International Journal on Advances in Intelligent Systems, vol 11 no 1 & 2, year 2018, http://www.iariajournals.org/intelligent_systems/

Deduction System for Decision Logic Based on Many-valued Logics

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***Abstract*—Rough set theory has been extensively used both as a mathematical foundation of granularity and vagueness in information systems and in a large number of applications. However, the decision logic for rough sets is based on classical bivalent logic; therefore, it would be desirable to develop decision logic for uncertain, ambiguous and inconsistent objects. In this study, a deduction system based on partial semantics is proposed for decision logic. We propose Belnap's four-valued semantics as the basis for three-valued and four-valued logics to extend the deduction of decision logic since the boundary region of rough sets is interpreted as both a non-deterministic and inconsistent state. We also introduce the consequence relations to serve as an intermediary between rough sets and many-valued semantics. Hence, consequence relations based on partial semantics for decision logic are defined, and axiomatization by Gentzen-type sequent calculi is obtained. Furthermore, we extend the sequent calculi with a weak implication to hold for a deduction theorem and also show a soundness and completeness theorem for the four-valued logic for decision logic.**

Keywords–rough set; decision logic; consequence relation; many-valued logic; sequent calculi.

I. INTRODUCTION

This research paper is an extended version of an earlier paper [1] presented at the IARIA Conference on SEMAPRO 2017. Pawlak introduced the theory of rough sets for handling rough (coarse) information [2]. Rough set theory is now used as a mathematical foundation of granularity and vagueness in information systems and is applied to a variety of problems. In applying rough set theory, decision logic was proposed for interpreting information extracted from data tables. However, decision logic adopts the classical two-valued logic semantics. It is known that classical logic is not adequate for reasoning with indefinite and inconsistent information. Moreover, the paradoxes of the material implications of classical logic are counterintuitive.

Rough set theory can handle the concept of approximation by the indiscernibility relation, which is a central concept in rough set theory. It is an equivalence relation, where all identical objects of sets are considered elementary. Rough set theory is concerned with the lower and upper approximations of object sets. These approximations divide sets into three regions, namely, the positive, negative, and boundary regions. Thus, Pawlak rough sets have often been studied in a three-valued logic framework because the third value is thought to correspond to the boundary region of rough sets [3][4].

On the contrary, in this paper, we propose that the interpretation of the boundary region is based on four-valued semantics rather than three-valued since the boundary region can be interpreted as both undefined and overdefined. For example, a knowledge base K of a Rough set can be seen as a theory *KB* whose underlying logic is L. *KB* is called *inconsistent* when it contains theorems of the form A and $\sim A$ (the negation of A). If *KB* is not inconsistent, it is called *consistent*. Our approach for a rough set proposes useful theory to handle such inconsistent information without system failure. In this study, non-deterministic features are considered the characteristic of partial semantics. Undetermined objects in the boundary region of rough sets have two interpretations of both undefinedness and inconsistency.

The formalization of both three-valued and four-valued logics is carried out using a consequence relation based on partial semantics. The basic logic for decision logic is assumed to be many-valued, in particular, three-valued or four-valued and some of its alternatives [5]. If such many-valued logics are used as a basic deduction system for decision logic, it can be enhanced to a more useful method for data analysis and information processing. The decision logic of rough set theory will be axiomatized using Gentzen sequent calculi and a four-valued semantic relation as basic theory. To introduce many-valued logic to decision logic, consequence relations based on partial interpretation are investigated, and the sequent calculi of many-valued logic based on them are constructed. Subsequently, many-valued logics with weak implication are considered for the deduction system of decision logic.

The deductive system of decision logic has been studied from the granule computing perspective, and in [6], an extension of decision logic was proposed for handling uncertain data tables by fuzzy and probabilistic methods. In [7], a natural deduction system based on classical logic was proposed for decision logic in granule computing. In [3], the sequent calculi of the Kleene and Łukasiewicz three-valued logics were proposed for rough set theory based on non-deterministic matrices for semantic interpretation. The Gentzen-type axiomatization of three-valued logics based on partial semantics for decision logic is proposed in [1]. The reasoning for rough sets is comprehensively studied in [8].

The paper is organized as follows. In Section II, we briefly review rough sets, the decision table, and decision logic. In Section III, Belnap's four-valued semantics is introduced as the basis of the semantics interpretation presented in the paper. In Section IV, we present a partial semantics model for rough sets and decision logic based on four-valued semantics, and some characteristics are presented. In Section V, an axiomatization using Gentzen sequent calculus is presented according to a consequence relation based on the previously discussed partial semantics. In Section VI, we discuss the extension of sequent calculi for many-valued logics with weak negation and implication to enable a deduction theorem. In Section V II, A soundness and completeness theorem is showed for a four-valued sequent calculus GC4. Finally, in Section VIII, a summary of the study and possible directions for future work are provided.

II. Rough Sets and Decision Logic

Rough set theory, proposed by Pawlak [2], provides a theoretical basis of sets based on approximation concepts. A rough set can be seen as an approximation of a set. It is denoted by a pair of sets called the lower and upper approximations of the set. Rough sets are used for imprecise data handling. For the upper and lower approximations, any subset X of U can be in any of three states according to the membership relation of the objects in U. If the positive and negative regions on a rough set are considered to correspond to the truth-value of a logical form, then the boundary region corresponds to ambiguity in deciding truth or falsity. Thus, it is natural to adopt a three-valued logic.

Rough set theory is outlined below. Let U be a non-empty finite set called a universe of objects. If R is an equivalence relation on U, then U/R denotes the family of all equivalence classes of R, and the pair (U, R) is called a Pawlak approximation space. A knowledge base K is defined as follows:

Definition 1. A knowledge base K is a pair $K = (U, R)$, where U is a universe of objects, and **R** is a set of equivalence relations on the objects in U.

Definition 2. Let $R \in$ **R** be an equivalence relation of the knowledge base $K = (U, R)$ and X any subset of U. Then, the lower and upper approximations of X for R are defined as follows:

$$\underline{R}X = \bigcup\{Y \in U/R \mid Y \subseteq X\} = \{x \in U \mid [x]_{\mathrm{R}} \subseteq X\},$$
$$\overline{R}X = \bigcup\{Y \in U/R \mid Y \cap X \neq 0\} = \{x \in U \mid [x]_{\mathrm{R}} \cap X \neq \emptyset\}.$$

Definition 3. If K = (U,R), R ∈ R, and X ⊆ U, then the R-positive, R-negative, and R-boundary regions of X with respect to R are defined respectively as follows:

$$POS_R(X) = \underline{R}X,$$
$$NEG_R(X) = U - \overline{R}X,$$
$$BN_R(X) = \overline{R}X - \underline{R}X.$$

Objects included in an R-boundary are interpreted as the truth-value gap or glut. The semantic interpretation for rough sets is defined later.

Here, we denote the language of rough sets.

A. Decision Tables

Decision tables can be seen as a special important class of knowledge representation systems and can be used for applications. Let $K = (U, A)$ be a knowledge representation system and $C, D \subset A$ be two subsets of attributes called condition and decision attributes, respectively.

A KR-system with a distinguished condition and decision attributes is called a decision table, denoted $T = (U, A, V, s)$ or in short DC, where U is a finite and nonempty set of objects, A is a finite and nonempty set of attributes, V is a nonempty set of values for $a \in A$, and s is an information function that assigns a value $U \times s_x : A \to V$ (for simplicity, the subscript x will be omitted), where $\forall x \in U$, and $\forall a \in C \cup D \subset A$.

Equivalence classes of the relations $IND(C)$ and $IND(D)$, a subset of A, are called condition and decision classes, respectively.

With every $x \in U$, we associate a function $dx : A \to V$, such that $d_x(a) = a(x)$ for every $a \in C \cup D$; the function d_x is called a decision rule (in T), and x is referred as a label of the decision rule d_x.

The decision rule d_x is *consistent* (in T) if for every $y \neq x$, $d_x|C = d_y|C$ implies $d_x|D = d_y|D$; otherwise the decision rule is *inconsistent*.

A decision table is *consistent* if all of its decision rules are consistent; otherwise the decision table is *inconsistent*. Consistency (inconsistency) sometimes may be interpreted as determinism (non-determinism).

U	a	b	c	d	e
1	1	0	2	2	0
2	0	1	1	1	2
3	2	0	0	2	2
4	1	0	2	2	0
5	1	0	2	0	1
6	2	2	0	1	1
7	2	1	1	1	2
8	0	1	1	0	1

TABLE I. Decision table

Proposition 1. *A decision table $T = (U, A, V, s)$ is consistent iff $C \Rightarrow D$, where C and D are condition and decision attributes.*

From Proposition 1, it follows that the practical method of checking the consistency of a decision table is by simply computing the degree of dependency between the condition and decision attributes. If the degree of dependency equals 1, then we conclude that the table is consistent; otherwise, it is inconsistent.

Consider Table I from Pawlak [2]. Assume that a, b, and c are condition attributes and d and e are decision attributes. In this table, for instance, decision rule 1 is inconsistent, whereas decision rule 3 is consistent. Decision rules 1 and 5 have the same condition, but their decisions are different.

B. Decision Logic

A decision logic language (DL-language) L is now introduced [2]. The set of attribute constants is defined as $a \in A$, and the set of attribute value constants is $V = \bigcup V_a$. The propositional variables are φ and ψ, and the propositional connectives are $\perp$, $\sim$, $\wedge$, $\vee$, $\to$ and $\equiv$.

Definition 4. The set of formulas of the decision logic language (DL-language) L is the smallest set satisfying the following conditions:

1) (a, v), or in short a_v, is an atomic formula of L.
2) If φ *and* ψ are formulas of the DL-language, then $\sim\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, and $\varphi \equiv \psi$ are formulas.

The interpretation of the DL-language L is performed using the universe U in $S = (U, A)$ of the Knowledge Representation System (*KR-system*) and the assignment function, mapping from U to objects of formulas. Formulas of the DL-language are interpreted as subsets of objects consisting of a value v and an attribute a.

Atomic formulas (a, v) describe objects that have a value v for the attribute a. Attribute a is a function from U to V, defined by $a(x) = s_x(a)$, where $x \in U$, and $s_x(a) \in V$. If let $s_x(a) = v$, then a can be viewed as a binary relation on U, such that for $\langle x, v\rangle \in U \times U, \langle a, v\rangle \in a$ if and only if $a(x) = v$. In this case, the atomic formula (a, v) can be denoted by $a(x, v)$, where x is a variable, and v is taken as a constant; they are all terms in U. Thus, (a, v) can be viewed as formula $a(x, v)$ which is an atomic formula. The semantics for *DL* is given by a model. For *DL*, the model is the KR-system $S = (U, A)$, which describes the meaning of symbols of predicates (a, v) in U, and if we properly interpret the formulas in the model, then each formula becomes a meaningful sentence, expressing the properties of some objects. An object $x \in U$ satisfies a formula φ in $S = (U, A)$, denoted $x \models_S \varphi$ or in short $x \models \varphi$, iff the following conditions are satisfied:

Definition 5. The semantic relations of a DL-language are defined as follows:

$x \models_S a(x, v)$ iff $a(x) = v$,
$x \models_S \sim\varphi$ iff $x \not\models_S \varphi$,
$x \models_S \varphi \vee \psi$ iff $x \models_S \varphi$ *or* $x \models_S \psi$,
$x \models_S \varphi \wedge \psi$ iff $x \models_S \varphi$ *and* $S \models_S \psi$,
$x \models_S \varphi \to \psi$ iff $x \models_S \sim\varphi \vee \psi$,
$x \models_S \varphi \equiv \psi$ iff $x \models_S \varphi \to \psi$ and $s \models_S \psi \to \varphi$.

If φ is a formula, then the set $|\varphi|_S$ defined as follows:

$$|\varphi|_S = \{x \in U \mid x \models_S \varphi\}$$

and will be called the meaning of the formula φ in S. The following properties are obvious:

Proposition 2. The meaning of an arbitrary formula satisfies the following:

$|\neg\varphi|_S = U - |\varphi|_S$,
$|\varphi \vee \psi|_S = |\varphi|_S \cup |\psi|_S$,
$|\varphi \wedge \psi|_S = |\varphi|_S \cap |\psi|_S$,
$|\varphi{\to}\psi|_S = (U - |\varphi|_S) \cup |\psi|_S$,
$|\varphi \equiv \psi|_S = |\varphi|_S \to |\psi|_S \cap |\varphi|_S \to |\psi|_S$.

Thus, the meaning of the formula φ is the set of all objects having the property expressed by the formula φ, or the meaning of the formula φ is the description in the *KR*-language of the set objects $|\varphi|$. A formula φ is said to be true in a KR-system S, denoted $\models_S \varphi$, iff $|\varphi|_S = U$, i.e., the formula is satisfied by all objects of the universe in the system S. Formulas φ and ψ are equivalent in S iff $|\varphi|_S = |\psi|_S$.

Proposition 3. The following are the simple properties of the meaning of a formula.

$\models_S \varphi$ *iff* $|\varphi|_S = U$,
$\models_S \sim\varphi$ *iff* $|\varphi|_S = \emptyset$,
$\varphi \to \psi$ *iff* $|\psi|_S \subseteq |\psi|_S$,
$\varphi \equiv \psi$ *iff* $|\psi|_S = |\psi|_S$.

To deal with deduction in *DL*, we need suitable axioms and inference rules. Here, the axioms will correspond closely to the axioms of classical propositional logic, but some specific axioms for the specific properties of knowledge representation systems are also needed. The only inference rule will be *modus ponens*. We will use the following abbreviations:

$$\varphi \wedge \sim\varphi =_{def} 0 \text{ and } \varphi \vee \sim\varphi =_{def} 1.$$

A formula of the form

$(a_1, v_1) \wedge (a_2, v_2) \wedge \ldots \wedge (a_n, v_n)$,

where $v_{ai} \in V_a, P = \{a_1, a_2, \ldots, a_n\}$, and $P \subseteq A$, is called a *P-basic formula* or in short *P*-formula. An atomic formula is called an *A-basic formula* or in short a basic formula.

Let $P \subseteq A$, φ be a *P*-formula, and $x \in U$. The set of all *A*-basic formulas satisfiable in the knowledge representation system $S = (U, A)$ is called the *basic knowledge* in S. We write $\sum\limits_s(P)$ to denote the disjunction of all *P*-formulas satisfied in S. If $P = A$, then $\sum\limits_s(A)$ is called the *characteristic formula* of S.

The knowledge representation system can be represented by a data table. Its columns are labeled by attributes, and its rows are labeled by objects. Thus, each row in the table is represented by a certain *A*-basic formula, and the whole table is represented by the set of all such formulas. In *DL*, instead of tables, we can use sentences to represent knowledge. There are specific axioms of *DL*:

1. $(a, v) \wedge (a, u) \equiv 0$ for any $a \in A, u, v \in V$, and $v \neq u$.
2. $\bigvee\limits_{v \in V_a} \equiv 1$ for every $a \in A$.
3. $\sim (a, v) \equiv \bigvee_{a \in V_a, u \neq v}(a, u)$ for every $a \in A$.

We say that a formula φ is derivable from a set of formulas Ω, denoted $\Omega\varphi$, iff it is derivable from the axioms and formulas of Ω by a finite application of modus ponens. Formula φ is a theorem of *DL*, denoted φ, if it is derivable from the axioms only. A set of formulas Ω is consistent iff the formula $\varphi \wedge \sim\varphi$ is not derivable from Ω. Note that the set of theorems of *DL* is identical with the set of theorems of classical propositional logic with specific axioms (1)–(3), in which negation can be eliminated.

Formulas in the KR-language can be represented in a special form called a *normal form*, which is similar to that in classical propositional logic. Let $P \subseteq A$ be a subset of attributes and let φ be a formula in the KR-language. We say that φ is in a P-normal form in S, in short in P-normal form, iff either φ is 0 or φ is 1, or φ is a disjunction of non-empty P-basic formulas in S. (The formula φ is non-empty if $|\varphi| \neq \emptyset$).

A-normal form will be referred to as normal form. The following is an important property in the *DL*-language.

Proposition 4. Let φ be a formula in a *DL*-language, and let P contain all attributes occurring in φ. Moreover, assume axioms (1)–(3) and the formula $\sum\limits_s(A)$. Then, there is a formula ψ in the P-normal form such that $\varphi \equiv \psi$.

Definition 6. A translation τ from the propositional constant L to an interpretation of a rough set language $\mathcal{L}_{\mathrm{RS}}$ of atomic expressions in the *KR*-system S is combined with $\neg, \vee, \wedge$ and $\rightarrow$ such that

$$\begin{aligned}
&\tau(|\varphi|_S) = |(a,v)|_S,\\
&\tau(|\sim\varphi|_S) = -\tau(|\varphi|_S),\\
&\tau(|\varphi \vee \psi|_S) = \tau(|\varphi|_S) \cup \tau(|\psi|_S),\\
&\tau(|\varphi \wedge \psi|_S) = \tau(|\varphi|_S) \cap \tau(|\psi|_S),\\
&\tau(|\varphi \rightarrow \psi|_S) = -\tau(|\varphi|_S) \cup \tau(|\psi|_S),\\
&\tau(|\varphi \equiv \psi|_S) =\\
&\qquad (\tau(|\varphi|_S) \cap \tau(|\psi|_S)) \cup (-\tau(|\varphi|_S) \cap -\tau(|\psi|_S)).
\end{aligned}$$

Let φ be an atomic formula of the DL-language, $R \in C \cup D$ an equivalence relation, X any subset of U, and a valuation v of propositional variables. Then, the truth-values of φ is defined as follows:

$$\|\varphi\|^v = \begin{cases} \mathbf{t} \ \textit{if} \ |\varphi|_S \subseteq POS_R(U/X) \\ \mathbf{f} \ \textit{if} \ |\varphi|_S \subseteq NEG_R(U/X) \end{cases}$$

This shows that decision logic is based on bivalent logic. In the next section, an interpretation of decision logic based on three-valued logics will be discussed.

III. Belnap's Four-Valued Logic

Belnap [9] first claimed that an inference mechanism for a database should employ a certain four-valued logic. The important point in Belnap's system is that we should deal with both incomplete and inconsistent information in databases. To represent such information, we need a four-valued logic since classical logic is not appropriate for the task. Belnap's four-valued semantics can in fact be viewed as an intuitive description of the internal states of a computer.

In Belnap's four-valued logic **B4**, four kinds of truth-values are used from the set **4** = $\{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$. These truth-values can be interpreted in the context of a computer, namely **T** means just told True, **F** means just told False, **N** means told neither True nor False, and **B** means told both True and False. Intuitively, **N** can be equated as $\emptyset$, and **B** as overdefined.

Belnap outlined a semantics for **B4** using logical connectives. Belnap's semantics uses a notion of *set-ups* mapping atomic formulas into **4**. A set-up can then be extended for any formula in **B4** in the following way:

$$\begin{aligned}
&s(A \ \& \ B) = s(A) \ \& \ s(B),\\
&s(A \ \vee \ B) = s(A) \ \vee \ s(B),\\
&s(\sim A) = \ \sim s(A).
\end{aligned}$$

Belnap also defined a concept of entailments in **B4**. We say that A entails B just in case for each assignment of one of the four values to variables, the value of A does not exceed the value of B in B4, i.e., $s(A) \leq s(B)$ for each set-up s. Here, $\leq$ is defined as $\mathbf{F} \leq \mathbf{B}$, $\mathbf{F} \leq \mathbf{N}$, $\mathbf{B} \leq \mathbf{T}$, $\mathbf{N} \leq \mathbf{T}$. Belnap's four-valued logic in fact coincides with the system of *tautological entailments* due to Anderson and Belnap [10]. Belnap's logic **B4** is one of the paraconsistent logics capable of tolerating contradictions. Belnap also studied the implications and quantifiers in **B4** in connection with question-answering systems. However, we will not go into detail here.
The structure that consists of these four elements and the five basic operators is usually called **B4**.

Designated elements and models: The next step in using **B4** for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{\mathbf{T}, \mathbf{B}\}$ since both values intuitively represent a formula known to be true. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff either a or b is in $\mathcal{D}$. From this point, various semantics notions are defined on **B4** as natural generalizations of similar classical notions.

IV. Rough Sets and Partial Semantics

Partial semantics for classical logic has been studied by van Benthem in the context of the *semantic tableaux* [11][12].

This insight can be generalized to study consequence relations in terms of a Gentzen-type sequent calculus. To handle an aspect of vagueness on the decision logic, the forcing relation for the partial interpretation is defined as a four-valued semantic.

As the proposed approach can replace the base bivalent logic of decision logic, alternative versions of decision logic based on many-valued logics are obtained.

The model $\mathcal{S}$ of decision logic based on four-valued semantics consists of a universe U for the language L and an assignment function s that provides an interpretation for L. For the domain $|\mathcal{S}|$ of the model $\mathcal{S}$, a subset is defined by $S = \langle S^+, S^- \rangle$. The first term of the ordered pair denotes the set of *n*-tuples of elements of the universe that *verify* the relation S, whereas the second term denotes the set of n-tuples that *falsify* the relation.

The interpretation of the propositional variables of L for the model $\mathcal{S}$ is given by $S_{\mathcal{S}} = \langle (S)^+_{\mathcal{S}}, (S)^-_{\mathcal{S}} \rangle$. An *interpretation function* for a domain $|\mathcal{S}|$ in the standard way as a function s with domain L such that $s(x) \in |\mathcal{S}|^n$ if S is a relation symbol. We need two interpretation functions for each model here; a model for partial logic for a predicate symbol is a triple $\langle |\mathcal{S}|, s^+, s^- \rangle$, where s^+ and s^- are interpretation functions for $|\mathcal{S}|$. The denotation of a relation symbol consists of those tuples for which it is *true* that they stand in the relation; the antidenotation consists of the tuples for which this is *false*. As before, truth and falsity are neither true nor false, or it may be both true and false that some tuple stands in a certain relation. The following definition is modified from [13].

Definition 7. Partial Relation: An *n*-ary *partial relation* S on the domain $|\mathcal{S}_1|, \ldots, |\mathcal{S}_n|$ is a tuple $\langle S^+, S^- \rangle$ of the relations $S^+, S^- \subseteq |\mathcal{S}_1| \times \ldots \times |\mathcal{S}_n|$. The relation S^+ is called S's *denotation*; S^- is called S's *antidenotation*, $|\mathcal{S}_1| \times \ldots \times |\mathcal{S}_n| / (S^+ \cup S^-)$ its gap, and $S^+ \cap S^-$ its glut. A partial relation is *coherent* if its glut is empty, *total* if its gap is empty, *incoherent* if it is not coherent and *classical* if it is both coherent and total. A unary partial relation is called a *partial set*.

Definition 8. Partial Operation for **4**: Let $S_1 = \langle S_1^+, S_1^- \rangle$ and $S_2 = \langle S_2^+, S_2^- \rangle$ be partial relations. Define

$$\begin{aligned}
-S_1 &:= \langle S_1^+, S_1^- \rangle \ (\textit{partial complementation}),\\
S_1 \cap S_2 &:= \langle S_1^+ \cap S_2^-, S_1^+ \cup S_2^- \rangle \ (\textit{partial intersection}),\\
S_1 \cup S_2 &:= \langle S_1^+ \cup S_2^-, S_1^+ \cap S_2^- \rangle \ (\textit{partial union}),\\
S_1 \subseteq S_2 &:= \langle S_1^+ \subseteq S_2^-, S_1^+ \subseteq S_2^- \rangle \ (\textit{partial inclusion}).
\end{aligned}$$

Partial inclusion means S_1 approximates S_2. Let A be some set of partial relations; then, following properties hold:

$$\bigcap A := \langle \bigcap \{S^+ \mid S \in A\}, \bigcup \{S^- \mid S \in A\}\rangle,$$
$$\bigcup A := \langle \bigcup \{S^+ \mid S \in A\}, \bigcap \{S^- \mid S \in A\}\rangle.$$

To handle three-valued and four-valued logic in a unified manner, we adopt the four-value interpretation by Belnap [9].

Let $\mathbf{4} = \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ be the truth-values for the four-valued semantics of L, where each value is interpreted as true, false, neither true nor false, and both true and false.

A model $\mathcal{S}$ determines a four-valued assignment v on atomic formula in the following way:

$$\|\varphi\|^v = \left\{ \begin{array}{c} \mathbf{T} \\ \mathbf{F} \\ \mathbf{N} \\ \mathbf{B} \end{array} \right\} \textit{if } |\varphi, {\sim}\varphi|_S \cap S = \left\{ \begin{array}{c} \{\varphi\} \\ \{{\sim}\varphi\} \\ \{\emptyset\} \\ \{\varphi, {\sim}\varphi\} \end{array} \right\}.$$

Then, the truth-values of φ on $S = (U, A)$ is defined as follows:

$$\|\varphi\|^v = \begin{cases} \mathbf{T} \textit{ if } |\varphi|_S \subseteq POS_R(U/X) \\ \mathbf{F} \textit{ if } |\varphi|_S \subseteq NEG_R(U/X) \\ \mathbf{N} \textit{ if } |\varphi|_S \not\subseteq POS_R(U/X) \cup NEG_R(U/X) \\ \mathbf{B} \textit{ if } |\varphi|_S \subseteq BN_R(U/X) \end{cases}$$

Definition 9 (Partial Model). A partial model for a propositional DL-language L is a tuple $\mathcal{M} = (\mathcal{T}, \mathcal{D}, \mathcal{O})$, where

- $\mathcal{T}$ is a non-empty set of truth-values.
- $\emptyset \subset \mathcal{D} \subseteq \mathcal{T}$ is the set of designated values.
- For every *n-ary connective* $\diamond$ of L, $\mathcal{O}$ includes a corresponding n-ary function $\widetilde{\diamond}$ from $\mathcal{T}^n$ to $\mathbf{4}$.

Let W be the set of *well-formed formulas* of L. A (legal) valuation in a Partial Model $\mathcal{S}$ is a function $V : W \to \mathbf{4}$ that satisfies the following condition:

$$V(\diamond(\psi_1, \cdots, \psi_n)) \in \widetilde{\diamond}(V(\psi_1), \cdots, V(\psi_n))$$

for every n-ary connective $\diamond$ of L and any $\psi_1, \cdots, \psi_n \in W$.

Let $\mathcal{V}_M$ denote the set of all valuations in the partial model $\mathcal{D}$. The notions of satisfaction under a valuation, validity, and consequence relation are defined as follows:

- A formula $\varphi \in W$ is satisfied by a valuation $v \in \mathcal{V}_M$, in symbols, $\mathcal{M} \models_v \varphi$, $v(\varphi) \in \mathcal{D}$.
- A sequent $\Sigma = \Gamma \Rightarrow \Delta$ is satisfied by a valuation $v \in \mathcal{V}_M$, in symbols, $\mathcal{M} \models_v \Sigma$, iff either v does not satisfy some formula in Γ or v satisfies some formula in Δ.
- A sequent Σ is valid, in symbols, $\models \Sigma$, if it is satisfied by all valuations $V \in \mathcal{V}_M$.
- The consequence relation on W defined by $\mathcal{M}$ is the relation $\mathcal{M} \vdash$ on sets of formulas in W such that, for any $T, S \subseteq W, T \vdash_{\mathcal{M}} S$ iff there exist finite sets $\Gamma \subseteq T, \Delta \subseteq S$ such that the sequent $\Gamma \Rightarrow \Delta$ is valid.

Definition 10. (Tarski truth definition for partial propositional logic) Let L be a set of propositional constants and let $v : P \to \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ be a (*valuation*) *function*.

$$\|p\|^v = v(p) \text{ if } p \in P$$

The truth-values of φ on the information system $S = (U, A)$ are represented by forcing relations as follows:

$$\|\varphi\|^v = \mathbf{T} \textit{ iff } \mathcal{M} \models^+_v \varphi \textit{ and } \mathcal{M} \not\models^-_v \varphi,$$
$$\|\varphi\|^v = \mathbf{F} \textit{ iff } \mathcal{M} \not\models^+_v \varphi \textit{ and } \mathcal{M} \models^-_v \varphi,$$
$$\|\varphi\|^v = \mathbf{N} \textit{ iff } \mathcal{M} \not\models^+_v \varphi \textit{ and } \mathcal{M} \not\models^-_v \varphi,$$
$$\|\varphi\|^v = \mathbf{B} \textit{ iff } \mathcal{M} \models^+_v \varphi \textit{ and } \mathcal{M} \models^-_v \varphi.$$

A semantic relation for the model $\mathcal{M}$ is defined following [11][14][13]. The truth and falsehood of a formula of the DL-language are defined in a model $\mathcal{M}$. The truth (denoted by $\models^+_v$) and the falsehood (denoted by $\models^-_v$) of the formulas of the decision logic in $\mathcal{M}$ are defined inductively.

Definition 11. The semantic relations of $\mathcal{M} \models^+_v \varphi$ and $\mathcal{M} \models^-_v \varphi$ are defined as follows:

$\mathcal{M} \models^+_v \varphi$ *iff* $\varphi \in M^+$,
$\mathcal{M} \models^-_v \varphi$ *iff* $\varphi \in M^-$,
$\mathcal{M} \models^+_v {\sim} \varphi$ *iff* $\mathcal{M} \models^-_v \varphi$,
$\mathcal{M} \models^-_v {\sim} \varphi$ *iff* $\mathcal{M} \models^+_v \varphi$,
$\mathcal{M} \models^+_v \varphi \vee \psi$ *iff* $\mathcal{M} \models^+_v \varphi$ *or* $\mathcal{M} \models^+_v \psi$,
$\mathcal{M} \models^-_v \varphi \vee \psi$ *iff* $\mathcal{M} \models^-_v \varphi$ *and* $\mathcal{M} \models^-_v \psi$,
$\mathcal{M} \models^+_v \varphi \wedge \psi$ *iff* $\mathcal{M} \models^+_v \varphi$ *and* $\mathcal{M} \models^+_v \psi$,
$\mathcal{M} \models^-_v \varphi \wedge \psi$ *iff* $\mathcal{M} \models^-_v \varphi$ *or* $\mathcal{M} \models^-_v \psi$,
$\mathcal{M} \models^+_v \varphi \to \psi$ *iff* $\mathcal{M} \models^-_v \varphi$ *or* $\mathcal{M} \models^+_v \psi$,
$\mathcal{M} \models^-_v \varphi \to \psi$ *iff* $\mathcal{M} \models^+_v \varphi$ *and* $\mathcal{M} \models^-_v \psi$.

The symbol $\sim$ denotes strong negation, in which $\sim$ is interpreted as true if the proposition is false.

Since validity in **B4** is defined in terms of truth preservation, the set of designated values is $\{\mathbf{T}, \mathbf{B}\}$ of $\mathbf{4}$. We assume that an interpretation of **B4** satisfies the following constraint.

Definition 12. Exclusion and Exhaustion:

Exclusion: model $\mathcal{M}$ is *exclusion iff* $S^+ \cap S^- = \emptyset$.

Exhaustion: model $\mathcal{M}$ is *exhaustion iff* $S^+ \cup S^- = S$.

The model $\mathcal{M}$ is *consistent* if and only if $S^+ \cap S^- = \emptyset$. The relational domains of general models are closed under the operations $\cap, \cup$.

The natural operation on the set of truth combinations $\mathbf{4} = \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ that we have defined in the previous section can be extended to the class of partial relations.

Definition 13. A model of **B4** for L is a pair $M = (S, |\cdot|)$, where S is a non-empty set, and $|\cdot|$ is an interpretation of a propositional symbol, with $|p| : S_n \to \mathbf{4}$ for any $p \in P_n, n \leq 0$.

Example 1. Suppose the decision table below where the condition and decision attributes are not considered.

$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$
Attribute: $C = \{c1, c2, c3, c4\}$
$c_1 = \{x_1, x_4, x_8\}, c_2 = \{x_2, x_5, x_7\}, c_3 = \{x_3\}$,
$c_4 = \{x_6\}$
$U/C = c_1 \cup c_2 \cup c_3 \cup c_4$
Any subset $X = \{x_3, x_6, x_8\}$

$POS_C(X) = c_3 \cup c_4 = \{x_3, x_6\}$
$BN_C(X) = c_1 = \{x_1, x_4, x_8\}$
$NEG_C(X) = c_2 = \{x_2, x_5, x_7\}$

The evaluation of the truth-values of the formulas is as follows:

If $|C_{c3}|_S \subseteq POS_C(X)$ *then* $\|C_{c3}\|^v = \mathbf{T}$,
If $|C_{c2}|_S \subseteq NEG_C(X)$ *then* $\|C_{c2}\|^v = \mathbf{F}$,
If $|C_{c2}|_S \nsubseteq POS_C(X) \cup NEG_C(X)$ *then* $\|C_{c2}\|^v = \mathbf{N}$,
If $|C_{c1}|_S \subseteq BN_C(X)$ *then* $\|C_{c1}\|^v = \mathbf{B}$.

Example 2. Consider Table I again. Assume that a, b, and c are condition attributes and d and e are decision attributes. Decision rules 1 and 5 are inconsistent. This means that 1 and 5 can be considered to have non-deterministic value, e.g., **N** or **B** respectively.

V. Consequence Relation and Sequent Calculus

Partial semantics in classical logic is closely related to the interpretation of the Beth tableau [12]. Van Benthem [11] suggested the relationship of the consequence relation to a Gentzen sequent calculus. We replace the bivalent logic of the decision logic with many-valued logics based on partial semantics.

A. Sequent Calculi for Many-valued Logics

We begin by recalling the basic idea of the Beth tableau. The Beth tableau proves $X \to Y$ by constructing a counterexample of X & $\sim Y$. The Beth tableaux has several partial features. For instance, there may be counterexamples even if a branch remains open. This insight led van Benthem [11] to work out partial semantics for classical logic.

Here, we describe a brief introduction of sequent calculi. For sequent calculi, formulas are constructed from the propositional variables and logical connectives, e.g., $\sim$, $\neg$, $\wedge$, $\vee$, *and* $\to$. Capital letters $A, B, \dots$ are used for formulas, and Greek capital letters Γ, Δ are used for finite sequences of formulas. A sequent is an expression of the form $\Gamma \Rightarrow A$. We introduce some concepts of sequent calculi. If a sequent $\Gamma \Rightarrow A$ is provable in a system S, then we write $S \vdash \Gamma \Rightarrow A$. A rule R of inference holds for a system S if the following condition is satisfied. For any instance of the following sequent of R, if $S \vdash \Gamma_i \Rightarrow A_i$ for all i, then $S \vdash \Delta \Rightarrow B$.

$$\frac{\Gamma_1 \Rightarrow A_1 \ \dots \ \Gamma_n \Rightarrow A_n}{\Delta \Rightarrow B}$$

Moreover, R is said to be derivable in S if there is a derivation from $\Gamma_1 \Rightarrow A_1, \dots, \Gamma_n \Rightarrow A_n$ to $\Delta \Rightarrow B$ in S.

To accommodate the Gentzen system to partial logics, we need some concepts of partial semantics. In the Beth tableau, It is assumed that V is a partial valuation function assigning the values 0 or 1 to an atomic formula p. We can then set $V(p) = 1$ for p on the left-hand side and $V(p) = 0$ for p on the right-hand side in an open branch of the tableau. To deal with an uncertain concept in many-valued semantics, we need to introduce the consequence relation [5]. *Pre* and *Cons* represent the sequent premise and conclusion, respectively, and 1 represents true and 0 false. First, we define the following concept of consequence relation C1.

(C1) for all V, if $V(Pre) = 1$, then $V(Cons) = 1$.

In C1, if *Pre* is evaluated as 1, then *Cons* preserves 1. Here, we define a classical Gentzen system.

Definition 14. The sequent calculus for the classical propositional logic CL is defined as follows:

Axiom: $A \Rightarrow A \ \ (ID)$

Sequent rules:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta, A}\ (Weakening)$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}\ (Cut)$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim A}\ (\sim R) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\sim A, \Gamma \Rightarrow \Delta}\ (\sim L)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge R) \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge L)$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee R) \qquad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee L)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim A, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to R)$$

$$\frac{\Gamma, \sim A \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\to L)$$

Theorem 5. The logic for C1 is axiomatized by the Gentzen sequent calculus CL.

Proof: See [12],[11],[15]. ■

Next, we define the sequent calculus GC1 for C1 that can be obtained by adding the following rules to CL without $(\sim R)$ such as CL$\backslash\{(\sim R)\}$, where, "$\backslash$" implies that the rule following "$\backslash$" is excluded:

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \sim\sim A}\ (\sim\sim R) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\sim\sim A, \Gamma \Rightarrow \Delta}\ (\sim\sim L)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim A, \sim B}{\Gamma \Rightarrow \Delta, \sim(A \wedge B)}\ (\sim\wedge R)$$

$$\frac{\sim A, \Gamma \Rightarrow \Delta \quad \sim B, \Gamma \Rightarrow \Delta}{\sim(A \wedge B), \Gamma \Rightarrow \Delta}\ (\sim\wedge L)$$

$$\frac{\Gamma \Rightarrow \Delta, \sim A \quad \Gamma \Rightarrow \Delta, \sim B}{\Gamma \Rightarrow \Delta, \sim (A \vee B)}\ (\sim \vee R)$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow \Delta}{\sim(A \vee B), \Gamma \Rightarrow \Delta}\ (\sim \vee L)$$

It is worth noting that the three-valued logic by Kleene has no tautology. Thus, to define a consequence relation, a tableau system for a three-valued logic is formalized [11] [15]. Then, the consequence relation C2 is defined as follows:

(C2) for all V, if $V(Pre) = 1$, then $V(Cons) \neq 0$.

C2 is interpreted as *exclusion*; then, the consequence relation C2 is regarded for Kleene's strong three-valued logic K_3. As the semantics for C2, we define the extension of the valuation function $V^{C2}(p)$ for an atomic formula p as follows:

$\mathbf{T} =_{def} V^{C2}(p) = 1 =_{def} V^{C2}(p) = 1$ *and* $V^{C2}(p) \neq 0$,
$\mathbf{F} =_{def} V^{C2}(p) = 0 =_{def} V^{C2}(p) = 0$ *and* $V^{C2}(p) \neq 1$,

$\mathbf{N} =_{def} V^{C2}(p) = \{\} =_{def} V^{C2}(p) \neq 1 \ and \ V^{C2}(p) \neq 0.$

The interpretation of C2 by the partial semantics is given as follows:

Definition 15. $\Gamma \models_s \varphi$ iff there is no φ that is not $\mathbf{F}$ under V^{C2} (in the three-valued $\{\mathbf{T}, \mathbf{F}, \mathbf{N}\}$) and for all $\gamma \in \Gamma$, γ is $\mathbf{T}$ under V^{C2}.

The Gentzen-type sequent calculus GC2 axiomatizes C2 [15][11]. We are now in a position to define GC2. For GC2, the principle of explosion (ex falso quodlibet (EFQ)), defined below, is added to TG1$\backslash\{(\sim L)\}$.

(EFQ) $A, \sim A \Rightarrow$

Definition 16. The sequent calculus GC2 is defined as follows:
GC2 := $\{(ID), (Weakening), (Cut), (EFQ), (\wedge R), (\wedge L), (\vee R), (\vee L), (\rightarrow R), (\rightarrow L), (\sim\sim R), (\sim\sim L), (\sim\wedge R), (\sim\wedge L), (\sim \vee R), (\sim \vee L)\}$.

GC2 can be interpreted as *truth preserving* with the matrix of a three-valued logic defined as $\langle\{T, F, N\}, \{T\}, \{\sim, \vee, \wedge, \rightarrow\}\rangle$. For the rule $(\sim L)$ obtained from (EFQ), GC2 and GC1 are equivalent.

Theorem 6. GC2 = GC1.

Proof: (EFQ) can be considered as $(\sim L)$; then, double negation and the de Morgan laws in GC2 are obtained. ■

In the classical interpretation of CL, the law of excluded middle (EM) holds but not in C2.

Then, the rule C2 for the Gentzen system is axiomatized as GC2.

Theorem 7. C2 can be axiomatized by the sequent calculus GC2.

Proof: See [11][15]. ■

Theorem 8. In the model for C2, $\mathcal{S}$, DL-language L, and formula φ, it is not the case that $\mathcal{M} \models_v^+ \varphi$ and $\mathcal{M} \models_v^- \varphi$ hold.

Proof: Only the proof for $\sim$ and $\wedge$ will be provided. It can be carried out by induction on the complexity of the formula. The condition of *consistent* implies that it is not the case that $\varphi \in \mathcal{S}^+$ and $\varphi \in \mathcal{S}^-$. Then, it is not the case that $\mathcal{M} \models_v^+ \varphi$ and $\mathcal{M} \models_v^- \varphi$.

$\sim$: We assume that $\mathcal{M} \models_v^+ \sim \varphi$ and $\mathcal{M} \models_v^- \sim \varphi$ hold. Then, it follows that $\mathcal{M} \models_v^+ \varphi$ and $\mathcal{M} \models_v^- \varphi$. This is a contradiction.

$\wedge$: We assume that $\mathcal{M} \models_v^- \varphi \wedge \psi$ and $\mathcal{M} \models_v^+ \varphi \wedge \psi$ hold. Then, it follows that $\mathcal{M} \models_v^+ \varphi$, $\mathcal{M} \models_v^+ \psi$ and either $\mathcal{M} \models_v^- \varphi$ or $\mathcal{M} \models_v^- \psi$. In either case, there is a contradiction. ■

Next, we provide another consequence relation with a different interpretation for the third-value below.

(C3) for all V, if $V(Pre) \neq 0$, then $V(Cons) = 0$.

C3 is interpreted as *exhaustion*, then the consequence relation C3 is for Logic for Paradox [16]. As the semantics for C3, we define the extension of the valuation function $V^{C3}(p)$ for an atomic formula p as follows:

$\mathbf{T} =_{def} V^{C3}(p) = 1 =_{def} V^{C3}(p) = 1 \ and \ V^{C3}(p) \neq 0,$
$\mathbf{F} =_{def} V^{C3}(p) = 0 =_{def} V^{C3}(p) = 0 \ and \ V^{C3}(p) \neq 1,$
$\mathbf{B} =_{def} V^{C3}(p) = \{1, 0\} =_{def} V^{C3}(p) = 1 \ and \ V^{C3}(p) = 0.$

The interpretation of C3 by the partial semantics is given as follows:

Definition 17. $\Gamma \models_v \varphi$ iff there is φ that is $\mathbf{T}$ under V^{C3} (in the three-valued $\{\mathbf{T}, \mathbf{F}, \mathbf{B}\}$) and for all $\gamma \in \Gamma$, γ is not $\mathbf{F}$ under V^{C3}.

The Gentzen sequent calculus GC3 is obtained from GC2, replacing EFQ with EM (*excluded middle*) as an axiom:

(EM) $\Rightarrow A, \sim A$

Definition 18. The sequent calculus GC3 is defined as follows:
GC3 := $\{(ID), (Weakening), (Cut), (EM), (\wedge R), (\wedge L), (\vee R), (\vee L), (\rightarrow R), (\rightarrow L), (\sim\sim R), (\sim\sim L), (\sim\wedge R), (\sim\wedge L), (\sim \vee R), (\sim \vee L)\}$.

Theorem 9. C3 can be axiomatized by the Gentzen calculus GC3.

Proof: GC3 can be obtained by deriving double negation and two de Morgan laws in GC3. The $(\sim R)$ rule can be provided as EM. ■

Next, we extend consequence relation C4 as follows:

(C4) *for all* V, *if* $V(Pre) \neq 0$, *then* $V(Cons) \neq 0$.

C4 is regarded as a four-valued logic since it allows for an inconsistent valuation. We are now in a position to define Belnap's four-valued logic **B4**.

As the semantics for GC4, Belnap's **B4** is adopted here. We define the extension of the valuation function $V^{C4}(p)$ for an atomic formula p as follows:

$\mathbf{T} =_{def} V^{C4}(p) = 1 =_{def} V^{C4}(p) = 1 \ and \ V^{C4}(p) \neq 0,$
$\mathbf{F} =_{def} V^{C4}(p) = 0 =_{def} V^{C4}(p) = 0 \ and \ V^{C4}(p) \neq 1,$
$\mathbf{N} =_{def} V^{C4}(p) = \{\} =_{def} V^{C4}(p) \neq 1 \ and \ V^{C4}(p) \neq 0,$
$\mathbf{B} =_{def} V^{C4}(p) = \{1, 0\} =_{def} V^{C4}(p) = 1 \ and \ V^{C4}(p) = 0.$

The interpretation of C4 by the partial semantics is given as follows:

Definition 19. $\Gamma \models_v \varphi$ *iff* there is no φ that is not $\mathbf{F}$ under V^{C4} (in **4**) and for all $\gamma \in \Gamma$, γ is not $\mathbf{F}$ under V^{C4}.

Definition 20. The sequent calculus GC4 is defined as follows:
GC4 := $\{(ID), (Weakening), (Cut), (\wedge R), (\wedge L), (\vee R), (\vee L), (\sim\sim R), (\sim\sim L), (\sim\wedge R), (\sim\wedge L), (\sim \vee R), (\sim \vee L)\}$.

Theorem 10. C4 can be axiomatized by the sequent calculus GC4.

Proof: GC4 can be obtained by deriving double negation and two de Morgan laws in GC4. The $(F\sim)$ rule can be provided as EM. ■

VI. Extension of Many-valued Semantics

We introduce three-valued logics and provide some relationship and properties between the consequence relations we denotated in the previous section.

Kleene's strong three-valued logic: Kleene proposed three-valued logics to deal with *undecidable* sentences in connection with recursive function theory [17]. Thus, the third truth-value can be interpreted as undecided in the strong Kleene logic K_3, which is of special interest to describe a machine's computational state. K_3 can give a truth value to a compound sentence even if some of its parts have no truth value. Kleene also proposed the weak three-valued logic in which the whole sentence is undecided if any component of a compound sentence is undecided.

The truth tables for K_3 are defined as follows:

$\sim$	T	F	N
	F	T	N

$\wedge$	T	F	N
T	T	F	N
F	F	F	F
N	N	F	N

$\vee$	T	F	N
T	T	T	T
F	T	F	N
N	T	N	N

$\rightarrow$	T	F	N
T	T	F	N
F	T	T	T
N	T	N	N

The implication $\rightarrow$ can be defined in the following way:

$$A \rightarrow B =_{def} \sim A \vee B$$

The axiomatization of K_3 by a Gentzen-type sequent calculus can be found in the literature [5].

Let $\models$ be the consequence relation of K_3. Then, we have the following Gentzen-type sequent calculus GK_3 for K_3, which contains an axiom of the form

$X \models Y \text{if} X \cap Y \neq \emptyset$

and the rules (Weakening), (Cut), and

$$A \models \sim\sim A, \qquad \sim\sim A \models A, \qquad A, \sim A \models,$$
$$A, B \models A \wedge B, \qquad A \wedge B \models A, \qquad A \wedge B \models B,$$
$$\sim A \models \sim(A \wedge B), \qquad \sim B \models \sim(A \wedge B),$$
$$\sim(A \wedge B) \models \sim A, \sim B.$$

GC2 is considered as Kleene's strong three-valued logic K_3. The implication of K_3 does not satisfy the deduction theorem. In addition, $A \rightarrow A$ is not a theorem in K_3.

Theorem 11. $\models_{K3} = \vdash_{C2}$, where $\models_{K3}$ denotes the consequence relation of K_3.

Proof: By induction on K_3 and GC2. It is easy to transform each proof of K_3 into GC2. The converse transformation can be also presented. ■

Łukasiewicz three-valued logic: Łukasiewicz's (1920) three-valued logic was proposed in order to interpret a future contingent statement in which the third truth-value can be read as indeterminate or possible. Thus, in Łukasiewicz's three-valued logic L_3, neither the law of excluded middle nor the law of non-contradiction holds. The difference between K_3 and L_3 lies in the interpretation of implication, as the truth table indicates.

It is also possible to describe the Hilbert presentation of L_3. Let $\supset$ be the Łukasiewicz implication. Then, we can show the following axiomatization of L_3 due to Wajsberg. It has been axiomatized by Wajsberg (1993) in [5] using a language based on $(\vee, \supset, \sim)$, the modus ponens rule and the following axioms:

(W1) $(p \supset q) \supset ((p \supset r) \supset (p \supset r))$,
(W2) $(\sim p \supset \sim q) \supset (q \supset p))$,
(W3) $(((p \supset \sim p) \supset p) \supset p)$.

They are closed under the rules of substitution and modus ponens. Unlike in K_3, $A \supset A$ is a theorem in L_3. It is noted, however, that the philosophical motivation of L_3 in connection with Aristotelian logic can be challenged. For a review of various three-valued logics, see Urquhart [5].

$\supset$	T	F	N
T	T	F	N
F	T	T	T
N	T	N	T

The definition of the semantic relation for the implication of L_3 is obtained by replacing the implication in Definition 11 with the following definition:

$\mathcal{M} \models_v^+ \varphi \rightarrow \psi \textit{ iff } \mathcal{M} \models_v^- \varphi \textit{ or } \mathcal{M} \models_v^+ \psi$ or
$(\mathcal{M} \not\models_v^+ \varphi \textit{ and } \mathcal{M} \not\models_v^- \varphi \textit{ and } \mathcal{M} \not\models_v^+ \psi \textit{ and } \mathcal{M} \not\models_v^- \psi)$.
$\mathcal{M} \models_v^- \varphi \rightarrow \psi \textit{ iff } \mathcal{M} \models_v^+ \varphi \textit{ and } \mathcal{M} \models_v^- \psi$.

Logic of Paradox (LP): Logic of Paradox (LP) has been studied by Priest [16], which is one of the paraconsistent logics excluding EFQ. As motivation for paraconsistent logics in general, LP can treat various logical paradoxes and Dialetheism, which is a philosophical position that admits some true contradictions. GC3 is taken as a sequent calculus of LP [18], and the truth table of LP can be obtained K_3's truth value **N** replaced with **B**.

The definition of the semantic relation for the implication of GC3 is obtained by replacing the implication in Definition 11 with the following definition.

$\mathcal{M} \models_v^+ \varphi \rightarrow \psi \textit{ iff } \mathcal{M} \not\models_v^+ \varphi \textit{ or } \mathcal{M} \not\models_v^- \psi$ or
$(\mathcal{M} \models_v^+ \varphi \textit{ and } \mathcal{M} \models_v^- \varphi \textit{ and } \mathcal{M} \models_v^+ \psi \textit{ and } \mathcal{M} \models_v^- \psi)$.
$\mathcal{M} \models_v^- \varphi \rightarrow \psi \textit{ iff } \mathcal{M} \models_v^+ \varphi \textit{ and } \mathcal{M} \models_v^- \psi$.

Belnap's four-valued logic: In section III, we have already seen the Belnap's four-valued logic. In addition to section III, we define the truth tables for $\sim$, $\wedge$, and $\vee$.

In this paper, the implication of B4 is defined with $\sim$ and $\vee$ and it does not hold for the rule of *modus ponens* because the *disjunctive syllogism* does not hold.

$\sim$	T	F	N	B
	F	T	N	B

$\wedge$	T	F	N	B
T	T	F	N	B
F	T	T	T	T
N	T	T	T	T
B	T	F	N	B

$\vee$	T	F	N	B
T	T	F	N	B
F	T	T	T	T
N	T	T	T	T
B	T	F	N	B

The aim of this paper is to present many-valued semantics for the decision logic. There are three candidates of consequence relations for the enhancement in the decision logic. GC2, which was discussed above, is interpreted as strong Kleene three-valued logic. The value of a proposition is neither true nor false in GC2. In this case, the designated value of GC2 is defined as $\{\mathbf{T}, \mathbf{N}\}$. GC3 is a paraconsistent logic, and its designated valued is defined as $\{\mathbf{T}, \mathbf{B}\}$. The paraconsistent logic does not hold for the principle of explosion (ex falso quodlibet); therefore, it is possible to interpret the consequence relation by C3. GC4 is obtained from C4 based on four-valued semantics and interpreted as both paracomplete and paraconsistent.

Here, we present the extended version of many-valued logics with weak negation $\neg$. Weak negation represents the lack of truth. The assignment of weak negation is defined as follows:

$$\|\neg\varphi\|_s = \begin{cases} \mathbf{T} \ \textit{if} \ \|\varphi\|_s \neq \mathbf{T} \\ \mathbf{F} \ \textit{otherwise} \end{cases}$$

Weak implication is defined as follows:

$$A \rightarrow_w B =_{def} \neg A \vee B$$

The assignment of weak implication is defined as follows:

$$\|A \rightarrow_w B\|^s = \begin{cases} \|B\|^s \ \textit{if} \ \|A\|^s \in \mathcal{D} \\ \mathbf{T} \ \textit{if} \ \|A\|^s \notin \mathcal{D} \end{cases}$$

We represent the truth tables for $\neg$ and $\rightarrow_w$ below.

$\neg$	T	F	N	B
	F	T	T	F

$\rightarrow_w$	T	F	N	B
T	T	F	N	B
F	T	T	T	T
N	T	T	T	T
B	T	F	N	B

The semantic relation for weak negation is as follows:

$$\begin{array}{lll} \mathcal{M} \models_v^+ \neg\varphi & \text{iff} & \mathcal{M} \not\models_v^+ \varphi, \\ \mathcal{M} \models_v^- \neg\varphi & \text{iff} & \mathcal{M} \models_v^+ \varphi. \end{array}$$

We try to extend many-valued logics with weak negation and weak implication. This regains some properties that some many-valued logics lack, such as the rule of modus ponens and the decision theorem. Obviously, L_3 recovers some properties that K_3 lacks and L_3's implication and weak implication has a close relationship.

Weak negation can represent the absence of truth. However, $\sim$ can serve as strong negation to express the verification of falsity. Note also that weak implication obeys the deduction theorem. This means that it can be regarded a logical implication. We can also interpret weak negation in terms of strong negation and weak implication:

$$\neg A =_{def} A \rightarrow_w \sim A$$

We define the sequent rules for $(\neg)$ and $(\rightarrow_w)$ as follows:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (\neg R) \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ (\neg L)$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow_w B}\ (\rightarrow_w R) \qquad \frac{B, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{A \rightarrow_w B, \Gamma \Rightarrow \Delta}\ (\rightarrow_w L)$$

GC2, GC3, and GC4 have additional rules of weak negation and weak implication, and we obtain GC2$^+$, GC3$^+$, and GC4$^+$. GC2$^+$ is the same as the extended Kleene logic EKL, that was proposed by Doherty [19] as the underlying three-valued logic for the non-monotonic logic and is provided with the deduction theorem.

GC4$^+$ is interpreted as both paracomplete and paraconsistent. This prevents the paradox of material implication of classical logic.

Here, it is observed that L_3 can be naturally interpreted in GC2$^+$. The Łukasiewicz implication can be defined as

$$A \supset B =_{def} (A \rightarrow_w B) \wedge (\sim B \rightarrow_w \sim A)$$

Next, we present the interpretation of weak negation for consequence relations C2, C3, and C4, which are interpreted as K_3, LP, and B4, respectively.

$$\|\neg A\|^{C2} = \begin{cases} \mathbf{F} \text{ if } \|A\| = \mathbf{T} \\ \mathbf{T} \text{ if } \textit{otherwise} \end{cases}$$

$$\|\neg A\|^{C3} = \begin{cases} \mathbf{T} \text{ if } \|A\| = \mathbf{F} \\ \mathbf{F} \text{ if } \textit{otherwise} \end{cases}$$

$$\|\neg A\|^{C4} = \begin{cases} \mathbf{F} \text{ if } \|A\| = \mathbf{T} \ \textit{or} \ \mathbf{B} \\ \mathbf{T} \text{ if } \|A\| = \mathbf{F} \ \textit{or} \ \mathbf{N} \end{cases}$$

We consider an application of weak negation for the interpretation of rough sets.

Example 3. Suppose the definition of a decision table is the same as Example 1.

$U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

Attribute: $C = \{c1, c2, c3, c4\}$
$c_1 = \{x_1, x_4, x_8\}, c_2 = \{x_2, x_5, x_7\}, c_3 = \{x_3\}$,
$c_4 = \{x_6\}$
$U/C = c_1 \cup c_2 \cup c_3 \cup c_4$

Any subset $X = \{x_3, x_6, x_8\}$

$POS_C(X) = c_3 \cup c_4 = \{x_3, x_6\}$

$BN_C(X) = c_1 = \{x_1, x_4, x_8\}$

$NEG_C(X) = c_2 = \{x_2, x_5, x_7\}$

The interpretation of the consequence relation C4 for weak negation in the decision table is defined as follows:

$\textit{If } |C_{c3}|_S \subseteq POS_C(X) \textit{ then } \neg\|C_{c3}\|^v = \mathbf{F}$,

$\textit{If } |C_{c2}|_S \subseteq NEG_C(X) \textit{ then } \neg\|C_{c2}\|^v = \mathbf{T}$,

$\textit{If } |C_{c2}|_S \not\subseteq POS_C(X) \cup NEG_C(X) \textit{ then } \neg\|C_{c2}\|^v = \mathbf{T}$,

$\textit{If } |C_{c1}|_S \subseteq BN_C(X) \textit{ then } \neg\|C_{c1}\|^v = \mathbf{F}$.

VII. Soundness and Completeness

The soundness and completeness theorem is shown for the sequent system GC4$^+$. Other systems can be adopted in a similar way for GC4$^+$. GC4$^+$, which was discussed above, is interpreted as one Belnap's four-valued logic B4 extended with weak negation and weak implication. The sequent calculus GC4$^+$ is defined as follows:

$$\begin{aligned}\text{GC4}^+ := \{&(ID),\ (Weakening),\ (Cut),\ (\wedge R),\ (\wedge L),\\ &(\vee R),\ (\vee L),\ ({\sim}{\sim}R),\ ({\sim}{\sim}L), ({\sim}\wedge R),\ ({\sim}\wedge L),\\ &({\sim}\vee R),\ ({\sim}\vee L),\ (\neg R),\ (\neg L),\ (\rightarrow_w R),\ (\rightarrow_w L)\}\end{aligned}$$

It is assumed that GC4$^+$ is the basic deduction system for decision logic obtained from C4 with weak negation and weak implication. This prevents the paradox of material implication of classical logic.

As the semantics for GC4$^+$, Belnap's **B4** is adopted here; we obey the definition of the valuation function $V^{C4}(p)$.

Lemma 12. The validity of the inference rules

1) The axioms of GC4$^+$ are valid.
2) For any inference rules of GC4$^+$ and any valuation s, if s satisfies all of the formulas of *Pre*, then s satisfies *Cons*.

Proof: 1) In GC4$^+$, the axiom (ID) and structural rules (weakening) and (cut) preserve validity.

For 2), the proof for $(\neg R)$, $(\rightarrow_w L)$, and $({\sim}\wedge L)$ will be provided.

$(\neg R)$:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (\neg R)$$

Suppose that $\models_{GC4^+} \Gamma, A \Rightarrow \Delta$. Then, either (1) $v(\gamma) \neq \mathbf{T}$ for some $\gamma \in \Gamma$ or $v(\delta) \neq \mathbf{F}$ for some $\delta \in \Delta$ or (2) $v(A) \neq \mathbf{T}$. If (1) holds, then clearly $\models_{GC4^+} \Gamma \Rightarrow \neg A, \Delta$ *iff* $\models^-_{GC4^+} \Gamma$ *or* $\models^+_{GC4^+} \Delta, \neg A$. If (2) holds, then from the definition of $\neg$, it follows that $v(\neg A) = \mathbf{T}$ and then $\models_{GC4^+} \Gamma \Rightarrow \neg A, \Delta$.

$(\rightarrow_w L)$:

$$\frac{B, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{A \rightarrow_w B, \Gamma \Rightarrow \Delta}\ (\rightarrow_w L)$$

Suppose that $\models_{GC4^+} B, \Gamma \Rightarrow \Delta$, and $\Gamma \Rightarrow \Delta, A$. Then, either (3) $v(\gamma) \neq \mathbf{T}$ for some $\gamma \in \Gamma$ or $v(\delta) \neq \mathbf{F}$ for some $\delta \in \Delta$ or (4) $v(B) \neq \mathbf{F}$ and $v(A) \neq \mathbf{T}$. If (3) holds, then clearly $\models_{GC4^+} A {\rightarrow_w} B, \Gamma \Rightarrow \Delta$. If (4) holds, then from the semantic relation of $\rightarrow_w$, it follows that $v(A {\rightarrow_w} B) \neq \mathbf{F}$ and again $\models_{GC4^+} A {\rightarrow_w} B, \Gamma \Rightarrow \Delta$.

$({\sim}\wedge L)$:

$$\frac{{\sim}A, \Gamma \Rightarrow \Delta \quad {\sim}B, \Gamma \Rightarrow \Delta}{{\sim}\,(A \wedge B), \Gamma \Rightarrow \Delta}\ ({\sim}\wedge L)$$

Suppose that $\models_{GC4^+} {\sim}A, \Gamma \Rightarrow \Delta$, and $\models_{GC4^+} {\sim}B, \Gamma \Rightarrow \Delta$. Then, either (5) $v(\gamma) \neq \mathbf{T}$ for some $\gamma \in \Gamma$ or $v(\delta) \neq \mathbf{F}$ for some $\delta \in \Delta$ or (6) $v({\sim}A) \neq \mathbf{F}$ or $v({\sim}B) \neq \mathbf{F}$. If (5) holds, then clearly $\models_{GC4^+} {\sim}(A \wedge B),\ \Gamma \Rightarrow \Delta$. If (6) holds, then from the definition of $\wedge$, it follows that $v(A \wedge B) \neq \mathbf{T}$, whence $v({\sim}(A \wedge B)) = \mathbf{T}$, and again $\models_{GC4^+} {\sim}(A \wedge B),\ \Gamma \Rightarrow \Delta$. ∎

Lemma 13 (Soundness of GC4$^+$). If $\vdash_{GC4^+}\ \Gamma \Rightarrow \Delta$ is provable in GC4$^+$, then $\models_{GC4^+} \Gamma \Rightarrow \Delta$.

Proof: If the sequent $\Gamma \Rightarrow \Delta$ is an instance of axiom (ID), then $\Gamma \Rightarrow \Delta$ is valid in GC4$^+$. By induction on the depth of a derivation of $\Gamma \Rightarrow \Delta$ in GC4$^+$, it follows, by Lemma 12, that the sequent $\Gamma \Rightarrow \Delta$ is valid in GC4$^+$. ∎

We are now in a position to prove the completeness of GC4$^+$. The proof below is similar to the Henkin proof described in Avron [3].

Theorem 14 (Completeness of GC4$^+$). The sequent calculus GC4$^+$ is *sound* and *complete* for $\models_{GC4^+}$.

Proof: Let us denote the provability in GC4$^+$ by $\vdash_{GC4^+}$. For any sequent Σ over the language of GC4$^+$,

$$\vdash_{GC4^+} \Sigma \text{ if } \Sigma \text{ has a proof in GC4}^+.$$

We have to prove that, for any sequent Σ over the language of GB4$^+$,

$$\models_{GC4^+} \Sigma \ \textit{iff}\ \vdash_{GC4^+} \Sigma.$$

The backward implication, representing the soundness of the system, follows immediately from Lemma 13. To prove the forward implication *completeness*, we argue by contradiction. Suppose Σ is a sequent such that $\nvdash_{GC4^+} \Sigma$. We shall prove that $\not\models_{GC4^+} \Sigma$. Let us assume that the inclusion and union of sequents are defined componetwise, i.e.,

$$(\Gamma' \Rightarrow \Delta') \subseteq (\Gamma'' \Rightarrow \Delta'') \text{ iff } \Gamma' \subseteq \Gamma'' \text{ and } \Delta' \subseteq \Delta'',$$
$$(\Gamma' \Rightarrow \Delta') \cup (\Gamma'' \Rightarrow \Delta'') = \Gamma', \Gamma'' \Rightarrow \Delta', \Delta''.$$

A sequent Σ_0 is called *saturated* if it is closed under all of the rules in GC4$^+$ applied backwards. More exactly, for any rule r in GC4$^+$ whose conclusion is contained in Σ_0, one of its premises must also be contained in Σ_0 (for a single premise rule, this means its only premise must be contained in Σ_0). For example, if $\Sigma_0 = (\Gamma_0 \Rightarrow \Delta_0)$ is saturated and $(A \rightarrow B) \in \Delta_0$, then in view of the rules $(\rightarrow R)$, we must have both ${\sim}A \in \Delta$ and $B \in \Delta$. In turn, if $(A \rightarrow B) \in \Gamma_0$, then in view of the rule $(\rightarrow L)$, we must have either ${\sim}A \in \Gamma$ or $B \in \Gamma$.

Let $\Sigma = (\Gamma \Rightarrow \Delta)$ be any sequent. We shall first prove that Σ can be extended to a saturated sequent $\Sigma^* = (\Gamma^* \Rightarrow \Delta^*)$, which is not provable in GC4$^+$. If Σ is already saturated, we are done. Otherwise, we start with the sequent Σ and expand it step by step by closing it under the subsequent rules of GC4$^+$ without losing the non-provability property. Specifically, we define a sequence $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ such that

1) $\Sigma_{i-1} \subseteq \Sigma_i$ for each $i \geq 1$,
2) Σ_i is not provable.

We take $\Sigma_0 = \Sigma_1 = \Sigma$; then, conditions 1 and 2 above are satisfied for $i = 1$. Assume that we have the constructed sequents $\Sigma_0, \Sigma_1, \ldots, \Sigma_k$ satisfying those conditions, and Σ_k is still not saturated. Then, there is a rule

$$r = \frac{\Pi_1 \ \cdots \ \Pi_l}{\Pi}$$

in GC4$^+$ such that $\Pi \subseteq \Sigma_k$ but $\Pi_i \not\subseteq \Sigma_k$ for $i = 1, \ldots, l$.

Since Σ_k is not provable, there must be an i such that $\Sigma_k \cup \Pi_i$ is not provable. Indeed, if $\Sigma_i \cup \Pi_i$ were provable for all $i, 1 \leq i \leq l$, then we could deduce $\Sigma_k \cup \Pi$ from the provable sequents $\Sigma_k \cup \Pi_i$, $i = 1, \ldots, l$, using rule r, which in view of $\Sigma_k \cup \Pi = \Sigma_k$ would contradict the fact that Σ_k is not provable. Thus, there is an $i_0, 1 \leq i_0 \leq l$, such that $\Sigma_k \cup \Pi_{i_0}$ is not provable, and we take $\Sigma_{k+1} = \Sigma_k \cup \Pi_{i_0}$. Obviously, the sequents $\Sigma_0, \Sigma_1, \ldots, \Sigma_{k+1}$ satisfy conditions 1 and 2 above.

Since all of the rules in GC4$^+$ have the subformula property, it is clear that after a finite number n of such steps, we will have added all possible premises of the rules r in GC4$^+$ whose conclusions are contained in the original sequent Σ or its descendants in the constructed sequence, obtaining a saturated extension $\Sigma^* = \Sigma_n$ of Σ, which is not provable in GC4$^+$.

Thus, we have

- $\Sigma^* = (\Gamma^* \Rightarrow \Delta^*)$ is closed under the rules in GC4$^+$ applied backwards,
- $\Gamma \subseteq \Gamma^*, \Delta \subseteq \Delta^*$,
- $\nvdash_{GC4^+} \Sigma^*$.

We use Σ^* to define a counter-valuation for Σ, i.e., a legal valuation v under the model of GC4$^+$ such that $v \not\models_{GC4^+} \Sigma$. For any propositional symbol $p \in P$ evaluated with the function defined in Definition 10, namely, we put:

$$v(p) = \begin{cases} \mathbf{T} \text{ if } p \in \Gamma \text{ and } p \notin \Delta \\ \mathbf{F} \text{ if } {\sim}p \in \Gamma \text{ and } {\sim}p \notin \Delta \\ \mathbf{B} \text{ if } \{p, {\sim}p\} \in \Gamma \\ \mathbf{N} \ \mathit{otherwise} \end{cases} \tag{1}$$

For the valuation for the strong negation in GC4$^+$, define the following:

$$v(\neg p) = \begin{cases} \mathbf{T} \text{ if } v(p) \in \{\mathbf{F}, \mathbf{N}\} \\ \mathbf{F} \text{ if } v(p) \in \{\mathbf{T}, \mathbf{B}\} \end{cases} \tag{2}$$

For any A, B of the set of all well-formed formulas of GC4$^+$,

$$v({\sim}A) = {\sim}v(A) \tag{3}$$

$$v(A \rightarrow_w B) = \begin{cases} \mathbf{T} \text{ if } v(A) \neq \mathbf{T} \text{ or } v(B) \neq \mathbf{F} \\ \mathbf{F} \text{ if } v(A) = \mathbf{T} \text{ and } v(B) = \mathbf{F} \\ \mathit{otherwise} \end{cases} \tag{4}$$

It is easy to see that v defined as above is a well-defined mapping of the formulas of GC4$^+$ into **4**. Indeed, as Σ^* is not provable in GC4$^+$, then by (1), $v(p)$ is uniquely defined for any propositional symbol p, whence by (2, 3), $v(\varphi)$ is uniquely defined for any well-formed formula.

Moreover, by (2, 3), v is a legal interpretation of the language of GC4$^+$ under the interpretation of GC4$^+$, for the interpretations of ${\sim}, \rightarrow_w$ under v are compliant with the truth tables of those operations for this interpretation.

As Σ^* is an extension of Σ, in order to prove that $\not\models_{GC4^+} \Sigma$, it suffices to prove that $\not\models_{GC4^+} \Sigma^*$. We should prove for any well-formed formulas φ,

$$\models_{GC4^+} \gamma \text{ for any } \gamma \in \Gamma^*, \ \not\models_{GC4^+} \delta \text{ for any } \delta \in \Delta^*. \tag{5}$$

Equation (5) is proved by structural induction on the formulas in $S = \Gamma^* \cup \Delta^*$.

We begin with literals in S, having the form of either p or ${\sim}p$, where $p \in P$. We have the following cases:

- $\varphi = p$. Then, by (1) and the fact that Γ^* and Δ^* are disjoint (for otherwise Σ^* would be provable), we have: $v(\varphi) \neq \mathbf{F}$ if $\varphi \in \Gamma^*$ and $v(\varphi) \neq \mathbf{T}$ if $\varphi \in \Delta^*$
- $\varphi = {\sim}p$. If $\varphi \in \Gamma^*$, then by (1), $v(p) \neq \mathbf{T}$, whence $v(\varphi) = {\sim}\mathbf{F} = \mathbf{T}$ by (3). In turn, if $\varphi \in \Delta^*$, then $\varphi \notin \Gamma^*$, whence $v(p) \neq \mathbf{F}$ and $v(\varphi) = {\sim}v(p) \neq \mathbf{T}$.
- $\varphi = \neg p$. If $\varphi \in \Gamma^*$, then by (1) $v(p) \neq \{\mathbf{T}, \mathbf{B}\}$, whence $v(\varphi) = \mathbf{T}$ by (2). In turn, if $\varphi \in \Delta^*$, then $\varphi \notin \Gamma^*$, whence $v(p) \neq \{\mathbf{F}, \mathbf{N}\}$ and $v(\varphi) = {\sim}v(p) = \mathbf{F}$.

Here, we define the rank ρ of formula φ by

$$\rho(p) = 1, \rho({\sim}\varphi) = \rho(\varphi) + 1, \rho(\varphi \rightarrow \psi) = \rho(\varphi) + \rho(\psi) + 1$$

Now we assume that the definition in (5) is satisfied for the formulas in S of rank up to n and suppose that $A, B \in S$ are at most of rank n. We prove that (5) holds for ${\sim}B$, $B \wedge C$ and $B \vee C$.

We begin with negation. Let $\varphi = {\sim}A$. As the case of $A = p \in P$ has already been considered, it remains to consider the following two cases:

- $A = {\sim}B$. Then, we have $\varphi = {\sim\sim}B$.
  - If $\varphi \in \Gamma^*$, then by rule $({\sim\sim}L)$, we have $B \in \Gamma^*$, since Σ^* is a saturated sequent. Hence, by inductive assumption, $v(B) = \mathbf{T}$, and by (3), $v(\varphi) = {\sim\sim}\mathbf{T} = \mathbf{T}$.
  - In turn, if $\varphi \in \Delta^*$, then by rule $({\sim\sim}R)$, we have $B \in \Delta^*$, whence by inductive assumption, $v(B) = \mathbf{F}$, and in consequence, $v(\varphi) = {\sim\sim}\mathbf{F} = \mathbf{F}$.
- $A = B \wedge C$. We again have two cases:
  - If $\varphi \in \Gamma^*$, then by rule $({\sim}\wedge L)$, we have ${\sim}B, {\sim}C \in \Gamma^*$ since Σ^* is saturated. Hence, by inductive assumption, $v(B) \neq \mathbf{T}$ and $v(C) \neq \mathbf{T}$ (because $v({\sim}B) \neq \mathbf{F}$ and $v({\sim}C) \neq \mathbf{F}$). Thus, by the truth table $v(B \wedge C) \neq \mathbf{T}$; therefore, $v(\varphi) = {\sim}\mathbf{F} = \mathbf{T}$.
  - If $\varphi \in \Delta^*$, then by rule $({\sim}\wedge R)$, we have either ${\sim}B \in \Delta^*$ or ${\sim}C \in \Delta^*$. By inductive assumption, this yields either $v(B) \neq \mathbf{T}$ or $v(C) \neq \mathbf{T}$. Thus, by the truth table, $v(B \wedge C) \neq \mathbf{T}$, whence $v(\varphi) = {\sim}\mathbf{T} = \mathbf{F}$.
- $A = B \vee C$. We again have two cases:
  - If $\varphi \in \Gamma^*$, then by rule $({\sim}\vee L)$, we have ${\sim}B, {\sim}C \in \Gamma^*$ since Σ^* is saturated. Hence, by inductive assumption, $v(B) \neq \mathbf{T}$ and $v(C) \neq \mathbf{T}$ (because $v({\sim}B) \neq \mathbf{F}$ and $v({\sim}C) \neq \mathbf{F}$). Thus, $v(B \vee C) \neq \mathbf{T}$, and $v(\varphi) = {\sim}\mathbf{F} = \mathbf{T}$.
  - If $\varphi \in \Delta^*$, then by rule $({\sim}\vee R)$ we have either ${\sim}B \in \Delta^*$ or ${\sim}C \in \Delta^*$. By inductive assumption, this yields either $v(B) \neq \mathbf{F}$ or $v(C) \neq \mathbf{F}$. Thus, $v(B \vee C) \neq \mathbf{F}$, whence $v(\varphi) \neq \mathbf{T} = \mathbf{F}$.

It remains to consider implication. Let $\varphi = A \rightarrow_w B$. We have the following two cases:

- $\varphi \in \Gamma^*$. Then, as Σ^* is saturated, by rule $(\rightarrow_w L)$, we have either $A \in \Delta^*$ or $B \in \Gamma^*$. In view of (1) and (3), and the fact that $\varphi \notin \Delta^*$, this yields either $v(A) \in$

$\{\mathbf{F}, \mathbf{N}\}$ or $v(B) \in \{\mathbf{T}, \mathbf{B}\}$. Thus $v(A \to_w B) \neq \mathbf{F}$, and $v(\varphi) = \mathbf{T}$.

- $\varphi \in \Delta^*$. Then, as Σ^* is saturated, by rules $(\to_w R)$ we have $A \in \Gamma^*$ and $B \in \Delta^*$. In view of (1) and (3), and the fact that $\varphi \notin \Gamma^*$, this yields $v(A) \in \{\mathbf{T}, \mathbf{B}\}$ and $v(B) \in \{\mathbf{F}, \mathbf{N}\}$, thus, $v(A \to_w B) \neq \mathbf{T}$, and $v(\varphi) = \mathbf{F}$.

Thus, (5) holds, and $\models_{GC4^+} \Sigma$, which ends the completeness proof. ■

GC4$^+$ may be one candidate for the extended version of decision logic that is needed to handle uncertain information and be tolerant to inconsistency.

VIII. Conclusion and Future Work

In this paper, we propose an extension of the decision logic of rough sets to handle uncertainty, ambiguity and inconsistent states in information systems based on rough sets. We investigate some properties of information system based on rough sets and define some characteristics of a certain relationship for the interpretation of truth values. We obtain some observations for a relationship between the interpretation with four-valued truth values and the regions defined with rough sets. To handle these characteristics we have introduced partial semantics with consequence relations for the axiomatization with many-valued logics and proposed a unified formulation of the decision logic of rough sets and many-valued logics. We also extend the language of many-valued logics with weak negation to enable the deduction theorem or the rule of modus ponens. We have shown that the system GC4$^+$ is sound and complete with Belnap's four-valued semantics.

In future work, the extension of language should be investigated, e.g., an operator to handle the granularity of objects or the uncertainty of a proposition, which is related to some kind of modal operators to recognize the crispness of objects. In this paper, we introduce rules of weak negation and weak implication to extend many-valued logics to handle a deduction system more usefully. To grasp the information state represented with information in detail, another extension of language should be investigated, such as modal type operators in a paraconsistent version of Łukasiewicz logic J3 [20]. Furthermore, we need to investigate another version of decision logics based on an extended version of rough set theories, e.g., the variable precision rough set (VPRS) [21]. VPRS models are an extension of rough set theory, which enables us to treat probabilistic or inconsistent information in the framework of rough sets. By these further investigations, a much more useful version of extended decision logic is expected for practical application and actual data analytics.

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