Optimal FIR Filters for DTMF Applications

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***Abstract*—A fast and robust analytical procedure for the design of high performance digital optimal band-pass finite impulse response filters for dual-tone multi-frequency applications is introduced. The filters exhibit equiripple behavior of the frequency response. The approximating function is based on Zolotarev polynomials. The presented closed form solution provides formulas for the filter degree and for the impulse response coefficients. Several examples are presented.**

***Keywords*-FIR filter; narrow band filter; dual-tone multi-frequency; iso-extremal approximation.**

I. Introduction

There are two basic tasks in the processing of dual-tone multi-frequency (DTMF) signals, namely the detection of DTMF frequencies and the removal of the DTMF frequencies in a broad band signal. The DTMF frequencies form two groups with four frequencies each. The lower group consists of frequencies 697, 770, 852 and 941 Hz while the higher group comprises sinusoids of 1209, 1336, 1477 and 1633 Hz. For the processing of DTMF signals, the infinite impulse response (IIR) filters are usually applied because of their lower number of coefficients. The IIR filters are usually part of the famous Goertzel procedure [1]. In the removal of DTMF frequencies in a broad band signal, the IIR filters produce substantial distortions of the output signal which appear near its flat region due to the group delay variation. This behavior is especially apparent, if pulse like components are present in the signal as demonstrated in [2]. In order to minimize these distortions in the processing of DTMF signals we propose the application of finite impulse response (FIR) filters which inherit a constant group delay. In order to maximize the discrimination of the DTMF sinusoids, the selective bands of the FIR filters should be as narrow as possible. In this paper we are focused upon the design of narrow optimal equiripple (ER) band-pass (BP) FIR filters for the DTMF decoding. They are optimal in terms of the shortest possible filter length related to the frequency specification. Note that the proposed filter design is based on formulas, i.e. no numerical procedures are involved. The presented closed form solution includes the degree equation and formulas for the robust evaluation of the impulse response coefficients of the filter.

II. Terminology

We assume a general FIR filter of type I represented by its impulse response $h(k)$ with odd length of $N = 2n+1$ coefficients and with even symmetry (1). In our further considerations we use the a-vector $a(k)$, which is related to the impulse response $h(k)$

$$a(0) = h(n)\ ,\ \ a(k) = 2h(n+k) = 2h(n-k)\,,\ k = 1 \dots n\ \ . \tag{1}$$

Further, we introduce an auxiliary real variable w

$$w = \frac{1}{2}(z + z^{-1})\big|_{z=e^{j\omega T}} = \cos(\omega T) = \cos\left(2\pi \frac{f}{f_s}\right), \tag{2}$$

where f_s is the sampling frequency. The transfer function of the filter is

$$\begin{aligned} H(z) &= \sum_{k=0}^{2n} h(k)\, z^{-k} \\ &= z^{-n}\left[h(n) + 2\sum_{k=1}^{n} h(n \pm k)\,\frac{1}{2}\left(z^{k} + z^{-k}\right)\right] \\ &= z^{-n}\sum_{k=0}^{n} a(k)\, T_k(w) = z^{-n} Q(w) \end{aligned} \tag{3}$$

where $T_k(w) = \cos(k \arccos(w))$ is Chebyshev polynomial of the first kind and $Q(w)$ is the real valued zero phase transfer function which we express using the a-vector in form of the expansion of Chebyshev polynomials

$$Q(w) = \sum_{k=0}^{n} a(k)\, T_k(w)\ . \tag{4}$$

The zero phase transfer function of an ER BP FIR filter is

$$Q(w) = \frac{Z_{p,q}(w,\kappa) + 1}{y_m + 1}\ , \tag{5}$$

where $Z_{p,q}(w,\kappa)$ represents the Zolotarev polynomial [8]. For illustration, the shape of a Zolotarev polynomial is shown in Fig. 1 and the corresponding frequency response is shown in Fig. 2.

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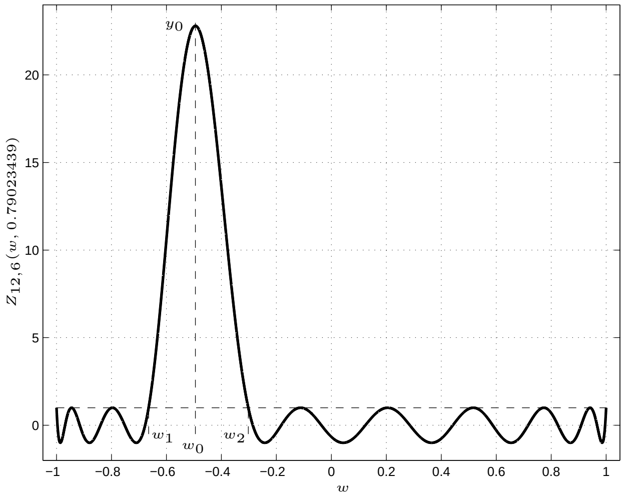

Fig. 1. Zolotarev polynomial $Z_{12,6}(w, 0.79023439)$.

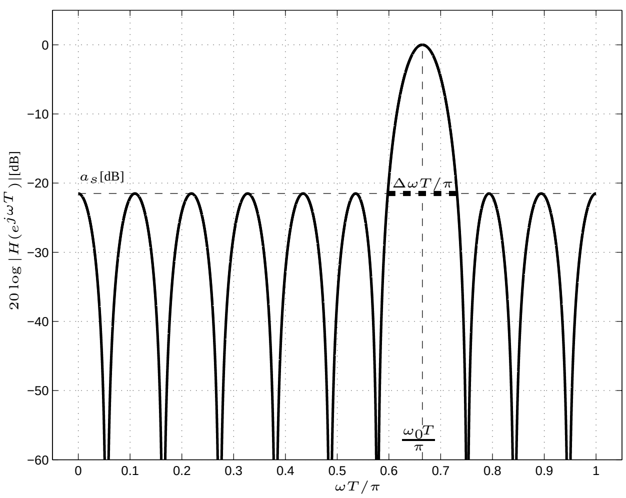

Fig. 2. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB] corresponding to the Zolotarev polynomial from Fig. 1.

III. Optimal Band-Pass FIR Filter

An optimal ER BP FIR filter (Fig. 2) is specified by the pass band frequency $\omega_0 T$ (or $f_0 = \omega_0 T f_s/2\pi$), width of the pass band $\Delta\omega T$ (or $\Delta f = \Delta\omega T f_s/2\pi$), attenuation in the stop-bands a_s[dB] and by the sampling frequency f_s. An approximation of the frequency response of a filter is based on the generating function. The generating function of an ER BP FIR filter is the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ which approximates a constant value in equiripple Chebyshev sense in two disjoint intervals $\langle -1, w_1\rangle$ and $\langle w_2, 1\rangle$ as shown in Fig. 1. The main lobe with the maximal value $y_0 = Z_{p,q}(w_0,\kappa)$ is located inside the interval (w_1, w_2). The notation $Z_{p,q}(w,\kappa)$ emphasizes the fact that the integer value p counts the number of zeros right from the maximum w_0 and the integer value q corresponds to the number of zeros left from the maximum w_0. The real value $0 \le \kappa \le 1$ is in fact the Jacobi elliptical modulus. It affects the maximum value y_0 and the width $w_2 - w_1$ of the main lobe (Fig. 1). For increasing κ the value y_0 increases and the main lobe broadens. The Zolotarev polynomial is usually expressed in terms of Jacobi elliptic functions [6]-[8]

$$Z_{p,q}(w,\kappa) = \frac{(-1)^p}{2} \times \left[\left(\frac{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))} \right)^n + \left(\frac{H(u + \frac{p}{n}\ \mathbf{K}(\kappa))}{H(u - \frac{p}{n}\ \mathbf{K}(\kappa))} \right)^n \right] . \tag{6}$$

The factor $(-1)^p/2$ appears in (6) as the Zolotarev polynomial alternates $(p+1)-$times in the interval $(w_2, 1)$. The variable u is expressed by the incomplete elliptical integral of the first kind $F(x|\kappa)$, namely

$$u = F\left(\mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right) \sqrt{\frac{1+w}{w + 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) - 1}} |\kappa \right) . \tag{7}$$

The function $H\left(u \pm (p/n)\ \mathbf{K}(\kappa)\right)$ is the Jacobi Eta function, $\mathrm{sn}(u|\kappa)$, $\mathrm{cn}(u|\kappa)$, $\mathrm{dn}(u|\kappa)$ are Jacobi elliptic functions and $\mathbf{K}(\kappa)$ is the quarter-period given by the complete elliptic integral of the first kind. The degree of the Zolotarev polynomial is $n = p+q$. A comprehensive treatise of Zolotarev polynomials was published in [8]. It includes the analytical solution of the coefficients of Zolotarev polynomials, the algebraic evaluation of the Jacobi Zeta function $\mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$ and of the elliptic integral of the third kind $\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\,\kappa)$. The position w_0 of the maximum value $y_0 = Z_{p,q}(w_0,\kappa)$ is

$$w_0 = w_1 + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right)} \mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) \tag{8}$$

where the edges of the main lobe are

$$w_1 = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa)|\kappa\right) \tag{9}$$

$$w_2 = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa)|\kappa\right) - 1\ . \tag{10}$$

The relation for the maximum value y_0

$$y_0 = \cosh 2n\left(\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - \Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)\right) \tag{11}$$

is useful in the normalization of Zolotarev polynomials. The degree of the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ is expressed by the degree formula

$$n \ge \frac{\ln(y_0 + \sqrt{y_0^2 - 1})}{2\sigma_m \mathrm{Z}(\frac{p}{n}\mathbf{K}(\kappa)|\kappa) - 2\Pi(\sigma_m, \frac{p}{n}\mathbf{K}(\kappa)|\kappa)}\ . \tag{12}$$

The auxiliary value σ_m in (11), (12) is given by the formula

$$\sigma_m = F\left(\arcsin\left(\frac{1}{\kappa\ \mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)}\sqrt{\frac{w_0 - w_s}{w_0 + 1}}\right)|\kappa\right) \tag{13}$$

$$= F\left(\arcsin\frac{\sqrt{\cos\left(2\pi\frac{f_0}{f_s}\right) - \cos\left[\frac{2\pi}{f_s}\left(f_0 + \frac{\Delta f}{2}\right)\right]}}{\kappa\ \mathrm{sn}\left(\frac{p}{n}\ \mathbf{K}(\kappa)|\kappa\right)\sqrt{1 + \cos\left(2\pi\frac{f_0}{f_s}\right)}}|\kappa\right) .$$

The Zolotarev polynomial $Z_{p,q}(w,\kappa)$ satisfies the differential equation

$$(1-w^2)(w-w_1)(w-w_2)\left(\frac{dZ_{p,q}(w,\kappa)}{dw}\right)^2 = n^2\left(1-Z_{p,q}^2(w,\kappa)\right)(w-w_0)^2 \ . \tag{14}$$

Based on the differential equation (14) we have developed an algorithm for the evaluation of the a-vector and of the impulse response $h(k)$ corresponding to the Zolotarev polynomial $Z_{p,q}(w,\kappa)$ in form of its expansion into Chebyshev polynomials (4). This algorithm is summarized in Table I. There are two goals in the filter design. The first one is to obtain the minimal filter length of N coefficients satisfying the filter specification. The second one is to evaluate its impulse response $h(k)$. In the standard design of an ER BP FIR filter, which is represented by the numerical Parks-McClellan procedure (e.g. the function *firpm* in Matlab), the exact filter length is not available because of no approximating function. Consequently, the filter length is not the result of the design, it is in fact an input argument in the Parks-McClellan procedure. The filter length is either estimated or successively adjusted in order to meet the filter specification, or it is obtained from empirical approximating formulas. Moreover, a successful design is not guaranteed in the Parks-McClellan procedure. The alternative design approach that we use here is based on the analytical design which we have developed for ER notch FIR filters and introduced in [3]. The available pass band frequencies which we denote f_Q are quantized because there is always an integer number of ripples (Fig. 2). That is why we additionally tune the actual pass band frequency f_Q of the initial filter to the specified value f_0. This tuning consists in multiplying the a-vector of the initial filter by a transformation matrix, resulting in the a-vector of the tuned filter (16) which exactly meets the specified pass band frequency. For the tuning, we present an efficient algebraic procedure which is a simplified version of that one introduced in [4]. The tuning procedure results in the zero phase transfer function of the tuned filter, which is

$$Q_t(w) = \sum_{k=0}^{n} a(k)\,T_k(\lambda w \pm \lambda') = \sum_{k=0}^{n} a(k) \sum_{m=0}^{k} \alpha_k(m) T_m(w). \tag{15}$$

Based on (15), the a-vector $a_t(k)$ of the tuned filter and the a-vector $a(k)$ of the initial filter are related by a triangular transformation matrix A

$$a_t(k) = [a_t(0)\ \ a_t(1)\ \ \cdots\ \ a_t(n)] = [a(0)\ \ a(1)\ \ \cdots\ \ a(n)]\times$$

$$\times\begin{bmatrix} \alpha_0(0) & 0 & 0 & 0 & \cdots & 0 \\ \alpha_1(0) & \alpha_1(1) & 0 & 0 & \cdots & 0 \\ \alpha_2(0) & \alpha_2(1) & \alpha_2(2) & 0 & \cdots & 0 \\ \alpha_3(0) & \alpha_3(1) & \alpha_3(2) & \alpha_3(3) & \cdots & 0 \\ \vdots & & & & & \vdots \\ \alpha_n(0) & \alpha_n(1) & \alpha_n(2) & \alpha_n(3) & \cdots & \alpha_n(n) \end{bmatrix} = a(k)A\,. \tag{16}$$

A fast algorithm for the evaluation of the coefficients $\alpha_k(m)$ is summarized in Tab. II. The presented tuning of the pass band frequency preserves the attenuation a_s[dB] in the stop bands.

IV. Design of the Optimal Band-Pass FIR Filter

Let us specify the optimal ER BP FIR filter by the pass band frequency f_0[Hz], width of the pass band Δf[Hz], sampling frequency f_s[Hz] and by the attenuation in the stop bands a_s[dB]. The design procedure reads as follows:
Calculate the Jacobi elliptic modulus κ

$$\kappa = \sqrt{1 - \frac{1}{\tan^2(\varphi_1)\tan^2(\varphi_2)}} \tag{17}$$

for the auxiliary values φ_1 and φ_2

$$\varphi_1 = \frac{\pi}{f_s}\left(f_0 + \frac{\Delta f}{2}\right) \ , \ \varphi_2 = \frac{\pi}{f_s}\left(\frac{f_s}{2} - f_0 + \frac{\Delta f}{2}\right). \tag{18}$$

Calculate the rational values $\frac{p}{n}$ and $\frac{q}{n}$

$$\frac{p}{n} = \frac{F(\varphi_1,\kappa)}{\mathbf{K}(\kappa)} \ , \ \frac{q}{n} = \frac{F(\varphi_2,\kappa)}{\mathbf{K}(\kappa)} \ , \tag{19}$$

where $\mathbf{K}(\kappa)$ is a complete elliptic integral of the first kind, which here represents the elliptic quarter-period. Determine the value y_0

$$y_0 = \frac{2}{10^{0.05 a_s[dB]}} \ . \tag{20}$$

Calculate the auxiliary value σ_m (13).
Calculate and round up the value n (12) which represents the degree $n = p + q$ of the Zolotarev polynomial $Z_{p,q}(w,\kappa)$.
Calculate the integer indices p and q of the Zolotarev polynomial $Z_{p,q}(w,\kappa)$

$$p = \left\updownarrow n\frac{F(\varphi_1,\kappa)}{\mathbf{K}(\kappa)} \right\updownarrow \ , \ q = \left\updownarrow n\frac{F(\varphi_2,\kappa)}{\mathbf{K}(\kappa)} \right\updownarrow \ . \tag{21}$$

The arrow brackets in (21) stand for rounding. For values p, q (21), κ (17) and y_0 (20) evaluate the a-vector $a(k)$ and the related impulse response $h(k)$ of the filter using the algebraical procedure summarized in Tab. I. Check the actual pass band frequency f_Q of the initial BP FIR filter

$$f_Q = \frac{f_s}{2\pi}\arccos\left[1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa),\kappa\right) + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa),\kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa),\kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa),\kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa),\kappa\right)\right]. \tag{22}$$

Because of the inherent quantization of the available pass band frequencies f_Q mentioned above, the actual pass band frequency f_Q usually slightly differs from the specified pass band frequency f_0. That is why we tune the quantized pass band frequency f_Q of the initial filter to the specified value f_0 using (16) and Tab. II. In our calculations, the Jacobi elliptic Zeta function $\mathrm{Z}(x,\kappa)$ in (8), (11), (12) and the incomplete elliptic integral of the first kind $F(x,\kappa)$ in (13) are evaluated

given	$p,\ q,\ \kappa,\ y_0$
initialization	$n = p + q\ ,\ w_1 = 1 - 2\,\mathrm{sn}^2\left(\frac{p}{n}\mathbf{K}(\kappa), \kappa\right)\ ,\ w_2 = 2\,\mathrm{sn}^2\left(\frac{q}{n}\mathbf{K}(\kappa), \kappa\right) - 1\ ,\ w_a = \frac{w_1 + w_2}{2}$
	$w_m = w_1 + 2\,\frac{\mathrm{sn}\left(\frac{p}{n}\mathbf{K}(\kappa), \kappa\right)\mathrm{cn}\left(\frac{p}{n}\mathbf{K}(\kappa), \kappa\right)}{\mathrm{dn}\left(\frac{p}{n}\mathbf{K}(\kappa), \kappa\right)}\mathrm{Z}\left(\frac{p}{n}\mathbf{K}(\kappa), \kappa\right)$
	$\alpha(n) = 1\ ,\ \alpha(n+1) = \alpha(n+2) = \alpha(n+3) = \alpha(n+4) = \alpha(n+5) = 0$
body	
(*for*	$m = n + 2$ *to* 3)
	$8c(1) = n^2 - (m+3)^2\ ,\ 4c(2) = (2m+5)(m+2)(w_m - w_a) + 3w_m[n^2 - (m+2)^2]$
	$2c(3) = \frac{3}{4}[n^2 - (m+1)^2] + 3w_m[n^2 w_m - (m+1)^2 w_a] - (m+1)(m+2)(w_1 w_2 - w_m w_a)$
	$c(4) = \frac{3}{2}(n^2 - m^2) + m^2(w_m - w_a) + w_m(n^2 w_m^2 - m^2 w_1 w_2)$
	$2c(5) = \frac{3}{4}[n^2 - (m-1)^2] + 3w_m[n^2 w_m - (m-1)^2 w_a] - (m-1)(m-2)(w_1 w_2 - w_m w_a)$
	$4c(6) = (2m-5)(m-2)(w_m - w_a) + 3w_m[n^2 - (m-2)^2]\ ,\ 8c(7) = n^2 - (m-3)^2$
	$\alpha(m-3) = \frac{1}{c(7)}\sum_{\mu=1}^{6} c(\mu)\alpha(m+4-\mu)$
(*end*	*loop on* m)
normalization	$s(n) = \frac{\alpha(0)}{2} + \sum_{m=1}^{n}\alpha(m)$
a-vector	$a(0) = (-1)^p\frac{\alpha(0)}{2s(n)}$, (*for* $m = 1$ *to* n) , $a(m) = (-1)^p\frac{\alpha(m)}{s(n)}$, (*end loop on* m)
impulse response	$h(n) = \frac{a(0)+1}{y_0+1}$, (*for* $m = 1$ *to* n) , $h(n \pm m) = \frac{a(m)}{2(y_0+1)}$, (*end loop on* m)

TABLE I

FAST ALGORITHM FOR EVALUATING THE a-VECTOR $a(k)$ AND THE IMPULSE RESPONSE $h(k)$.

given	$f_Q,\ \ f_0,\ \ f_s,\ \ k$
initialization	if $f_Q < f_0\ :\ \lambda = \frac{\cos\left(2\pi\frac{f_Q}{f_s}\right) - 1}{\cos\left(2\pi\frac{f_0}{f_s}\right) - 1},\ s = 1,$ else : $\lambda = \frac{\cos\left(2\pi\frac{f_Q}{f_s}\right) + 1}{\cos\left(2\pi\frac{f_0}{f_s}\right) + 1},\ s = -1$
	$\lambda' = 1 - \lambda\ \ ,\ \ \alpha_k(k+1) = \alpha_k(k+2) = \alpha_k(k+3) = 0\ \ ,\ \ \alpha_k(k) = \lambda^k$
body	
(*for* $\mu = -3\ ...\ k-4$)	
	$\alpha_k(k-\mu-4) =$
	$\{$
	$-2s\left[(\mu+3)(2k-\mu-3) - \frac{\lambda'}{\lambda}(k-\mu-3)(2k-2\mu-7)\right]\alpha_k(k-\mu-3)$
	$+2\frac{\lambda'}{\lambda}(k-\mu-2)\ \alpha_k(k-\mu-2)$
	$+2s\left[(\mu+1)(2k-\mu-1) - \frac{\lambda'}{\lambda}(k-\mu-1)(2k-2\mu-1)\right]\alpha_k(k-\mu-1)$
	$+\mu(2k-\mu)\ \alpha_k(k-\mu)$
	$\}\ /\ (\mu+4)(2k-\mu-4)$
(*end loop on* μ)	

TABLE II

FAST ALGORITHM FOR EVALUATING THE COEFFICIENTS $\alpha_k(m)$ OF TRANSFORMATION MATRIX A.

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by the arithmetic-geometric mean [7]. The Jacobi elliptic integral of the third kind $\Pi(x, y, \kappa)$ in (12) is evaluated by a fast procedure proposed in [3].

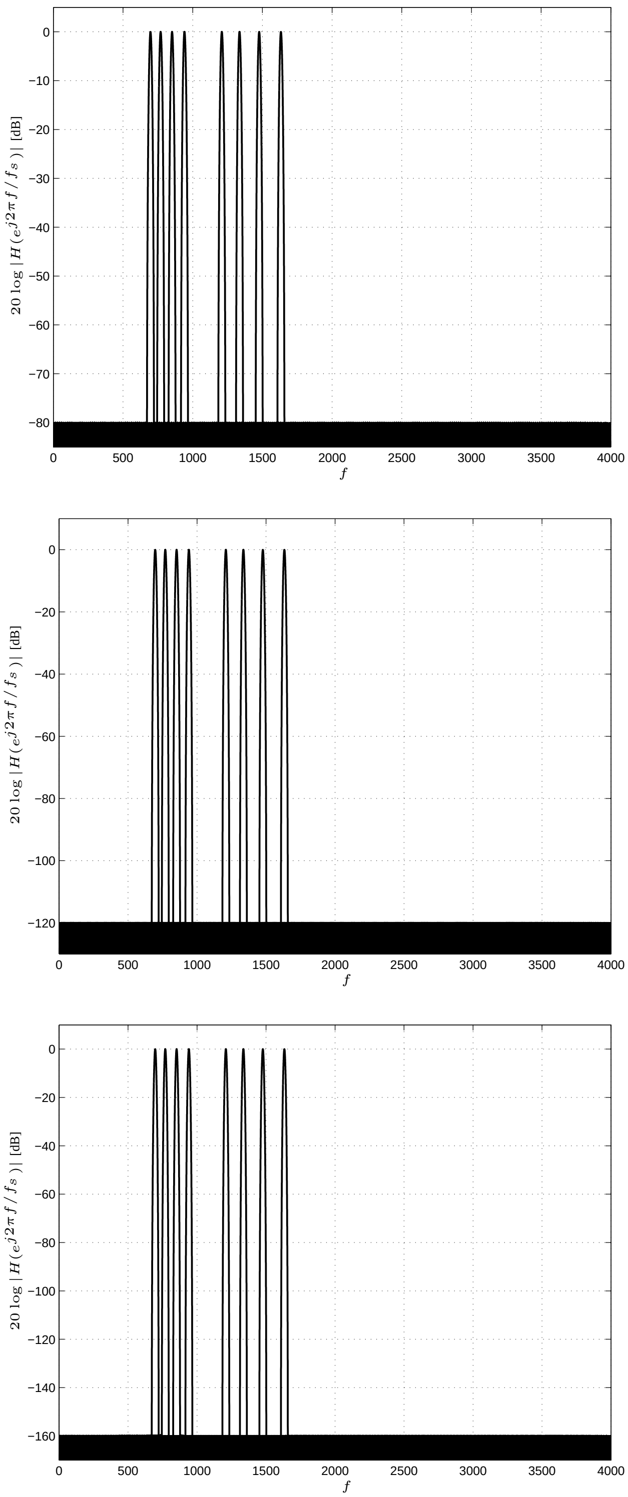

Fig. 3. Amplitude frequency responses $20 \log |H(e^{j2\pi f/f_s})|$ [dB].

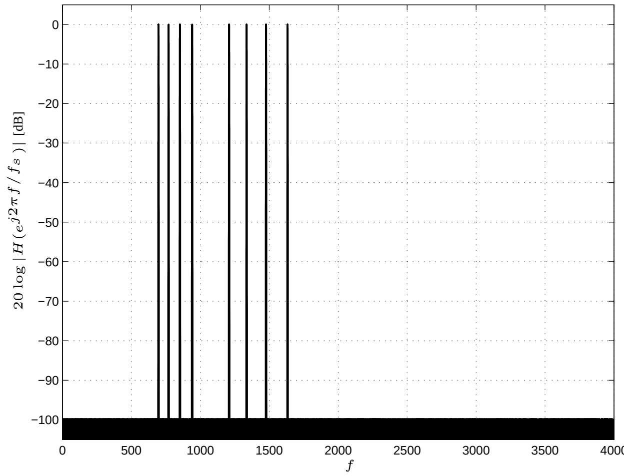

Fig. 4. Amplitude frequency responses $20 \log |H(e^{j2\pi f/f_s})|$ [dB] of the filters with $N = 13141$ coefficients.

V. Examples of Design

Let us design three sets of band pass FIR filters specified by the DTMF frequencies $f_0 = 697, 770, 852, 941, 1209, 1336, 1477, 1633$ Hz, width of the pass bands $\Delta f = 50$Hz, sampling frequency $f_s = 8000$Hz and with the attenuations in the stop bands $a_s = -80$dB, -120dB and -160dB.
Using the presented design procedure, we get filter lengths $N = 1083$ coefficients for $a_s = -80$dB, 1551 coefficients for $a_s = -120$dB and 2019 coefficients for $a_s = -160$dB. The corresponding amplitude frequency responses are shown in Fig. 3. In order to demonstrate the remarkable selectivity of the ER BP FIR filters and the robustness of the presented design procedure, let us design the DTMF filters with very narrow pass band of $\Delta f = 5$Hz, sampling frequency $f_s = 8000$Hz and with the attenuation in the stop bands $a_s = -100$dB. The filter length is $N = 13141$ coefficients. The amplitude frequency responses are shown in Fig. 4.

VI. Conclusion and Future Work

We have presented a fast and robust procedure for the design of optimal equiripple narrow band-pass FIR filters for DTMF applications. In contrast to the established numerical design procedure the proposed methodology solves the approximation problem and provides a formula for the degree of the filter and formulas for the evaluation of the coefficients of the impulse response of the filter. Our future activity will include an efficient implementation of the DTMF filters using digital signal processors.

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