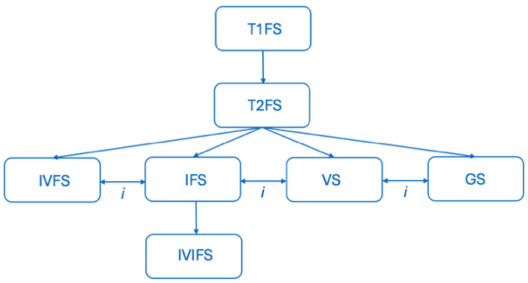


Figure 13. Mathematical Equivalence among particular GFS [118]

Lu illuminates the notions that VS "is more natural than an IFS for merging fuzzy" values [59]. A classic example involves the merging of three fuzzy values, such as 0.3/*m*, 0.7/*m*, and 0.9/*m*; the resultant VS expression would be [0.3, 0.9]/*m*, wherein the lower bound of the membership *m* is 0.3 and the upper bound is 0.9. In contrast, the IFS expression would be less intuitive with 0.6 for the degree of membership, 0.2 for the degree of non-membership, and 0.2 for the hesitation margin. Furthermore, Lu notes that ascertaining the Median Membership (M_m) and/or the Imprecision Membership (M_i) is more intuitively gleaned via VS, such as depicted in Figure 14, which is a variation of Lu's figure.

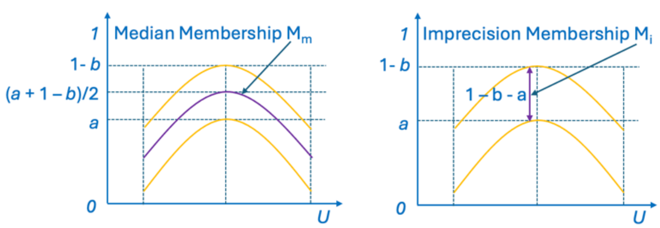


Figure 14. Delineating the Median and Imprecision Membership via VS, where y axis denotes "Membership Functions" and the x axis denotes "Data Objects" [59]

Likewise, delineating the hesitation region for VS is much more straightforward, such as shown in Figure 15. In essence, VS is more intuitive for delineating the Support Region, Opposition Region, and the Hesitation Region (wherein there is neither support nor opposition).

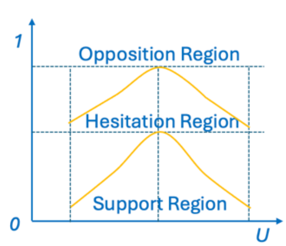


Figure 15. Delineating the Hesitation Region for VS (as contrasted to IFS)

As Lu discusses, "in the literature, the notions of IFSs and VSs are regarded as equivalent," in the sense that an IFS is isomorphic to a VS [59]. This is useful for transitioning among IVFS, IFS, VS, and GS [118]. Next, there is also a distinction between VS and GS. For example, Alkhazaleh states that a VS "is defined by a truth-membership function … and a false-membership function" [60]. Lu adds to this by asserting that "interval-based membership is used in a VS" and that "the interval-based membership generalization" in VS "is more expressive in capturing vagueness of data" [59]. On the other hand, GS leverages the notion of Grey Numbers (GNs), and Khuman notes that "the generalised" GN "can cater to both discrete and continuous data" [61]. Although there is the obvious connection between GS and GN, they are distinguished by the fact that a GS captures uncertainty about membership to the involved set while a GN denotes uncertainty about the actual value of a particular quantity. Sifeng provides a generalization for both: "the information that grey is often associated with is information that can be described as being partially known and partially unknown, which in actuality is a common occurrence of uncertain systems" [62]. Along this vein, Shaker and Moore-Clingenpeel reference the epistemological (pertaining to the theory of knowledge) constructs of the "known knowns" (KK), "known unknowns" (KU), and "unknown unknowns" (UU) that were infamously popularized by Donald Rumsfeld. The extrapolated quad chart, which also references the writings of Shaker and McGregor, is shown in Table XXVI [63][64].

TABLE XXVI. EPISTEMOLOGICAL CONSTRUCTS [63][64]

Known Knowns (KK) "Things we are aware of and understand"	*Known Unknowns (KU)* "Things we are aware of and do not understand"
Unknown Knowns (UK) "Things we are not aware of, but understand"	*Unknown Unknowns (UU)* "Things we are not aware of and do not understand"

With regards to Sifeng's referencing of "partially known" and "partially unknown," this could pertain to the KK and KU for the former and KU, UK, and UU for the latter. As previously stated in the prologue of Section II (and as exhibited in Tables XI, XII, XVI, XVII and the broader categorizations of Table XIV), despite the varied RMs that might be utilized, an inconsistency/contradiction could occur due to phenomenon, such as the Ellsberg Paradox, which can be described as "ambiguity aversion" — a predilection to avert alternatives whose prospects are unknown; citing the logic of Lang ("zero exposure to ambiguity can be optimal") as well as the assertions of Ellsberg, Machina, and others, ambiguity aversion "violate[s] both the key rationality axioms and classic models of choice under uncertainty" [65]. For a number of cases, ambiguity aversion segues to sub-optimal DM [65][66]. As a case in point, Jia's findings show that experimental participants preferred KK over KU/UK/UU even when the options with U might be more favorable (it is not clear whether KK/KU prevailed over UK/UU). In any case, Jia does indeed note that "participants who learned about the Ellsberg Paradox were more tolerant of ambiguity, yet ambiguity aversion was not completely abolished" [67]. Coleman summarizes matters as: "The Ellsberg paradox is often cited as evidence for unknowable 'ambiguity' versus computable 'risk' and a refutation of the Savage axioms regarding expected utility maximization" [68][69]. Chen describes the Ellsberg paradox more simply, as "people prefer betting on known (objective) probabilities rather than unknown (subjective) probabilities" [70]. Weber describes it as "in *ambiguity* over time, the eventual outcome is known, but the length of time before the outcome will occur is *uncertain* [71]. In Table I of Section I, it is clear that ambiguity and uncertainty are indeed different. Yet, despite these differences, definitions of the "Ellsberg paradox" demonstrate how these terms are often used interchangeably (with a lack of distinction made between the two). Exemplars of this are shown in Table XXVII.

TABLE XXVII. EXEMPLAR UTILIZATION OF AMBIGUITY (A) VERSUS UNCERTAINTY (U) IN VARIED DEFINITIONS OF THE ELLSBERG PARADOX

Researcher	***Remarks***
Binmore	"Experimental results on the Ellsberg paradox typically reveal behavior that is commonly interpreted as *ambiguity* aversion" [72].
Leopold	"Smarter in the Long-Term: Diminishing *Ambiguity* Aversion in a Repeated Ellsberg Urn Task" [73].
Chen	*"Ambiguity* Aversion: The Ellsberg paradox shows that people prefer betting on known (objective) probabilities rather than unknown (subjective) probabilities" [74].
Halevy and Feltkamp	"The Ellsberg paradox demonstrates that people's belief over *uncertain* events might not be representable by subjective probability" [75].
Segal	"Measuring Nonmonetary Utilities in *Uncertain* Choices: The Ellsberg Urn" [76].
Joreno-Jimenez and Vargas	"we present a method to deal with *uncertainty*, which considers Ellsberg's objections" [77].
Jabarian and Lazrus	"55% of the subjects prefer avoiding *ambiguity* even when it means choosing dominated risky options – what we call the Two-Ball Ellsberg Paradox" [78].

Perhaps, the preferred embodiment for the use of the cited terms (ambiguity, uncertainty), with regards to the Ellsberg paradox, is as follows. In essence, the Ellsberg paradox alludes to the notion that people have a predilection towards *choices with less uncertainty*, thereby running counter to the Expected Utility Theory (EUT) (e.g., a rational cost-benefit analysis) by subscribing to *ambiguity aversion*.

Apart from the Ellsberg paradox, there is also the Machina paradox (wherein the preference ranking may change although the underlying probabilities are equivalent). Aerts notes that the Machina and Ellsberg paradoxes run counter to the EUT [79]. Likewise, the Allais paradox (where DM are unlikely to consistently make rational decisions under CDC) runs counter to the EUT [80]. Both the Machina and Allais paradoxes run counter to Savage's Independence Axiom (IA). Ferrari-Toniolo proclaims that the IA is the "most demanding axiom" as pertains to EUT [81]; Blavatskyy explains IA simply: "The independence axiom postulates that"…the DM's… "preferences between two lotteries are not affected by mixing both lotteries with the same third lottery (in identical proportions)" with the extension of this being that the DM "does not necessarily prefer the same choice alternative when repeatedly presented with the same choice set" [82]. Overall, the Ellsberg, Machina, and Allais paradoxes all run counter to EUT and highlight DM behavior that deviates from rational behavior, such as is shown in Table XXVIII.

TABLE XXVIII. EXEMPLAR TYPES OF PARADOXES WITH TYPE OF EFFECTS

Type of Paradox	***Type of Effect***	***Description***
Ellsberg	Ambiguity Aversion	DMs have a predilection to be ambiguity-averse and tend towards choices with known calculable risks rather than those with unknown incalculable risks.
Machina	Inconsistent Preference Rankings	Accurate preference ranking methods are problematic amidst A and U; this has high impact for AI models that are trained to learn and predict preferences.*
Allais	Certainty Effect	DMs have a predilection for more certain outcomes over probabilistic outcomes.

Jim refers to this as "each of these models is trained from a common base model to predict the…preferences of a single individual," Aldoseri refers to this as "machine learning algorithms learn, make predictions, and improve their performance over time" [83][84]. From Table XXVIII and the discussions leading up to this point, the key take-away, taking the cue on inconsistent preference rankings, is: (1) with regards to non-monotonic phenomenon, human behavior may not be able to be captured and the involved AIS may actually adopt "the opposite" position (from Section IIB) [34]; (2) "the shift of the involved variables from a monotonic to a non-monotonic paradigm can be quite unexpected and occur more frequently than anticipated/desired" for AIS (from Section IIC) [2]; and (3) as report by Chen, with regards to inconsistent preference rankings, "most state-of-the-art preference-tuned models achieve a ranking accuracy of less than 60% on common preference datasets," which basically equates to "preference learning algorithms do *not* learn preference rankings," "existing reference models *rarely have* correct rankings," and "preference learning *rarely corrects* incorrect rankings" [85]. Also, interestingly and ironically, leaning the other way towards monotonic does not seem to be viable either; for example, "enforcing a strict monotonic paradigm can segue to an unnatural rigidity and/or incorrect/irrelevant responses by the" AIS (from Section IIC). A robust counterpoising seems to be the key.

III. Experimentation

Counterpoising mechanisms, such as that of a robust MMSO mechanism, have been previously experimented upon, as shown in Table XXIX.

TABLE XXIX. Exemplar DOIs and paper titles pertaining to the notions of MADM/MODM SM/OM

Facet	*DOI*
MADM/MODM SM/OM (MMSO)	• A Prospective Monotonic/Non-Monotonic Transition Zone Impediment for Concept Model-Centric Artificial Intelligence Systems • 10.1109/IAICT65714.2025.11100570 •10.1109/AIIoT65859.2025.1110531 • http://dx.doi.org/10.2139/ssrn.5291883 • http://dx.doi.org/10.2139/ssrn.5291881 • http://dx.doi.org/10.2139/ssrn.5291879 • 10.1109/ICAIIC64266.2025.10920828 • http://dx.doi.org/10.2139/ssrn.5183492 • 10.1109/IAICT62357.2024.10617473 • 10.1109/GEM61861.2024.10585580 • 10.1109/AIIoT61789.2024.10579033 • http://dx.doi.org/10.2139/ssrn.4984663 • 10.1109/ICSGTEIS60500.2023.10424230 • 10.1109/ICPEA56918.2023.10093212

As previously discussed in Section IID, "depending upon the RM/amalgam of RMs/overall RM pathway initially selected, such as by the LHM, the related DM point/DMIP will likely impact the downstream behavior within the MNTZ in terms of a preponderance of monotonic or non-monotonic behavior." Consequently, the selection is non-trivial. To assist in this regard, the MMSO could be quite useful. After all, as pertains to preference ranking, it is a central component of MMSO, which is the mainstay of MCDM. For MCDM, preference ranking impacts the specification of alternatives, the defining of criteria, the assigning of weights, and the sorting of alternatives, among other items.

Homing in on the aforementioned delineation of criteria, assignment of weights, and the ranking of alternatives, the Shapley value has been found to be quite instrumental. It can facilitate the analysis of interactions between the various criteria, thereby providing invaluable insights during the ascertainment of apropos criteria weights, as well as enhance various ranking methods. As Qin puts it, the Shapley value "is widely used for" "feature importance analysis" [86]. To further this, B. Rozemberczki asserts that, "the Shapley value is used to measure the contributions of input features to the output of a machine learning model" [87]. In essence, the Shapley value can be construed as helping to gauge the overall significance of each criterion. Given this pivotal role, it seemed prudent to explore *the Shapley value in terms of its prospective value within the MNTZ for the ultimate purpose of helping to better counterpoise MR/NMR via a better counterpoised MMSO*. For this reason, explorations of a bespoke IbSOA approach were undertaken.

Given the various counterpoising within the discussed AIS, enhanced context is a mainstay of this paper, and this was alluded to in [1], which this paper is rooted upon. Axiomatically, to enhance context for an AIS, ML2 is vital. Hence, in the case of this paper, interstitial analyses are crucial for ascertaining "whether the prospective ML learnings are of potential benefit" [1]. In terms of theoretical foundations, it is posited that "Borgonovo's glocal notion can help bridge the gap, and the significance of the [Optimal Shapley-Nondominated Solution] OSNS segueing to an Optimal Shapley-Owen-Nondominated Solution (OSONS)" paradigm is underscored in [1] as well as articulated by various researchers within this arena, such as Casajus, Lopez, Beal, and others [88][89][90]. Fundamentally, the referenced Owen value nicely captures the intricate interactions between/among the constituent members of the involved feature set as well as "extends the Shapley value (which well captures the individual feature contributions) in a consistent fashion" [1]. Yet, OSONS is predicated upon the referenced IbSOA, which encompasses "temporal-centric [Finite-Change Shapley-Owen] FCSO values, ["Squared Cohorts" Shapley-Owen] SCSO values, and [Generic Shapley-Owen] GSO values/[Variance-Based Shapley-Owen] VBSP values" [1]. In addition, to operationalize the IbSOA, it should be conjoined with the previously referenced MMSO mechanism.

Based upon the experimentation referenced in Table XXIX, it seemed prudent to further investigate those areas promulgated by Wu, Wang, Hua, and others. By way of example, among others, Wu and Wang have been proponents of "Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) OM in conjunction with SHAP," and this seemed favorable for MMSO experimentation [91][92]. As another example, among others, Hua has been a proponent of "the PROMETHEE OM with SHAP" [1]. Again, this seemed promising for MMSO experimentation as well. As noted in [1], the experimentation assessed the referenced favorable/promising avenues, and an output, among others, centers upon the notion that it seems well advised to utilize "an OM (e.g., CRiteria Importance through Intercriteria Correlation or CRITIC) to first, derive the criteria weights and second, use a complementary pairing for the ensuing ranking (e.g., TOPSIS, PROMETHEE)" [1]. The reader should note that a substantive portion of the balance of Section III (particularly Sections IIIA and IIIB) is derived from [1], and that this journal paper is an invited extended version of that paper (i.e., [1]).

A. *Experimental Testbed*

For a tasked ML to well learn atop another ML in an ML2 sense, it is important to mitigate against inadvertent spawning (i.e., *Spawn Reduction*). Accordingly, enhanced context and understanding at the interstices can be central to the necessary ML2 mechanism to substantially decrease the "spawning of further non-convex MINLP (e.g., from the transformation pathways of non-convex MINLP to convex MILP)" [1]. For the experimentation in [1] and herein, "SDP solvers were implemented aboard GNU's Not Unix (GNU) Octave (a "numerical computation platform" that is "under the GNU [General Public License] (GPL) v3 license" and is generally "compatible with the likes of MATLAB") along with a myriad of Octave Forge packages" [1][93]. As noted in [1] and [93], "'the source code was modified in the lab environment' so as to implement accelerants for the referenced SDP solvers to quickly address the various involved convex optimization problems described herein;" also, as noted in [1] and [93], "GPLv3 avoids the issue of tivoization (the instantiation of a system that incorporates software under the terms of a copyleft software license but leverages hardware restrictions or digital rights management to prevent users from running modified versions of the software on the involved hardware)" [93]. The experimental "testing was conducted using a variety of open-source software packages, such as Automatic Differentiation Model Builder (ADMB) (for non-linear statistical modeling) and Interior Point OPTimizer (IPOPT) (for large-scale nonlinear optimization)" [1][93]. Additional "promising software packages, such as LOQO (like IPOPT, it is based upon the interior-point method) and Sparse Nonlinear OPTimizer (SNOPT) (it leverages Sequential Quadratic Programming or SQP for resolving large-scale non-linear optimization problems) were examined, but they were not utilized given their licensing caveats" [1].

It had been discussed in [1] and [the second DOI of Table IV] that a particular numerical implementation of Continuous Wavelet Transforms (CWTs), aboard a **C**onstriction Factor (CF)-**P**article Swarm Optimization (PSO)-**R**obust Convex Relaxation (RCR)-**L**ong Short-Term Memory (LSTM)-**D**eep Convolutional Neural Network DCNN (CPRLD) architectural paradigm, well contributes to System Transparency Explainability & Accountability (STEA) by way of the intrinsic "successive convolutional layers (which contain the cascading of ever smaller 'CWT-like' convolutional filters)" [1][the second DOI of Table IV]. As noted in [1], "the referenced CPRLD construct handled the various transformation pathways" alluded to in Figure 2 previously "(e.g., convex approximations, series of convex relaxations, etc.), and the architectural implementation for this paper was unique in that a ML2 paradigm was implemented for *Spawn Reduction* (SR_2 on SR_1)," such as depicted in Figure 16.

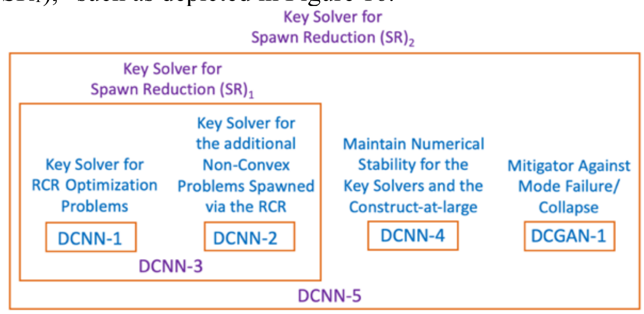


Figure 16. CPRLD Architectural Construct with a ML2 (SR_2 on SR_1) Spawn Reduction paradigm [1]

As discussed in [1], "in terms of implementation details, a DCNN-centric instantiation was chosen for the requisite sufficient balance of reduced computational complexity along with sufficient robustness to be fit for purpose. The assigned tasks of the various DCNN are labeled accordingly in Figure 16. For example, as DCNN-1 was tasked with being the key solver for the involved convex optimization problems, it required a high degree of numerical stability, and PyTorch version 0.4.1 was selected; DCGAN-1 leveraged a "forward stable" TensorFlow-based Deep Learning (DL) Convolutional GAN (DCGAN) implementation to be able to well address the potentiality of mode collapse/mode failure (a phenomenon that may occur when adversarial GANs, which are being trained in tandem, are either unable to converge or undergo an anomalous convergence)" [the second DOI of Table IV].

B. *Experimental Construct*

As pertains to the involved experimental construct, which is based upon Figure 4 of [1], such as is now shown in Figure 17, "prior experimentation aspects used as presets are reflected in blue font while current experimental elements are shown in purple font" [1].

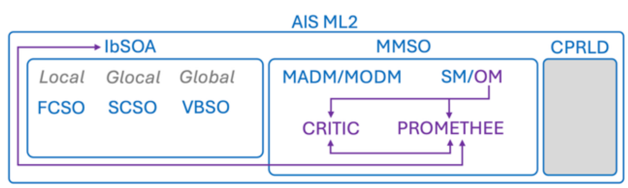


Figure 17. Conjoining of IbSOA and MMSO

For additional context, "t-" elements (e.g., t-FCSO, t-SCSO, t-GSO, t-VBSO) of the b-SHAP can be extrapolated, and these relate to Borgonovo's work…which more fully considers "Kotthoff's emphasis on temporal-sensitive/temporal-centric Shapley values" [94]. As described in [1], "the OM of CRITIC was utilized as a preset for deriving the criteria weights, and the OMs of PROMETHEE, TOPSIS, and ÉLimination Et Choix Traduisant la REalité (ELECTRE) were utilized for the subsequent rankings. Initial selections and avoidances, among others, were based upon the following rationale. For example, PROMETHEE was known to be 'easily… understood' and interpretable, so it was selected for testing" [95][96]. Along this vein, [fuzzy] VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) was not selected, as it was known to be less interpretable and "less explainable than other more intuitive methods" [97].

In general, selections were made in adherence with the principles for high efficacy ML2. In addition, there were other technical considerations. As discussed in [1] as well as those papers pertaining to related experimentation, such as shown in Table XXX, the issue of "Rank Reversal" (RR), wherein ranking results might change when the method changes or when the set of alternatives changes (leading to inconsistent and/or inaccurate results), was also investigated (as RR affects several of the methodologies)

TABLE XXX. EXEMPLAR DOIS PERTAINING TO THE NOTIONS OF RANK REVERSAL (RR)

Facet	*DOI*
RR	• 10.1109/IAICT65714.2025.11100570 • 10.1109/AIIoT65859.2025.11105315 • http://dx.doi.org/10.2139/ssrn.5291883 • http://dx.doi.org/10.2139/ssrn.5291881

As noted in [1], "the select OMs experimented with were known to be the most resistant to RR (yet are still subject to the phenomenon), and preliminary results" are reflected in Figure 18 [98]. "The key for the chart is as follows. First, the referenced 'select OMs' of this Section IIIC are self-evident: ELECTRE, TOPSIS, and PROMETHEE" [1] "Second, these 'select OMs' were benchmarked by execution time (E), sensitivity (S), performance under uncertainty (U), validity (V), and interpretability (I)" [1]. "Third, the aforementioned were benchmarked against classical SHAP (c-SHAP), as well as the b-SHAP approach described within this paper" [1]. "Using the CPRLD as a preset, collectively, this forms the basis" of the mechanisms described herein. The relative values were normalized against a scale of one to ten for ease of comparison" [1].

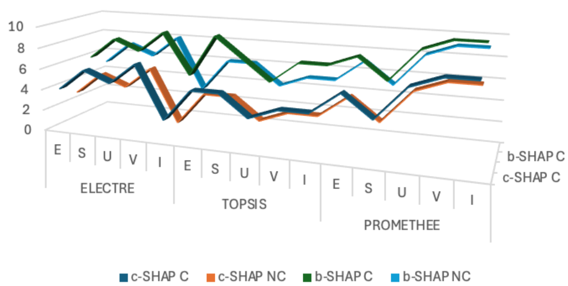


Figure 18. Preliminary Results from b-SHAP/select OM Benchmarking

Also, "the V and I were higher for PROMETHEE than for TOPSIS or ELECTRE. The E for TOPSIS was notably higher than that of the others, but the computational complexity is known to be less, and the performance under conditions of U was weaker than that of the others; the performance of PROMETHEE under conditions of U were seemingly better than ELECTRE and TOPSIS, in that order. Overall, the performance of b-SHAP was better than that of c-SHAP across the board for the range of E, S, U, V, I (for all the "select OMs" of ELECTRE, TOPSIS, and PROMETHEE). Hence, the b-SHAP-PROMETHEE amalgam (along with the CRITIC, CPRLD, etc. presets) exhibits promise" [1].

C. *Discussion re: IbSOA*

As noted in the first DOI of Table XXIX, "the myriads of interplays among local, glocal, and global is clear, as a transformation and/or sequence of transformations can lead from one to another" [94]. For example, what Mase deemed to be the [local] *Baseline Shapley* (i.e., what equates to the average of the *FCSO values* function under uncertainty) can be readily transformed to the [global] *VBSO values* [94][99]. Likewise, the [glocal] *SCSO values* can be transformed to the *VBSO values*. These interplays, among others, demonstrate how "additional insights into the [ML] model behavior" are possible [99].

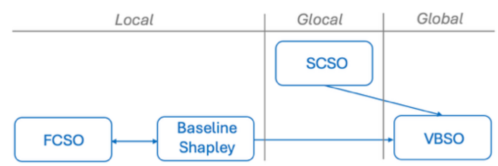


Figure 19. Derived from Figure 1 of [94]; the interplay among Local (FCSO and Baseline Shapley), Glocal (SCSO), and Global (VBSO)

This is akin to the Figure 13 interplays, and the successful operationalization of Figure 17 is contingent upon a phenomenon, which has been previously investigated, such as exhibited in Table XXXI — Robust Convex Relaxations (RCR).

TABLE XXXI. EXEMPLAR DOIS PERTAINING TO THE NOTIONS OF ROBUST CONVEX RELAXATIONS (RCR)

Facet	*DOI*
RCR	• A Prospective Monotonic/Non-Monotonic Transition Zone Impediment for Concept Model-Centric Artificial Intelligence Systems • 10.1109/IAICT65714.2025.11100570 • 10.1109/AIIoT65859.2025.11105315 • http://dx.doi.org/10.2139/ssrn.5291883 • http://dx.doi.org/10.2139/ssrn.5291881 • http://dx.doi.org/10.2139/ssrn.5291879 • http://dx.doi.org/10.2139/ssrn.5183492 • 10.1109/IBDAP62940.2024.10689701 • 10.1109/IAICT62357.2024.10617473 • 10.1109/GEM61861.2024.10585580 • 10.1109/AIIoT61789.2024.10579033 • http://dx.doi.org/10.2139/ssrn.4984663 • http://dx.doi.org/10.2139/ssrn.4679251

	• 10.1109/ICSGTEIS60500.2023.10424230 • http://dx.doi.org/10.2139/ssrn.4679260 • 10.1109/ICPEA56918.2023.10093212 • 10.1109/CyMaEn57228.2023.10051057 • 10.1109/CyMaEn57228.2023.10050946 • 10.1109/OETIC57156.2022.10176215 • http://dx.doi.org/10.2139/ssrn.4287248 • https://ssrn.com/abstract=4219298

By leveraging the described RCR approach, the notion of relaxations/convex relaxations can be applied so as to attain "Relaxed Isomorphisms"/"Isomorphic Relaxations." In this way, should IsoP be required, the DMP can be more computationally tractable. Mancinska references this approach as do Atserias and Maneva. Aflalo asserts "that for friendly graphs, the convex relaxation is guaranteed to find the exact isomorphism or certify its inexistence" [100][101][102].

The starting impetus of this paper was to address SDM, particularly SDM-A/U, and the LHM was discussed as a meaningful option. As discussed, the main components of the LHM include the ICSM2 and MAM. The ICSM2 has a list of questions to address (e.g., whether the involved set is UnS, EqS, EquivS, OrS, POSET, UnS-Iso, POSET-Iso, or IV), such as previously described in Table V. The MAM also has a list of questions to answer (e.g., how much time is available, what and how much information is available, as this might dictate that type of initial RM is most feasible and what is the probability that the involved is PDMP, ADMP, or NPDMP), such as shown in Figure 20, and re-expressed in Figure 21 in terms of N+V or IsoP pathways.

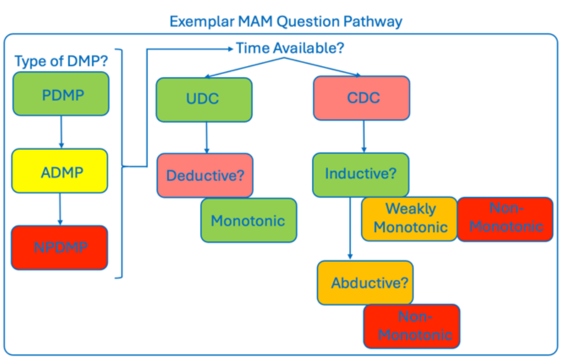


Figure 20. MAM Exemplar Question Pathway

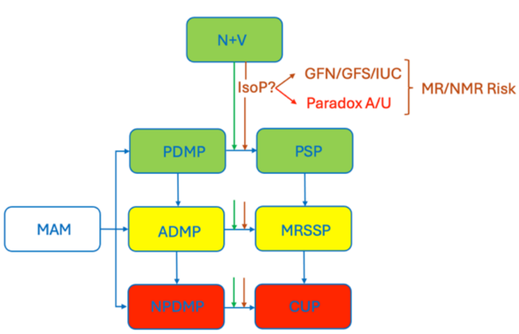


Figure 21. N+V or IsoP Pathways; IsoP has MR/NMR Risk

It should be remembered that Deductive is achieved iteratively (not front-loaded) while Inductive and Abductive are front-loaded. In addition, the CAIR-A also has the challenge of ascertaining what primary RMs and secondary RMs might be involved, and this was previously presented in Table XVII.

As noted in the 1st DOI of Table VIII, "the establishment of an Inherent Uncertainty Construct (IUC)…is central, and crucial to this is the leveraging of T2FS and Zadeh's Fuzzy Systems Theory (a.k.a., IUC-1a) along with the consideration of Rough-Fuzzy Set (a.k.a., IUC-2a), which is an extension of IUC-1a and Pawlak's Rough Set (a.k.a., IUC-1b); after all, IUC-2a can well accommodate the notion of an affiliation, "but not necessarily absolute inclusion" [103]. Furthermore, "Deng's Grey Systems Theory (a.k.a., IUC-2b) can enhance the precision of IUC-2a."

TABLE XXXII. COMPONENTS OF THE INHERENT UNCERTAINTY CONSTRUCT (IUC)

Short-Form Acronym	***Long-Form Name***
IUC-1a	Leverages T2FS and Zadeh's Fuzzy Systems Theory (FST).
IUC-1b	Pawlak's Rough Set (RS).
IUC-2a	Rough-Fuzzy Set (RFS) can well accommodate the notion of an affiliation, "but not necessarily absolute inclusion" [103]. It is an extension of IUC-1a and IUC-1b.
IUC-2b	Deng's Grey Systems Theory (GST) can contend with systems with incomplete or uncertain information. For RWS, the uncertainty can stem from continuous data that may absent, exhibit intricate relationships, be noisy, etc. GST leverages GSs (which rely upon GNs), which are numbers with known upper and lower boundaries, but for which the precise value is unknown within those boundaries; this embodiment is able to concurrently represent of both discrete and continuous data simultaneously) to enhance the precision of IUC-2a. GST also leverages discontinuous Grey Sets, which are specifically geared to handling discontinuous data (wherein only partial information is available) or data with uncertainty. It should be remembered that although a discontinuous function may have a continuous domain, it may have breaks/gaps/jumps or points where it is not defined. In contrast, a discrete function has distinct and disparate values (e.g., integers).
IUC-3	Information Entropy Methods, whose strength resides in ascertaining "unknown attribute weights" [1st DOI of Table VIII][104].

"If the relationship/membership (e.g., entity, attribute, etc.) is discontinuous, IUC-2b can be leveraged; otherwise, given a continuous/continuous alternative paradigm, then other Probability [& statistics] Systems Theory (PST) approaches might be utilized, such as Information Entropy Methods (IEMs) (a.k.a., IUC-3), whose strength resides in ascertaining ‘unknown attribute weights’” [second DOI of Table IV][1st DOI of Table VIII][104]. Furthermore, “whether the relationship/membership is discontinuous or continuous (e.g., pulsed, rather than continuous), it can still be construed as a Relationship/Membership Stream (RMS) — a key constituent component of the IUC that is summarized in” Figure 22 below [1st DOI of Table VIII]:

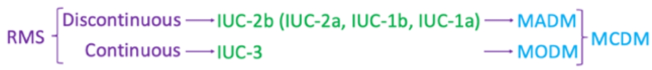


Figure 22. RMS Paradigms for the IUC [1st DOI of Table VIII]

The following Sections IIIC1 through IIIC6 stem from the 4th DOI of Table IV.

1) Nondominated Solution (NS)

Wu had noted the opportunity of transforming FN-related Fuzzy Optimization and Decision Making (FODM) problems to “Scalar Optimization Problem[s]” (SOPs), which can be efficiently resolved to segue to the nondominated solution, wherein “no one objective function can be improved without” a concurrent degradation to “the other objectives,” and “the OSNS can then be ascertained.” [3rd DOI of Table IV][105].

2) Optimal Corresponding (OC) GLfSFN-based membership function

Interestingly, “regarding ‘best-fix approximation[s],’ Lakshmana had reported on the efficacy of the “approximations of general non-linear FNs’ by way of higher-order linearized Generalized ‘f’-gonal FN/SFN forms, ‘such as *Tr*iangular, *T*r*ap*ezoidal,’ as well as *Pe*ntagonal, *H*e*x*agonal, *Hep*tagonal, *O*ctago*n*al, etc.” [3rd DOI of Table IV]; “these can be re-expressed as “G*Tr*FN, G*Tp*FN, G*Pe*FN, G*Hx*FN, G*Hp*FN, G*On*FN, etc., respectively” [3rd DOI of Table IV]. “According to Velu and Ramalingam, ‘best-fit approximations’ can be improved ‘when higher-order piecewise linear’ FNs are utilized to approximate ‘non-linear information’” [3rd DOI of Table IV][106]. “Along this vein, Augustin asserts that, as one example, G*Hp*FN ‘can represent more intricate and nuanced degrees of uncertainty’ since ‘certain apropos ‘f’-gonal FN/SFN forms’ are quite good at ‘preserving ambiguity’” [3rd DOI of Table IV][107]. “Ban, another advocate of this principle” is satisfied even with “*Tr*iangular, *T*r*ap*ezoidal, and semi-*T*r*ap*ezoidal for the ‘preserv[ing]…and weight[ing]’ of ambiguity” [3rd DOI of Table IV][108]. “The pathways for deriving the OSNS (a Multi-Objective Decision Making or MODM problem) and the selection of the ‘f’-gonal FN/SFN form (a FODM and MCDM problem) are informed by the ICSM2.”

3) Preferred OCGLfSFN

“A goal of the involved [Metaheuristic Algorithm] MA (for which the MAM is responsible) is to ascertain the OCGL*f*SFN. Prior to segueing to this OC form, there is a Precursor (P-) non-OC form (i.e., P-GL*f*SFN); for example, Augustin acknowledged…G*Hp*FN for its ability to ‘represent more intricate and nuanced degrees of uncertainty’ while Ban favored G*Tr*FN, G*Tp*FN, and semi-G*Tp*FN for the preservation of ambiguity and weighted ambiguity” [3rd DOI of Table IV][116][117]. “Whatever the preferred form, the choice of the precursor Generalized ‘f’-gonal FN/SFN form (e.g., P-GL*f*SFN) affects the efficacy of the utilized ‘defuzzification method’ (i.e., ‘the transformation of a FN/SFN into a crisp form’)” [3rd DOI of Table IV]. “The significance of this center upon the intricacy that ‘as the LHM contends with the counterpoising of’ A/U, the precursor non-OC form, which best preserves ambiguity, is likely to be optimal for facilitating/deriving the OCGLfSFN.”

4) The Ranking of FNs/SFNs

Significantly, “there are numerous ‘ranking methods for the discussed pre-cursor [non-OC form] Generalized ‘f’-gonal FN/SFN form (e.g., P-GL*f*SFN), and the appropriate selection” is central” [3rd DOI of Table IV]. For example, Velu and Ramalingam noted that ‘a ranking method which works very well for’ G *H*e*x*agonal FNs/SFNs ‘may have some shortfalls when it is extended for’ G *O*ctago*n*al FNs/SFNs” [3rd DOI of Table IV][105]. “Similarly, ‘a ranking method which works very well for’ G *O*ctago*n*al FNs/SFNs might have ‘some shortfalls when it is used for’ *Tr*iangular or *T*r*ap*ezoidal FNs/SFNs” [3rd DOI of Table IV][109]. “In any case, the ranking mechanism (facilitated by the ACWS) informs the precursor non-OC to final OC form.”

5) Similarity Measure (SimM) Challenge for FN/SFN

With regards ranking methods, the underpinning measures typically involve various SimMs. Gogoi & Chutia noted that while there are a myriad of methods (each with advantages/drawbacks), a universally accepted ‘silver bullet’ SimM for ascertaining the similarity between FNs/SFNs ‘does not necessarily exist’” [3rd DOI of Table IV][110]. “They also noted that a ‘literature survey reveals that most of the’ SimM ‘are being developed based upon’ the following parameters: “geometric distances, height, area, perimeter, ‘Center of Gravity (COG),’ ‘Radius of Gyration (ROG),’ etc.” [3rd DOI of Table IV][110]. It was noted in [110] that for various studies, with the exception ‘of Hejazi et al. (2011),’ certain ‘glass ceiling’ SimM methods (e.g., “’failing to give reasonable similarity between pairs of FNs when one FN is identical for both the pairs) ‘*are being carried forward’ into contemporary works*” [3rd DOI of Table IV][110]. This is reminiscent of our prior finding that certain bugs/issues in various frameworks/libraries/toolkits, such as made available via assorted developer platforms, were being carried forward into various projects/papers. To aggravate matters, FNs are simply a special case of a FS, and ‘beyond FS, there’ are other FS variations; these include the IFS, Pythagorean Fuzzy Set (PFS), and Neutrosophic Fuzzy Set (NFS)” [3rd DOI of Table IV][111][112]. “The IFS, which is often leveraged for ‘coalition decision-making,’ is comprised of constituent elements that ‘have both membership function u and non-membership function v, such that $u + v <=1$, and hesitation margin h, such that $u +$

$v + h = 1$'" [3rd DOI of Table IV]. "Other situations are better addressed by PFS, 'wherein $u + v >= 1$ (or $u + v <= 1$) and $u^2 + v^2 + h^2 = 1$'" [3rd DOI of Table IV][111]. "Yet other cases are better handled by NFS, which combines "FS with NS" [3rd DOI of Table IV][112]; "delving into this, Das notes that while FS addresses "uncertainty" by the utilization of 'membership grade,' Smarandache's NS tackles 'uncertainty using truth, indeterminacy, and falsity member grades'" [3rd DOI of Table IV][112]. "Furthermore, Ashraf, Gundogdu & Kahraman, Mahmood, etc. have 'contributed to the [overall] notion of…SFS, which is the generalized structure over' the referenced FS (e.g., IFS, PFS, and NFS)" [3rd DOI of Table IV][113].

6) SimMs and Distance Measures (DMs) for SFS/T-SFS

"Various SimM approaches have been adapted for the SFS ecosystem, as noted by Zhang, and Wei observes, by way of example, that a plethora of 'SimMs for SFS based on the cosine and cotangent function have been' put forth" [3rd DOI of Table IV][114]. "Likewise, certain combinatorial, such as 'Jaccard, Exponential, Square root cosine for SFS,' etc., have been employed as pragmatic implementations of SimMs" [3rd DOI of Table IV][115]. "With regards to DMs, Donyatalab and others have examined 'Minkowski, Minkowski-Hausdorff, Weighted Minkowski and Weighted Minkowski-Hausdorff distances for SFSs'" [3rd DOI of Table IV]. "Overall, there have been numerous SimM and DM advances, and among these, researchers, such as Wu, have 'focused on the T-SFS,' which is a 'specific case of NS' (a.k.a., 'n-hyper SFS')" [3rd DOI of Table IV][114][115]. According to Wu, T-SFS is quite adept in contending with "uncertainty information" and "can handle information that SFSs…cannot process" [3rd DOI of Table IV][115]. Accordingly, the SimMs/DMs of T-SFS show promise for higher efficacy.

IV. Concluding Remarks

For a CAA to successfully operationalize a high efficacy NLG and engage in unscripted dialogues, it needs to have RDM; for a CAA to learn/tune from each and every engagement, it needs to not only have an efficient ML mechanism, but it also needs to have a high performance ML2 mechanism. With these two facets in hand, the CAA is more likely to be able to engage in real-time conversation without any undesired delays in performance. The benchmark as to whether the aforementioned is successfully actualized is predicated upon whether the CAA responses are "consistent, coherent, and valid." As noted in Section I, this is non-trivial to achieve. Delving deeper (from the high-level references to NLG and RDM), it was necessary, for the purposes of this paper, to first address SDM (specifically, SDM-A/U for RWS).

While the longer-term impetus of this paper is to address SDM-A/U for AIS/AICDS/AS (e.g., AUVs, USVs, UAVs, etc.) as well as other applications of SAE 5, the proxy application of CAIR was selected to better understand the multi-stage challenges of SDM-A/U. For SDM-A/U, at each DM point, a choice needs to be made as to whether a "full resolution," "partial resolution," or "no resolution" action will be taken. In contrast to a single-stage DM, a key challenge of SDM is to optimize the overall outcome over the full range of multi-stage DM points.

One approach, among others, for tackling the SDM-A/U challenge resides in the leveraging of an LHM, which carefully considers – concurrently – the notions of A/U. The LHM is powered by, among other key modules, an ICSM2 and a MAM; for the purposes of the CAA use case and CAIR application discussed in this paper, a CAIR-A was added as well. The ICSM2 starts with an initial set of considerations and ascertainments (UnS, EqS, EquivS, OrS, POSET, UnS-Iso, POSET-Iso, IV, etc.) prior to proceeding to an IsoP, whose computational complexities can vary greatly. Of course, the pre-IsoP approach is quickly handled while the IsoP determination and approach is also contingent upon how much time is available. For example, while a full IsoP approach might be selected under UDC, a relaxed IsoP (r-IsoP) might be opted for under CDC or in-between CDC and UDC. The choice of approach is heavily contingent upon the CW mechanism of the ACWS, which remains a central element of not only the entire described apparatus, but also both ICSM2 and MAM operations in particular.

There exists a mutual reinforcement interplay between ACWS and MMSO, and the overarching MCDM rubric (which tends to have a myriad of conflicting objectives or criteria, particularly for RWS) can be contextualized by a non-dominated solution (which illuminates a number of best-scenario trade-offs; however, it does not indicate a single optimal solution). It had previously been determined that OSNS (particularly the extended OSONS version) can be useful in this regard, and it has been found to be of value-added proposition for AIS not only in the area of better understanding trade-offs but has also been acknowledged for its value in the areas of I&E (under the STEA rubric) and feature importance analysis. For this extrapolated journal paper, it should be remembered from [1] varied SHAP approaches differ in their local and global efficacies. It should also be remembered that CW/ACWS ranks the criteria (i.e., CW/ACWS prioritizes what is most important before the alternatives are even reviewed), wherein criteria are the factors utilized to assess, compare, and contrast the set of alternatives. Each of the alternatives are characterized by features. Interestingly, IbSOA can leverage the clustering of features into coalitions to facilitate computational efficiency. Section IB2 reviews the notion of "updating a heuristic" and notes that, contrary to popular opinion, heuristics may segue to sub-optimal paradigms more frequently than anticipated; in essence, the notion of simply using a heuristic as an accelerant is far too simplistic and may be disappointing. Of significance, the "updating of a heuristic" may necessitate sufficient STEA (and IbSOA may be useful in this regard) so as to facilitate ML2; overall, heuristics for RWS should be able to: (1)self-recognize that the heuristic is not qualified to handle the incoming information and needs to refer the matter back to the higher-order hyper-heuristic, (2) mitigate against the ongoing prospective brittleness of the heuristic itself, and (3) have the capability to operationalize the involved [meta]heuristic-algorithm amalgam in an advantageous fashion. Section IB2 culminates with the overarching equation of this paper: ASL-IMC⇢RMMI, wherein ASL equates to AIS⇢SDM⇢LHM, IMC equates to ICSM2+MAM+CAIR-A, and RMMI equates to RDM ⇢MR/NMR⇢MNTZ⇢IbSOA. Of course, IbSOA is captured within the first part of the title of this paper: "Interstitial b-SHAP-Owen Amalgam."

IbSOA is put forth as a central item for the second part of the title of this paper: "Enhancement of Artificial Intelligence System-Centric Sequential Decision-Making," and Section II had opened with the notion that "the AIS approach towards DMP, such as SDM, may involve a variety of" RMs. Each of the varied RMs has certain monotonic and/or non-monotonic tendencies, and depending upon a paradigm of UDC or CDC, the MAM may opt for certain RM pathways and proceed in stages depending upon the time remaining and/or any time adjustments that may occur along the SDM pathway. The MAM supports the ICSM2, particularly when it is determined that IsoP needs to occur. Along this vein, the ICSM2 operationalizes the IsoP comparison via the use of GFN/GFS-IUC and Paradoxes-A/U. By working in conjunction, the ICSM2 and MAM combinatorial form a powerful engine for LHM. The addition of the CAIR-A module enhances the combinatorial by serving as a precursor function; the CAIR-A endeavors to ascertain what primary, secondary, etc. RMs might be involved. The involved CAIR-related CW/ACWS-MMSO pre-sort tends to be of value-added proposition for RWS SDM paradigms. The "A" facet of the CAIR-A Module is an orchestration mechanism that leverages the MAM (which supports the LHM) in conjunction with a Birnbaum Importance assignment; it should be remembered that the Birnbaum importance measure is often leveraged as a Lower-Level Heuristic (LLH) for use in resolving Component Assignment Problem (CAP) (which centers upon optimally placing components at various positions within an AIS to maximize reliability).

To clarify this point, if N+V in it of itself does not suffice (and IsoP is required), then there is an accompanying increased MR/NMR skewing risk that should be mitigated. In particular, as pertains to the MNTZ, it has been noted consistently throughout this paper and the specified apriori examinations that there exists a tendency for spawning to the NP-Hard side (e.g., non-monotonic, non-polynomial, and even non-continuous functions). Roughly speaking, the problem with this is that LLMs struggle with emulating human behavior on the non-monotonic side, and oftentimes, the AIS (in this case, the CAA) will take the opposite stance as what the human would choose. Cognizant of the range of paradoxes (e.g., Ellsberg, Machina, Allais, etc.) that influence this paradigm, due consideration should be paid regarding the associated preference ranking, and the related issue of RR has been previously scrutinized. While it seems that focusing on certain RMs (e.g., deductive) might seem computationally more tractable, it would preclude more RWS RM approaches (e.g., probabilistic, temporal, and graph-based). Yet these parenthetical approaches, among others, can segue to non-monotonic unexpectedly. Moreover, the desired related CBR and inductive approaches are at much higher risk for segueing to non-monotonic. Finally, some of the other desired more sophisticated approaches (e.g., analogical, abductive) squarely reside in the non-monotonic. Depending upon the RM pathway opted for, the related DM will likely impact the downstream behavior within the MNTZ in terms of a skewing towards monotonic or non-monotonic behavior. In essence, while the gamut of isomorphic cases can be construed to be analogies, not all analogies are construed to be isomorphic. This particular determination is central because if IsoP can be avoided, then the expenditure of an unknown amount of computation resources can by bypassed.

The significance of the IbSOA is that, fundamentally, the referenced Owen value nicely captures the intricate interactions between/among the constituent members of the involved feature set and, as noted in [1], "extends the Shapley value (which well captures the individual feature contributions) in a consistent fashion." These pertinent insights into CAA behavior at the MNTZ, as noted by [2], "can potentially be quite meaningful for elevating CAIR-related coherence and consistency (with the concomitant validity)." After all, as [2] further articulates, "maintaining coherence and monotonicity is non-trivial, as the involved AIS might discern connections (particularly those that are non-monotonic) within the evolving dataset. In the context of CAIR, non-monotonic aspects can arise as incoming information can re-contextualize and/or contradict matters."

As noted in [2], the effectiveness of particular RWS implementations, such as that of CAA, "is often predicated upon consistency and reliability" (along with the associated notions of coherency and validity), and this aspect is the essence of CAIR. As noted by [2], the CAA should, ideally, provide 'human-like conversations' by comprehending user intent, maintaining context, and putting forth pertinent responses' consistent with 'the principle of CAIR;' in essence, a key tenet of CAIR is that responses provided by the CAA remain steady in their validity not only during the course of the involved interaction, but also over the course of the multi-turn conversation. In summary, this paper posits that the CAIR-A notion isof VAP to the ongoing challenge of conversational coherence and RDM. Future work will involve more quantitative experimentation, and as discussed in Section IIB, the issues of provenance/pedigree, as pertains to RM selection, regarding the "who" and the "where" (as contrasted to simply the "what" and "when" the information is made available) will be further explored.

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