Consensus Problem in Stochastic Network Systems with Switched Topology, Noise and Delay

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***Abstract*—This paper deals with the problem of achieving consensus in decentralized stochastic network with switched topology and noise and delays in measurements. To solve the consensus problem of the group of interacting agents it was supposed to use the stochastic approximation type algorithm with the step-size non-decreasing to zero. Simulation results show the quality of the algorithm.**

***Index Terms*—consensus problem; stochastic networks; discrete systems; network systems.**

I. INTRODUCTION

The problems of control and distributed interaction in dynamical networks attracted more and more attention during last decade. A number of survey papers [1], [2], monographs [3], [4], [5], special issues of journals [6], [7], [8] and edited volumes [9], [10] are published. An interest is driven by applications to multiprocessor networks, transportation networks, production networks, coordinated control of motion of flying vehicles, submarines and mobile robots, distributed systems of control of power networks, complex crystal lattices, and nanostructured objects. In the presence of stochastic disturbances and noise, the stochastic gradient-like (stochastic approximation) methods have been used [11], [12], [13], [14], [15], [16].

Despite of large number of publications, satisfactory solutions are obtained only for a restricted class of problems by now. Such factors as nonlinearity of agent dynamics switching topology, noisy and delayed measurements may significantly complicate the solution. Additional important factors are limited transmission rate in the channels and quantizing (discretization) phenomenon. In presence of various disturbing factors, asymptotically exact consensus may be hard to achieve, especially in time-varying environment [17]. It those cases, approximate consensus problems should be examined. In [18], the approximate consensus problem in multi-agent stochastic systems with noisy information about the current state of the nodes and randomly switched topology for agents with nonlinear dynamics is considered.

This work is an extension of [18] to the case of multi-agent systems with delays in measurements. Following [18], we adopt an approach to analysis of stochastic multi-agent systems based on using the averaged models of system dynamics: the so-called method of averaged (discrete or continuous) models [19], [20], [21], [22], [23], [24].

The paper is organized as follows. In Section II, the basic concepts are introduced and an approximate mean-square consensus problem is posed. In Section III, the basic assumptions are described. The main results are presented in Section IV. In Section V, an example of computer network is given and the simulation results are provided. Section VI presents the conclusion.

II. PRELIMINARIES: CONSENSUS PROBLEM ON GRAPHS

Consider a dynamic network of a set of agents (nodes) $N = \{1,2,\dots,n\}$.

Graph (N,E) is defined by N and set edges E. Define the *set of neighbors* of node i as $N^i = \{j : (j,i) \in E\}$, i.e. the set of nodes with edges incoming to i. Associate with each edge $(j,i) \in E$ a weight $a^{i,j} > 0$ and denote *adjacency matrix (or connectivity matrix)* $A = [a^{i,j}]$ of graph, denoted hereinafter as $\mathscr{G}_A$ (hereinafter the index of variables shows the corresponding number of nodes). Define the *weighted in-degree* of node i as the i-th row sum of A: $d^i(A) = \sum_{j=1}^n a^{i,j}$.

Endow each node $i \in N$ at time $t = 0,1,2\dots,T$ with a time-varying state $x_t^i \in \mathbb{R}$ with dynamics

$$x_{t+1}^i = x_t^i + f^i(x_t^i, u_t^i), \tag{1}$$

where $f^i(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are some functions that depend on the states in the previous time x_t^i and on control actions $u_t^i \in \mathbb{R}$.

We consider the network (multi-agent) system consisting of dynamic agents with inputs u_t^i, outputs $y_t^{i,i}$ and states x_t^i.

Nodes i and j *agree* in a network at time t if and only if $x_t^i = x_t^j$.

The consensus problem is the agreement of all nodes in network, i.e., we have to find a control protocol that drives all states to the same constant steady-state values: $x_t^i = x_t^j \quad \forall i,j \in N, i \neq j$.

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We assume that the structure of links of the dynamic network is modeled by a sequence of directed graphs $\{(N, E_t)\}_{t\geq 0}$, where $E_t \subset E$ change in time. If $(j,i) \in E_t$, then we say that node i at time t obtains information from the node j for the purposes of feedback control. Denote A_t as adjacency matrix corresponding to E_t; $E_{\max} = \{(j,i) : \sup_{t\geq 0} a_t^{i,j} > 0\}$ is the maximum set communication links.

To form its control strategy each node uses its own state (possibly noisy)

$$y_t^{i,i} = x_t^i + w_t^{i,i}, \tag{2}$$

and if $N_t^i \neq \emptyset$, noisy measurements of its neighbors states

$$y_t^{i,j} = x^j_{t-d_t^{i,j}} + w_t^{i,j},\ j \in N_t^i, \tag{3}$$

where $w_t^{i,i}, w_t^{i,j}$ is the noise, $0 \leq d_t^{i,j} \leq \bar{d}$ is integer-valued delay, $\bar{d}$ is a maximal delay.

Since the system starts working at $t = 0$ so implicit requirement to set of neighbors would be: $j \in N_t^i \Rightarrow t - d_t^{i,j} \geq 0$. We put $w_t^{i,j} = 0$ for all other pairs of i, j and denote $\bar{w}_t \in \mathbb{R}^{n^2}$ as a vector (matrix $n \times n$ which is written in rows as a vector) consisting of elements $w_t^{i,j}$, $i, j \in N$.

The control algorithm (protocol), called the *local voting protocol*, is given by

$$u_t^i = \alpha_t \sum_{j\in \bar{N}_t^i} b_t^{i,j}(y_t^{i,j} - y_t^{i,i}), \tag{4}$$

where $\alpha_t > 0$ are step-sizes of control protocol, $b_t^{i,j} > 0\ \forall j \in \bar{N}_t^i$. We set $b_t^{i,j} = 0$ for other pairs i, j and denote $B_t = [b_t^{i,j}]$ as the matrix of control protocol.

For the vector or matrix M denote the Frobenius norm: $||M|| = [Tr(M^{\mathrm{T}}M)]^{1/2}$, where $Tr(\cdot)$ is a trace (sum of the diagonal elements) of matrix. In some cases for matrix A the vector norm (square root of the sum of the squares of all its elements) will be used, which we denote as $||A||_2$.

The n nodes to achieve *asymptotic mean square consensus* if $\mathrm{E}||x_t^i||^2 < \infty, t = 0, 1, \ldots,\ i \in N$ and there exists a random variable $x^\star$ such that $\overline{\lim}_{t\to\infty} \mathrm{E}||x_t^i - x^\star||^2 = 0$ for $i \in N$.

The n nodes to achieve *ε-consensus* if $\mathrm{E}||x_t^i||^2 < \infty,\ i \in N$, and there exists a random variable $x^\star$ such that $\mathrm{E}||x_t^i - x^\star||^2 \leq \varepsilon$ for all $i \in N$.

III. MAIN ASSUMPTIONS

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space. Let E be symbol of mathematical expectation and E_x be conditional expectation under the condition x.

In the formulation of further results, we assume that the following conditions are satisfied.

A1. $\forall i \in N$ functions $f^i(x,u)$ are Lipschitz in x and u: $|f^i(x,u) - f^i(x',u')| \leq L_1(L_x|x - x'| + |u - u'|)$, the growth rate is bounded: $|f^i(x,u)|^2 \leq L_2(L_c + L_x|x|^2 + |u|^2)$, and for any fixed x the function $f^i(x,\cdot)$ is such that $\mathrm{E}_x f^i(x,u) = f^i(x, \mathrm{E}_x u)$;

Remark. A typical case when this condition holds is the case when $f^i(x,u)$ is linear in control.

A2. **a)** $\forall i \in N, j \in N^i$ the noises $w_t^{i,j}$ are centered, independent and have bounded variance: $E(w_t^{i,j})^2 \leq \sigma_w^2$.

b) $\forall i \in N, j \in N^i$ the appearances of variable edges (j,i) in the graph $\mathscr{G}_{A_t}$ are independent random events with probability $p_a^{i,j}$ (i.e., matrices A_t are independent, identically distributed random matrices).

c) $\forall i \in N, j \in N^i$ weights $b_t^{i,j}$ in the control protocol are bounded random variables: $\underline{b} \leq b_t^{i,j} \leq \bar{b}$ with probability 1, and there exist limits $b^{i,j} = \lim_{t\to\infty} \mathrm{E} b_t^{i,j}$.

d) $\forall i \in N, j \in N^i$ there exists a finite quantity $\bar{d} \in \mathbb{N}$: $d_t^{i,j} \leq \bar{d}$ with probability 1 and integer-valued delay $d_t^{i,j}$ — independent, identically distributed random variables taking values $k = 0, \ldots, \bar{d}$ with probability $p_k^{i,j}$.

Moreover, all of these random variables and matrices are independent of each other and their components have a limited variance.

If $\bar{d} > 0$ we add new nodes to the current network topology $n\bar{d}$. We add new "fictitious" agents with states at time t equal to the corresponding states of the "real" agents at the previous $\bar{d}$ time: $t-1, t-2, \ldots, t-\bar{d}$.

Denote $\bar{n} = n(\bar{d}+1)$. Matrix $A_{\max}$ of size $\bar{n} \times \bar{n}$ is denoted as:

$$a_{\max}^{i,j} = p_{j\div\bar{d}}^{i,j \bmod \bar{d}} p_a^{i,j \bmod \bar{d}} b^{i,j \bmod \bar{d}},\ i \in N,\ j = 1, 2, \ldots, \bar{n},$$

$$a_{\max}^{i,j} = 0,\ i = n+1, n+2, \ldots, \bar{n},\ j = 1, 2, \ldots, \bar{n}.$$

Here, the operation mod is a remainder of the division, and $\div$ is division without a remainder.

Note that if $\bar{d} = 0$ so this definition of network topology (of matrix $A_{\max}$ of size $n \times n$) is as follows

$$a_{\max}^{i,j} = p_a^{i,j} b^{i,j},\ i \in N,\ j \in N.$$

If we consider the sequence of random matrices $\bar{A}_t$ with elements that define the connections at time t, then all of them are identically distributed and the matrix $A_{\max}$ is in fact its expectation (averaging).

We assume that the following condition is satisfied for the network topology matrix:

A3. Graph $(N, E_{\max})$ has a spanning tree, and for any edge $(j,i) \in E_{\max}$ among the elements $a_{\max}^{i,j}, a_{\max}^{i,j+n}, \ldots, a_{\max}^{i,j+\bar{d}n}$ of the matrix $A_{\max}$ there exists at least one non-zero.

For $t = 1, 2, \ldots$ we define an increasing sequence of σ-algebras of probability of events $\tilde{\mathscr{F}}_t$, generated by random elements $A_1, \ldots, A_{t-1}$; $d_1^{i,j}, \ldots, d_{t-1}^{i,j}$, $b_1^{i,j}, \ldots, b_{t-1}^{i,j}, w_1^{i,j}, \ldots, w_t^{i,j}$, $i, j \in N$, and $\mathscr{F}_t = \sigma\{\mathscr{F}_t^A, A_t; b_t^{i,j}, d_t^{i,j}, i, j \in N\}$.

For a random variable Q and σ-algebra of probability event $\mathscr{F}$ we use the notation $\mathrm{E}_{\mathscr{F}} Q$ for the conditional expectation Q with respect to σ-algebra $\mathscr{F}$.

Note that the random variables $\bar{x}_t$ are measurable with respect σ-algebra $\mathscr{F}_{t-1}$, i.e. $\mathrm{E}_{\mathscr{F}_{t-1}} \bar{x}_t = \bar{x}_t$.

IV. ANALYSIS OF THE CLOSED LOOP SYSTEM DYNAMICS

Denote $\bar{x}_t = [x_t^1; \ldots; x_t^n]$. Let $\bar{x}_t \equiv 0$ for $-\bar{d} \leq t < 0$, and denote $\bar{X}_t \in \mathbb{R}^{n\bar{d}}$ as extended state vector $\bar{X}_t = [\bar{x}_t, \tilde{x}_{t-1}, \ldots, \tilde{x}_{t-\bar{d}}]$, where $\tilde{x}_{t-k}$ is vector consisting of such x_{t-k}^i that $\exists j \in N^i\ \exists k' \geq k : p_{k'}^{i,j} > 0$, i.e. this value with positive probability involved in

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the formation of at least one of the controls. To simplicity, we assume that so introduced an extended state vector is $\bar{X}_t = [\bar{x}_t, \bar{x}_{t-1}, \ldots, \bar{x}_{t-\bar{d}}]$, i.e. it includes all the components with all kinds of delays not exceeding $\bar{d}$.

Rewrite the dynamics of the nodes in vector-matrix form:

$$\bar{X}_{t+1} = U\bar{X}_t + F(\alpha_t, \bar{X}_t, \bar{w}_t), \tag{5}$$

where U is the following matrix of size $\bar{n} \times \bar{n}$:

$$U = \begin{pmatrix} I & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix}, \tag{6}$$

where I is the identity matrix of size $n \times n$, and $F(\alpha_t, \bar{X}_t, \bar{w}_t) : \mathbb{R} \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{n^2} \to \mathbb{R}^{\bar{n}}$ — vector function of the arguments:

$$F(\alpha_t, \bar{X}_t, \bar{w}_t) =$$

$$= \begin{pmatrix} \cdots \\ f^i(x_t^i, \alpha_t \sum_{j \in \bar{N}_t^i} b_t^{i,j}((x^j_{t-d_t^{i,j}} - x_t^i) + (w_t^{i,j} - w_t^{i,i}))) \\ \cdots \\ 0_{n\bar{d}} \end{pmatrix}, \tag{7}$$

containing a non-zero components only on the first n places.

Consider the corresponding (5) averaged discrete model

$$\bar{Z}_{t+1} = U\bar{Z}_t + G(\alpha_t, \bar{Z}_t),\ \bar{Z}_0 = \bar{X}_0, \tag{8}$$

where

$$G(\alpha, \bar{Z}) = G\left(\alpha, \begin{matrix} z^1 \\ \vdots \\ z^{n(\bar{d}+1)} \end{matrix}\right) = \begin{pmatrix} \cdots \\ f^i(z^i, \alpha s^i(\bar{Z})) \\ \cdots \\ 0_{n\bar{d}} \end{pmatrix}, \tag{9}$$

$$s^i(\bar{Z}) = \sum_{j \in N^i} p_a^{i,j} b^{i,j}((\sum_{k=0}^{\bar{d}} p_k^{i,j} z^{j+kn}) - z^i) =$$

$$= -d^i(A_{\max}) z^i + \sum_{j=1}^{\bar{n}} a_{\max}^{i,j} z^j,\ i \in N.$$

It turns out that the trajectory of solutions of the initial system $\{\bar{X}_t\}$ from (5) at time t are close in mean square sense to the average trajectory of the discrete system (8).

Theorem *1:* If conditions **A1**, **A2** are satisfied, **then** there exists $\tilde{\alpha}$ such that for $0 < \alpha_t \leq \bar{\alpha} < \tilde{\alpha}$ the following inequality holds:

$$\mathrm{E} \max_{0 \leq t \leq T} ||\bar{X}_t - \bar{Z}_t||^2 \leq c_1 \tau_T e^{c_2 \tau_T^2} \bar{\alpha}, \tag{10}$$

where $\tau_T = 2^{\bar{d}}(\alpha_0 + \alpha_1 + \ldots + \alpha_{T-1})$, $c_1, c_2 > 0$ are some constants:

$$c_1 = 8n\left(\tilde{c} + \hat{c}(\frac{nL_2L_c + \bar{\alpha}^2\tilde{c}}{c_3} + ||\bar{X}_0||^2)e^{T\ln(c_3+1)}\right),$$

$$\tilde{c} = n^2 L_1^2 \bar{b}^2 \sigma_w^2,\ c_2 = 2^{1-\tilde{d}} L_1^2 (\frac{L_x}{\underline{\alpha}} + 2\bar{\alpha}^2 ||\mathscr{L}(A_{\max})||_2^2),$$

$$c_3 = \tilde{d} + L_x(2^{1+\tilde{d}/2}L_1 + L_2) + \bar{\alpha}c',\ \hat{c} = 2L_1^2 n(n-1)\bar{b}^2,$$

$$c' = 2^{1+\tilde{d}/2} L_1 ||\mathscr{L}(A_{\max})||_2 + \bar{\alpha}(L_2||\mathscr{L}(A_{\max})||_2^2 + \hat{c}),$$

$\underline{\alpha} = \min_{1 \leq t \leq T} \alpha_t$, $\tilde{d} = 0$ if $\bar{d} = 0$, or $\tilde{d} = 1$ if $\bar{d} > 0$.

Note that in case without delays in the measurement ($\tilde{d} = 0$) and if $L_x = 0$ then constant c_3 which is defined in Theorem 1 is estimated by the value proportional to $\bar{\alpha}$ and therefore constant c_1 is estimated by the value proportional to τ_T, which corresponds to the previously obtained results for this case from [21], [19].

Proof:

Denote

$$v_t = F(\alpha_t, \bar{X}_t, \bar{w}_t) - G(\alpha_t, \bar{X}_t). \tag{11}$$

By condition **A2** averaging with respect to σ-algebras $\mathscr{F}_t^d$ and $\mathscr{F}_t$ yields $\mathrm{E}_{\mathscr{F}_t} v_t = 0$.

To proof Theorem 1, the following facts will be useful.

Proposition *1:*

$$||U\bar{X}||^2 \leq 2^{\bar{d}}||\bar{X}||^2,\ \ldots,\ ||U^{\bar{d}}\bar{X}||^2 \leq 2^{\bar{d}}||\bar{X}||^2,\ \ldots,\ ||U^k\bar{X}||^2 \leq$$
$$\leq 2^{\bar{d}}||\bar{X}||^2,$$

Proof: By the definition of matrix U it is easy to obtain the first inequality, and the rest we get by induction on k and by the following equality

$$\forall k > \bar{d}\ U^k = U^{\bar{d}} = \begin{pmatrix} I & 0 & 0 & \ldots & 0 \\ I & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{12}$$

■

Proposition *2:* By assumptions **A2** the following inequality holds

$$\mathrm{E} \max_{1 \leq t \leq T} ||\sum_{i=1}^t v_t||^2 \leq 4n \sum_{t=1}^T \mathrm{E}||v_t||^2.$$

Proof:

Under the conditions **A2** random elements v_t are martingale differences, i.e., they are centered with respect to the conditional averaging of the background: $\mathrm{E}_{\mathscr{F}_{t-1}} v_t = 0$. So, Lemma 1 from section 3 of [25] is applicable. The dimension of vectors v_t is $n\bar{d}$, but since only the first n components of vectors v_t are nonzero, then it is possible to use in the estimation the value of n instead of $n\bar{d}$. ■

Proposition *3:* Let the sequence of numbers $\mu_t \geq 0$, $t = 0, 1, \ldots, T$ satisfies the inequalities

$$\mu_{t+1} \leq \bar{\alpha} c_1 \tau_t + c_2 2^{\bar{d}} \tau_t \sum_{k=1}^t \gamma_k \mu_k,\ c_1, c_2 \geq 0,$$

then

$$\mu_t \leq c_1 \tau_t e^{c_2 \tau_t^2} \bar{\alpha}.$$

Proof: Statement of Proposition follows directly from the corresponding result in [26] ■

Proposition *4:* [18] For $\bar{z} \in \mathbb{R}^n$ and matrix $A_{\max}$ the following inequality holds $\sum_{i=1}^n (\sum_{j \in N^i} a_{\max}^{i,j} z^j)^2 \leq ||A_{\max}||_2^2 ||\bar{z}||^2$.

Proposition *5:* [18] $||\bar{s}(\bar{z})||^2 \le 2||\mathscr{L}(A_{\max})||_2^2||\bar{z}||^2$.

Proposition *6:* [18] If **A2** is satisfied then $s^i(\bar{x}) = \frac{1}{\alpha_t}\mathrm{E}_{\mathscr{F}_{t-1}} u_t^i$ and the following inequality holds $\frac{1}{\alpha_t^2}\mathrm{E}_{\mathscr{F}_{t-1}} {u_t^i}^2 \le (n-1)\bar{b}^2||\bar{x}_t - x_t^i\underline{1}||^2 + n\bar{b}^2\sigma_w^2,\ i \in N$.

Proposition *7:* By assumptions **A1**, **A2** yields

$$\mathrm{E}||\bar{X}_t||^2 \le (\frac{2nL_2 + \bar{\alpha}^2\tilde{c}}{c_3} + ||\bar{X}_0||^2)e^{t\ln(c_3+1)}.$$

Proof: We write equation (5) as

$$\bar{X}_{t+1} = U\bar{X}_t + G(\alpha_t, \bar{X}_t) + v_t. \quad (13)$$

For the squared norm of $\bar{X}_{t+1}$ we have

$$||\bar{X}_{t+1}||^2 = ||U\bar{X}_t + G(\alpha_t,\bar{X}_t)||^2 + 2(U\bar{X}_t + G(\alpha_t,\bar{X}_t))^{\mathrm{T}}v_t + ||v_t||^2. \quad (14)$$

Taking the conditional expectation of both parts of (14) on σ-algebra $\mathscr{F}_{t-1}$ (i.e. for fixed $\bar{X}_t$) by the centrality of v_t we obtain

$$\mathrm{E}_{\mathscr{F}_{t-1}}||\bar{X}_{t+1}||^2 = ||U\bar{X}_t + G(\alpha_t,\bar{X}_t)||^2 + \mathrm{E}_{\mathscr{F}_{t-1}}||v_t||^2 \le$$
$$\le 2||U\bar{X}_t||^2 + 2||G(\alpha_t,\bar{X}_t)||^2 + \mathrm{E}_{\mathscr{F}_{t-1}}||v_t||^2. \quad (15)$$

By the form of v_t and Lipschitz in u of functions $f^i(u)$ (by **A1**) for $||v_t||^2$ we have

$$||v_t||^2 = \sum_{i\in N}|f^i(x_t^i, \alpha_t \sum_{j\in \bar{N}_t^i} b_t^{i,j}(x^j_{t-d_t^{i,j}} - x_t^i + w_t^{i,j} - w_t^{i,i}))-$$
$$-f^i(x_t^i, \alpha_t s_t^i(\bar{X}_t))|^2 \le L_1^2||\bar{u}_t - \alpha_t^2\bar{s}_t||^2.$$

Under the conditions **A2**, random variables $\mathrm{E}_{\mathscr{F}_{t-1}}u_t^i,\ i \in N$ satisfy the conditions of Proposition 6

$$\mathrm{E}_{\mathscr{F}_{t-1}}||v_t||^2 = \alpha_t^2L_1^2(2n(n-1)\bar{b}^2||\bar{X}_t||^2 + n^2\bar{b}^2\sigma_w^2). \quad (16)$$

Consistently evaluating all three summands on the right hand side of (15) and taking into account the results of Propositions 1, 5 and 6, we deduce

$$\mathrm{E}_{\mathscr{F}_t}||\bar{X}_{t+1}||^2 \le 2^{\tilde{d}}||\bar{X}_t||^2 + 2^{1+\tilde{d}/2}||\bar{X}_t||L_1(L_x||\bar{X}_t|| + \alpha_t||\bar{s}||)+$$
$$+L_2(nL_c + L_x||\bar{X}_t||^2 + \alpha_t^2||\bar{s}||^2) + \alpha_t^2L_1^2(2n(n-1)\bar{b}^2||\bar{X}_t||^2+$$
$$+n^2\bar{b}^2\sigma_w^2) \le (2^{\tilde{d}} + 2^{1+\tilde{d}/2}L_1L_x + L_2L_x + \alpha_t 2^{1+\tilde{d}/2}L_1||\mathscr{L}(A_{\max})||_2+$$
$$+\alpha_t^2(L_2||\mathscr{L}(A_{\max})||_2^2 + 2n(n-1)L_1^2\bar{b}^2))||\bar{X}_t||^2 + nL_2L_c+$$
$$+\alpha_t^2n^2L_1^2\bar{b}^2\sigma_w^2 \le \bar{c} + \bar{c}_3||\bar{X}_t||^2,$$

where $\bar{c} = nL_2L_c + \alpha_t^2\tilde{c},\ \bar{c}_3 = c_3 + 1$.

By taking unconditional expectation of both parts of this inequality and consistently iterating on t, we obtain Proposition 7

$$\mathrm{E}||\bar{X}_t||^2 \le \bar{c} + \bar{c}_3\mathrm{E}||\bar{X}_{t-1}||^2 \le \bar{c} + \bar{c}\bar{c}_3 + \bar{c}_3^2\mathrm{E}||\bar{X}_{t-2}||^2 \le$$
$$\le \bar{c}(1 + \bar{c}_3 + \bar{c}_3^2 + \ldots + \bar{c}_3^{t-1}) + \bar{c}_3^t||\bar{X}_0||^2 \le \bar{c}\frac{\bar{c}_3^t - 1}{c_3} + \bar{c}_3^t||\bar{X}_0||^2 \le$$
$$\le \left(\frac{\bar{c}}{c_3} + ||\bar{X}_0||^2\right)\bar{c}_3^t \le (\bar{c}_4 + ||\bar{X}_0||^2)e^{t\ln\bar{c}_3},$$

$\bar{c}_4 = \bar{c}/c_3$. ■

Let us turn to the proof of Theorem 1. By iterating equation (5) for $t, t-1, \ldots t-d+1$ we obtain

$$\bar{X}_{t+1} = U\bar{X}_t + G(\alpha_t,\bar{X}_t) + v_t =$$
$$= U^2\bar{X}_{t-1} + UG(\alpha_{t-1},\bar{X}_{t-1}) + G(\alpha_t,\bar{X}_t) + Uv_{t-1} + v_t = \quad (17)$$
$$= \cdots = U^{t+1}\bar{X}_0 + \sum_{k=0}^{t}U^{t-k}G(\alpha_k,\bar{X}_k) + \sum_{k=0}^{t}U^{t-k}v_k.$$

Similarly we obtain

$$\bar{Z}_{t+1} = U^{t+1}\bar{X}_0 + \sum_{k=0}^{t}U^{t-k}G(\alpha_k,\bar{Z}_k). \quad (18)$$

Let us estimate $||\bar{X}_t - \bar{Z}_t||^2,\ t = 1,\ldots,T$. By subtracting (18) from (17) and squaring the result we obtain

$$||\bar{X}_t - \bar{Z}_t||^2 = ||\sum_{k=1}^{t}U^{t-k}v_k + \sum_{k=1}^{t}U^{t-k}(G(\alpha_k,\bar{X}_k) - G(\alpha_k,\bar{Z}_k))||^2 \le$$
$$\le 2||\sum_{k=1}^{t}U^{t-k}v_k||^2 + 2||\sum_{k=1}^{t}U^{t-k}(G(\alpha_k,\bar{X}_k) - G(\alpha_k,\bar{Z}_k))||^2 \le$$
$$\le 2||\sum_{k=1}^{t}U^{t-k}v_k||^2 + 2\frac{\tau_t}{2^{\bar{d}}}\sum_{k=1}^{t}\frac{1}{\alpha_t}||U^{t-k}(G(\alpha_k,\bar{X}_k) - G(\alpha_k,\bar{Z}_k))||^2. \quad (19)$$

For the summands in the second sum of (19) using Propositions 5, 1 and Lipschitz condition $f^i(\cdot,\cdot)$ (assumption **A1**) we obtain

$$||U^{t-k}(G(\alpha_k,\bar{X}_k) - G(\alpha_k,\bar{Z}_k))||^2 \le 2^{\bar{d}}L_1^2\sum_{i=1}^{n}(L_x|x_k^i - z_k^i|+$$
$$+\alpha_k|s(x_k^i) - s(z_k^i)|)^2 \le 2^{1+\bar{d}}L_1^2\sum_{i=1}^{n}L_x|x_k^i - z_k^i|^2 + \alpha_k^2 s(x_k^i - z_k^i)^2 \le$$
$$\le 2^{1+\bar{d}}L_1^2(L_x + 2\alpha_k^2||\mathscr{L}(A_{\max})||_2^2)||\bar{X}_k - \bar{Z}_k||^2$$

We take expectation of both parts of (19) and denote $\mu_T = \max_{0\le t\le T}\mathrm{E}||\bar{X}_t - \bar{Z}_t||^2$. By applying Proposition 2 to the first summand and obtained above estimate of the second summand we obtain

$$\mu_T \le 2^{3+\bar{d}}n\sum_{k=1}^{T}\mathrm{E}||v_k||^2 + 2\tau_T L_1^2\sum_{k=1}^{t}(\frac{L_x}{\underline{\alpha}} + 2\alpha_k||\mathscr{L}(A_{\max})||_2^2)\mu_k. \quad (20)$$

To estimate $\mathrm{E}||v_k||^2$ by using previously obtained relation (16) and the result of Proposition 7 we deduce

$$\mathrm{E}||v_k||^2 \le \alpha_k^2(\tilde{c} + \hat{c}(\bar{c}_4 + ||\bar{X}_0||^2)e^{k\ln(c_3+1)})$$

and hence

$$2^{3+\bar{d}}n\sum_{k=1}^{T}\mathrm{E}||v_k||^2 \le \bar{\alpha}8n\tau_T(\tilde{c} + \hat{c}(\bar{c}_4 + ||\bar{X}_0||^2)e^{T\ln(c_3+1)}). \quad (21)$$

By the following relation $2^{\bar{d}}\sum_{k=1}^{t}\alpha_k^2 \le \bar{\alpha}2^{\bar{d}}\sum_{k=1}^{t}\alpha_k = \bar{\alpha}\tau_t$, considering estimates (21) from (20), we have

$$\mathrm{E}\mu_T \le \bar{\alpha}c_1\tau_T + c_2\tau_T 2^{\bar{d}}\sum_{k=1}^{T}\alpha_k\mathrm{E}\mu_k. \quad (22)$$

From last inequality (22) by applying Proposition 3 we get the conclusion of Theorem 1. ■

Theorem *2:* Let the conditions **A1**, **A2** be satisfied; $0 < \alpha_t \leq \bar{\alpha}$; in averaged discrete system (8) $\frac{\varepsilon}{4}$-consensus is achieved for time T and for constants c_1, c_2 from Theorem 1 the following estimate holds

$$c_1 \tau_T e^{c_2 \tau_T^2} \bar{\alpha} \leq \frac{\varepsilon}{4},$$

then ε-consensus is achieved in stochastic discrete system (5) at time t.

Proof: Denote $x^\star$ as consensus value of discrete system (8). From the first group of conditions of Theorem 2 the conditions of Theorem 1 hold. From other conditions of Theorem 2 and the result of Theorem 1 we obtain

$$\mathrm{E}||\bar{X}_t - x^\star \underline{1}||^2 \leq 2\mathrm{E}||\bar{X}_t - \bar{Z}_t||^2 + 2||\bar{Z}_t - x^\star \underline{1}||^2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

■

Consider an important particular case $\forall i \in N\ f^i(x,u) = u$ and $\alpha_t = \alpha = const$, in which the discrete averaged system (8) has the form:

$$\bar{Z}_{t+1} = (I - ((I-U) - \mathscr{L}(\alpha A_{\max})))Z_t. \qquad (23)$$

Theorem *3:* If conditions **A2**, **A3** are satisfied; $\alpha_t = \alpha > 0$; $f^i(x,u) = u$ for any $i \in N$ and condition $\alpha < \frac{1}{d_{\max}}$ for matrix $A_{\max}$ is satisfied, **then** asymptotic mean square consensus for n nodes in averaged discrete system (23).

Moreover if $\frac{\varepsilon}{4}$-consensus is achieved for the time $T(\frac{\varepsilon}{4})$ in averaged discrete system (23) and there exist $T > T(\frac{\varepsilon}{4})$ for which the parameter α provides the condition

$$\bar{C}_1 e^{\bar{C}_2} \alpha \leq \frac{\varepsilon}{4},$$

$$\bar{C}_1 = 8n\left(\tilde{c} + \hat{c}(\frac{\alpha^2 \tilde{c}}{c_3} + ||\bar{X}_0||^2)e^{T\ln(c_3+1)}\right)\tau_t,$$

$$\bar{C}_2 = 2^{2-\tilde{d}}\alpha^2||\mathscr{L}(A_{\max})||_2^2,\ \tilde{c} = n^2\bar{b}^2\sigma_w^2,\ \hat{c} = 2n(n-1)\bar{b}^2\tau_t^2,$$

$$c_3 = 2^{1+\tilde{d}} + 2\alpha^2(||\mathscr{L}(A_{\max})||_2^2 + \hat{c}),$$

where $\tilde{d} = 0$ if $\bar{d} = 0$, or $\tilde{d} = 1$ if $\bar{d} > 0$.

then ε-consensus at time $t:\ T(\frac{\varepsilon}{4}) \leq t \leq T$ is achieved in stochastic discrete system (5).

Proof:

The result of Theorem 3 is derived from Theorem 2.

All amounts in rows of elements of the matrix $\bar{\mathscr{L}} = (I-U) - \mathscr{L}(\alpha A_{\max})$ are equal to zero and, moreover, all the diagonal elements are positive and equal to the absolute value of the sum of all the other elements in the row, which are negative. Hence the matrix $\bar{\mathscr{L}}$ is the Laplacian of a graph and a vector of 1's $\underline{1}$ is the right eigenvector corresponding to zero eigenvalue.

By condition **A3**, the graph corresponding to the Laplacian $\bar{\mathscr{L}}$ has a spanning tree. By condition **A3** graph of the first n nodes has a spanning tree. And units on $(n+1)$-th diagonal consistently connect $\bar{n}$-th node with $(\bar{n}-\bar{d})$-th node, $(\bar{n}-1)$-th node with $(\bar{n}-\bar{d}-1)$-th and so on. Hence asymptotic consensus is achieved in such a discrete system since the condition $\alpha < \frac{1}{d_{\max}}$ holds by the assumptions of Theorem 3.

To satisfy the conditions of Theorem 2 it remains to show that the constants $\bar{C}_1$ and $\bar{C}_2$ are the same as the corresponding constants from Theorem 1. It follows from the fact that in this case $L_1 = L_2 = 1$, $L_x = L_c = 0$.

■

Note that in [11], under certain assumptions similar to the conditions of Theorem 3, the necessary and sufficient condition for achieving mean square consensus in case when step-sizes α_t tending to zero was proved. More general case of the form of functions $f^i(x_t^i, u_t^i)$ and step-sizes α_t not tending to zero were considered above.

V. EXAMPLE

To illustrate the theoretical results we give an example the computer network.

We consider the system of separation the same type of jobs between different agents for parallel computing with feedback. Denote $N = \{1,\ldots,n\}$ as a set of intelligent agents, each of which serves the incoming requests a first-in-first-out queue. Jobs are received at different times and on different nodes.

At any time t state of agent i, $i = 1,\ldots,n$ is described by two characteristics:

- q_t^i is queue length of the atomic elementary jobs of the node i at time t;
- p_t^i is a productivity of the node i at time t.

The dynamics of each agent are described by

$$q_{t+1}^i = q_t^i - p_t^i + z_t^i + u_t^i;\ i \in N,\ t = 0,1,\ldots,T, \qquad (24)$$

where z_t^i is the new job received by node i at time t, u_t^i is the result of information redistribution between agents, which is obtained by using the selected protocol of information redistribution. In the dynamics we assume that $\sum_i u_t^i = 0$, $t = 0,1,2,\ldots$.

We assume that to form the control strategy each agent $i \in N$ at time t can receive from its neighbors $j \in N_t^i$ the following information:

- the noisy observations about its queue length
$$y_t^{i,i} = q_t^i + w_t^{i,i}, \qquad (25)$$
- the noisy and delayed observations about its neighbors queue length, if $N_t^i \neq \emptyset$
$$y_t^{i,j} = q^j_{t-d_t^{i,j}} + w_t^{i,j},\ j \in N_t^i, \qquad (26)$$
where $w_t^{i,j}$ are noises, $0 \leq d_t^{i,j} \leq \bar{d}$ is integer-valued delay, $\bar{d}$ is a maximal delay,
- the information about its productivity p_t^i and about its neighbors productivity $p_t^j,\ j \in N_t^i$.

In the stationary case from all possible options for all job redistribution, which are not distributed by the time t, then minimum operation time of the system corresponds to

$$q_t^i/p_t^i = q_t^j/p_t^j,\ \forall i,j\ \in N \qquad (27)$$

So if we take $x_t^i = q_t^i / p_t^i$ as a state of agent i in dynamic network, then the control gain — to achieve consensus in network — will correspond to the optimal job redistribution between agents in the stationary case [27]. Let the fraction $\frac{q_t^i}{p_t^i}$ denote *the load* of agent i at time t. Thus, it is enough to consider the problem of how to keep the equal load of all agents in the network.

Assume that $p_t^i \neq 0 \forall\, i$. Consider the control protocol (4), where $\forall\, i \in N$, $\forall\, t$ denote $\bar{N}_t^i = N_t^i$ and $b_t^{i,j} = p_t^j / p_t^i, \, , j \in N_t^i$.

As an example of such system consider the simulation for the computer network consisting of six computing agents.

We set the initial queue lengths, the productivities of agents and some initial network topology. Let p_j^t be constant $\forall t$.

For the considered case the dynamics of closed loop system (24) with local voting protocol (4) is as follows:

$$x_{t+1}^i = x_t^i - 1 + z_t^i / p_t^i + \alpha_t \sum_{j \in N_t^i} b_t^{i,j} (y_t^{i,j} / p_t^j - y_t^{i,i} / p_t^i). \qquad (28)$$

where α_t are step-sizes of control protocol, $y_t^{i,j}$ noisy and delayed observation about j-th agents queue length, z_t^i is the new job received by agent i at time t.

In Fig. 1, we can see the system operation in nonstationary case with local voting protocol (4). It means that new jobs can come to different nodes during the system work. We can see that the income of new jobs do not affect to the quality of the system work. It is a big advantage of the algorithm.

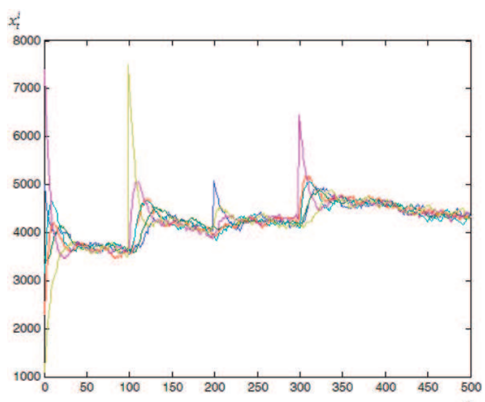

Fig. 1. The dynamics of the agents x_t^i for nonstationary case.

VI. CONCLUSION

In this paper, an approximate consensus problem for networks of nonlinear agents with switching topology, noisy and delayed measurements was studied. In contrast to the existing stochastic approximation-based control algorithms (protocols) local voting protocols with nonvanishing step size are proposed. Nonvanishing (e.g., constant) step size ensures better transients in the time-invariant case and provides bounded error in the case of time-varying loads and agent states. The price to pay is replacement of the almost sure or mean square convergence with an approximate one. To analyze dynamics of the closed loop system the so-called method of the averaged models is used. It allows to reduce complexity of the closed loop system analysis. In the paper new upper bounds for mean square distance between initial system and its approximate average model are proposed. The proposed upper bounds are used to obtain conditions for approximate consensus achievement.

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