Comb Filters for Communication Technology

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***Abstract*—An extension of the design of digital equiripple comb FIR filters is presented. We introduce the fifth type of comb FIR filter which complements the existing four standard types. The design runs from the filter specifications through the degree formula to the impulse response coefficients, which are evaluated by an efficient recursive algorithm. One example is included.**

***Keywords*-comb filter; FIR filter; recursive algorithm; fast design; robustness;**

I. Introduction

Comb FIR filters are often used in the digital processing of communication signals. They are mainly used in the suppression of unwanted spectral components in the signal, e.g. for the attenuation of induced power line interferences. The standard design of a comb FIR filter starts with a prototype filter which is usually a low pass or high pass filter. The comb FIR filter is obtained by inserting of a finite number of zeros between the values of the impulse response of the prototype [1]. In [2], [3] we have demonstrated an efficient design of optimal equiripple comb FIR filters which outperforms the standard design both in terms of speed and robustness. All the design procedures [1] - [3] lead to the comb FIR filters of four types (Fig. 1) with available notch frequencies summarized in Tab. I. In this paper we introduce the design of comb FIR filter of another type (Fig. 2). We call it comb5 FIR filter. The comb5 FIR filter cannot be designed using design procedures [1] - [3]. The reason for this is that the frequency response of the comb5 FIR filter is not obtained by the repetition of a LP or HP prototype filter resulting from the interleaving of the impulse response of the prototype filter by zeros. Moreover, it cannot be derived from the types 1-4, e.g. using frequency transformation. The organization of the paper is as follows. In Section II the impulse response of the filter is defined. In Section III we summarize available positions of notch frequencies. In Section IV the idea of the comb5 FIR filter is introduced. In Section V we present the design of the comb5 FIR filter. Finally, Section VI illustrates the practical design of the comb5 FIR filter.

II. Impulse Response

We assume the impulse response $h(k)$ with odd length $N = 2\nu + 1$ coefficients and with even symmetry

$$a(0) = h(\nu)\ \ ,\ \ a(k) = 2h(\nu+k) = 2h(\nu-k)\,,\ k = 1 \dots \nu\,. \tag{1}$$

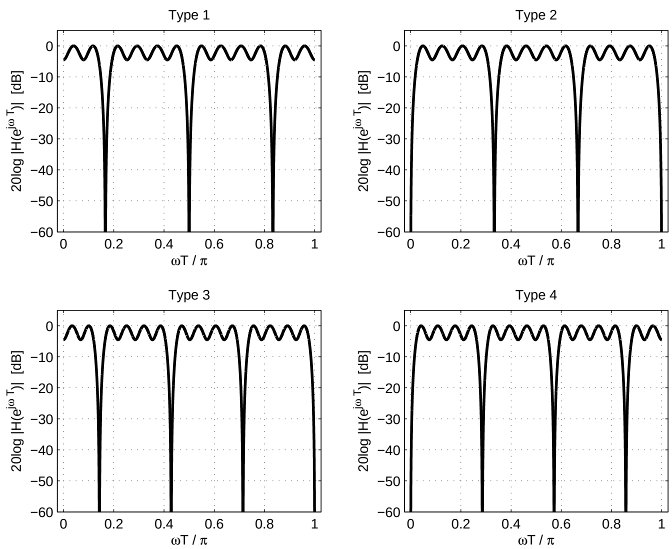

Fig. 1. Four types of comb FIR filter [3].

TABLE I
Notch frequencies.

Type 1 — r even Marginal : none Non-marginal : $\omega_i = (2i+1)\frac{\pi}{r}$ $i = 0, \dots, \frac{r}{2} - 1$	Type 2 — r even Marginal : $\omega = 0,\ \omega = \pi$ Non-marginal : $\omega_i = 2i\frac{\pi}{r}$ $i = 1, \dots, \frac{r}{2} - 1$
Type 3 — r odd Marginal : $\omega = \pi$ Non-marginal : $\omega_i = (2i+1)\frac{\pi}{r}$ $i = 0, \dots, \frac{r-3}{2}$	Type 4 — r odd Marginal : $\omega = 0$ Non-marginal : $\omega_i = 2i\frac{\pi}{r}$ $i = 1, \dots, \frac{r-1}{2}$

The transfer function of the filter is

$$H(z) = \sum_{k=0}^{2\nu} h(k)\, z^{-k} = z^{-\nu}\left[h(\nu) + 2\sum_{k=1}^{\nu} h(\nu \pm k)\frac{(z^k + z^{-k})}{2}\right]$$
$$= z^{-\nu}\sum_{k=0}^{\nu} a(k)\, T_k(w) = z^{-\nu} Q(w) \tag{2}$$

where $T_n(x)$ is Chebyshev polynomial of the first kind

$$T_n(x) = \begin{cases} \cos\left(n \arccos(x)\right) & \text{if} |x| \le 1 \\ \cosh\left(n\,\text{arccosh}(x)\right) & \text{otherwise} \end{cases} \quad . \tag{3}$$

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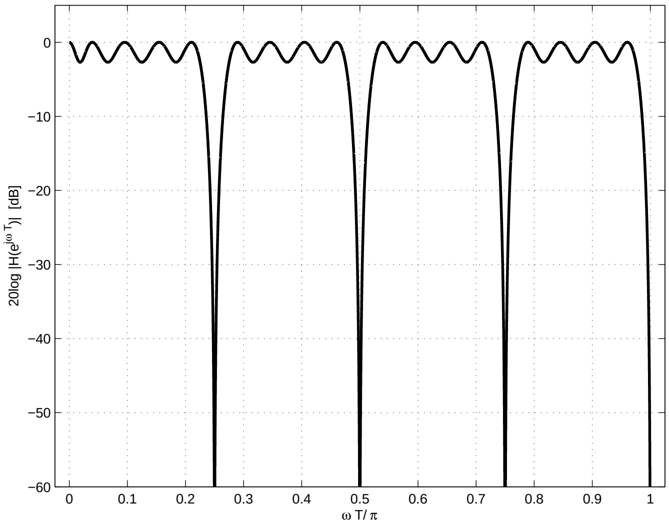

Fig. 2. Comb5 FIR filter.

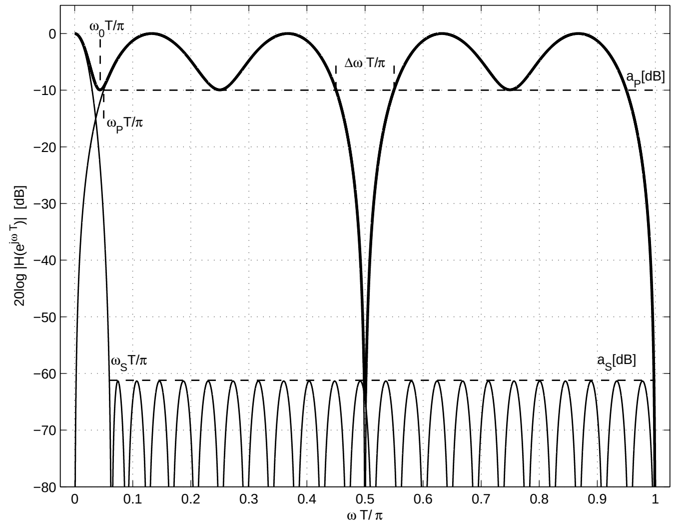

Fig. 3. Forming of the frequency response of comb5 FIR filter.

The function $Q(w)$ represents a polynomial in the variable $w = \frac{1}{2}(z + z^{-1})$ which on the unit circle $z = e^{j\omega T}$ reduces to the real valued zero phase transfer function $Q(w)$ of the real argument

$$w = \cos(\omega T)\,. \tag{4}$$

III. Notch Frequencies

There are two types of notch bands. First, the marginal notch bands are located at the notch frequencies $\omega T = 0$ and $\omega T = \pi$. The width of the marginal notch band(s) is $\Delta\omega T/2$. Second, the non-marginal notch bands are located inside the frequency interval $(0, \pi)$. The width of the non-marginal notch band(s) is $\Delta\omega T$. Four types of comb FIR filters are generated containing either none, or one, or two marginal notch bands (Fig. 1, Tab. I). The comb5 FIR filter (Fig. 2) introduced here consists of one marginal notch band at $\omega T = \pi$ and of non-marginal notch band(s) inside the interval $(0, \pi)$.

IV. The Comb5 FIR Filter

The comb5 FIR filter (Fig. 2) is in fact a comb FIR filter of type 2 with compensated attenuation at and near the frequency $\omega T = 0$. Its frequency behavior is obtained by the parallel combination of a comb FIR filter of the type 2 and of a DC-pass FIR filter which is an inverted DC-notch FIR filter [4]. In the discrete time domain, the impulse response $h(k)$ of the comb5 FIR filter is the sum of the impulse response $h_{C2}(k)$ of the comb FIR filter of the type 2 and of the impulse response $h_{DC}(k)$ of the DC-pass FIR filter. Consequently the design of the comb5 FIR filter is a co-design of the two partial FIR filters. The approximation theory behind the partial filters was treated in [3] and [4] in detail. Here we are focused on the practical aspects of the design.

V. Design Procedure

The design procedure of the comb5 FIR filter consists of two basic steps. In the first step we design the comb FIR filter of the type 2 with the zero phase transfer function [3]

$$Q(w) = 1 - \frac{1 + (-1)^n T_n\left[\dfrac{T_r(w) - \kappa^2}{1 - \kappa^2}\right]}{1 + (-1)^n \cosh\left(n\,\mathrm{acosh}\dfrac{\kappa^2 + 1}{\kappa^2 - 1}\right)}\ . \tag{5}$$

The impulse response $h_{C2}(k)$ corresponding to the zero phase transfer function (5) consists of $N = 2rn + 1$ coefficients. The design of the comb FIR filter of the type 2 proceeds as follows:

- Specify the number b of non-marginal notch bands, the width of the non-marginal notchbands $\Delta\omega T$, the maximal attenuation in the pass bands a_p [dB] and the minimal attenuation at notch frequencies a_s [dB] (Fig. 3).
- Evaluate the degree n of the outer Chebyshev polynomial $T_n()$ in (5)

$$n \geq \frac{\mathrm{acosh}\dfrac{1 + 10^{0.05 a_p[dB]}}{1 - 10^{0.05 a_p[dB]}}}{\mathrm{acosh}\dfrac{1 + \kappa^2}{1 - \kappa^2}} \tag{6}$$

where κ^2 is

$$\kappa^2 = \frac{1 - \cos\left(r\dfrac{\Delta\omega T}{2}\right)}{1 + \cos\left(r\dfrac{\Delta\omega T}{2}\right)} \tag{7}$$

and

$$r = 2(b + 1)\ . \tag{8}$$

- Evaluate the impulse response $h_{c1}(k)$ of the comb filter of the type 1 (Tab. II).
- Convert the impulse response $h_{c1}(k)$ of the comb FIR filter of type 1 into the impulse response $h_{c2}(k)$ of the comb FIR filter of type 2 using matrix multiplication

TABLE II
EVALUATION OF THE IMPULSE RESPONSE $h_{C1}(k)$.

given	$n,\ r$ (integer values) , $\kappa^2 \neq 0$ (real value)
initialization	$a(n) = \dfrac{1}{(1-\kappa^2)^n}$, $a(n+k) = 0$ for $k = 1, 2, 3$, $h(k) = 0$ for $k = 0, \ldots, 2nr$
body (*for* $k = 1$ ... n)	$a(n-k) = \Big\{ -\frac{1}{2}\left[(2n-k+1)(k-1) + \kappa^2(n+1-k)(2(n-k)+1)\right] a(n+1-k) - \kappa^2(n+2-k)\ a(n+2-k) + \frac{1}{2}\left[(2n-k+3)(k-3) + \kappa^2(n+3-k)(2(n-k)+7)\right] a(n+3-k) + \frac{1}{4}(2n+4-k)(k-4)\ a(n+4-k) \Big\} \Big/ \frac{1}{4}k(2n-k)$
(*end loop on* k)	$a(0) = \dfrac{a(0)}{2}$, $C = \cosh\left(n \cosh \dfrac{\kappa^2+1}{\kappa^2-1}\right)$
impulse response	$h_{C1}(rn) = 1 - \dfrac{1+(-1)^n a(0)}{1+(-1)^n C}$
(*for* $k = 1$ ... n)	$h_{C1}(r(n \pm k)) = -(-1)^n \dfrac{1}{2} \dfrac{a(k)}{1+(-1)^n C}$ (*end loop on* k)

TABLE III
EVALUATION OF THE IMPULSE RESPONSE $h_{DC}(k)$.

given	$n = n_{DC}$ (integer value), λ (real value)
initialization	$A(n) = \lambda^n$, $A(n+1) = A(n+2) = A(n+3) = 0$
body (*for* $k = 2$ ... $n+1$)	$A(n+1-k) = \Big\{ 2\left[(k-1)(2n+1-k) - (\lambda'/\lambda)(n+1-k)(2n+1-2k)\right] A(n+2-k) + 4\,(\lambda'/\lambda)(n+2-k)\ A(n+3-k) - 2\left[(k-3)(2n+3-k) - (\lambda'/\lambda)(n+3-k)(2n+7-2k)\right] A(n+4-k) + (k-4)(2n+4-k)\ A(n+5-k) \Big\} \Big/ k(2n-k)$
(*end loop on* k)	
impulse response	$h_{DC}(n_{DC}) = \dfrac{\frac{A(0)}{2}+1}{T_n(2\lambda-1)+1}$
(*for* $k = 1$ ... n)	$h_{DC}(n_{DC} \pm k) = \dfrac{1}{2} \dfrac{A(k)+1}{T_n(2\lambda-1)+1}$ (*end loop on* k)

$$h_{c2}(k) = \sum_{m=0}^{2rn} \Lambda(k-rn, l-rn)\, h_{c1}(m)\ ,\ k = 0, \dots, 2rn \tag{9}$$

using the matrix where

$$\Lambda(x,y) = \begin{cases} \cos\left(y\dfrac{\pi}{r}\right) & \text{if } x = y \\ \dfrac{2\sin\left(y\dfrac{\pi}{r}\right)}{\pi(y-x)} & \text{if } x \neq y \text{ and } |x-y| \text{ is even} \\ 0 & \text{otherwise} \quad . \end{cases} \tag{10}$$

In the second step, we design the DC-pass FIR filter based on the zero phase transfer function [4]

$$Q_{DC}(w) = 1 - \frac{T_{n_{DC}}(\lambda\, w + \lambda - 1) + 1}{T_{n_{DC}}(2\lambda - 1) + 1} \quad . \tag{11}$$

An important issue is the proper choice of the stop-band frequency $\omega_s T$ of the DC-pass FIR filter. This choice guarantees the value of the ripple at $\omega_0 T$ equal to the values of remaining ripples in the pass bands of the comb5 FIR filter (Fig. 3). We have no formula for the calculation of the value $\omega_s T$. Consequently we calculate the value $\omega_s T$ numerically by minimizing of the function

$$\left| |H_{C2}(e^{j\omega_0 T})| + |H_{DC}(e^{j\omega_0 T}) - a_p| \right| \quad . \tag{12}$$

The design of the DC-pass FIR filter continues as follows:

- Calculate the real parameter λ

$$\lambda = \frac{1}{1 - \sin^2 \dfrac{\omega_s T}{2}} \quad . \tag{13}$$

- Evaluate the degree n_{DC} of the zero phase transfer function (11)

$$n_{DC} \geq \frac{\text{acosh}\dfrac{1 + 10^{0.05 a_s[dB]}}{1 - 10^{0.05 a_s[dB]}}}{\text{acosh}\dfrac{1 + \sin^2 \dfrac{\omega_s T}{2}}{1 - \sin^2 \dfrac{\omega_s T}{2}}} \quad . \tag{14}$$

- Evaluate the impulse response $h_{DC}(k)$ of the length $2n_{DC} + 1$ coefficients of the DC-pass FIR filter (Tab. III).
- Finally, calculate the impulse response $h(k)$ of the comb5 FIR filter

$$h(k) = h_{C2}(k) + h_{DC}(k) \quad . \tag{15}$$

VI. Example of the Design

Design a comb5 FIR filter with 9 non-marginal notch bands of the width $\Delta\omega T = \pi/100$, maximal attenuation in the pass bands $a_p = -3$ dB and minimal attenuation in the stop bands $a_s = -60$ dB.
We get $r = 20$ using (Tab. I) and (8), $\kappa^2 = 0.02508563$ (7), $n = 7.67506483 \rightarrow n = 8$ (6). Using the recursive algorithm (Tab. II) we get the impulse response $h_{C1}(k)$ of the equiripple comb FIR filter of type 1. The impulse response $h_{C2}(k)$ of the equiripple comb FIR filter of type 2 is obtained from $h_{C1}(k)$ using (9), (10). The length of the impulse response $h_{C2}(k)$ is 321 coefficients. The actual attenuation in the pass bands is $a_{p\ act} = -2.7029$ dB. The amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the comb FIR filter of type 2 is shown in Fig. 4. The value $\omega_s T = 0.00668686\pi$ of the DC-pass FIR filter was evaluated numerically (12). Further, we get $\lambda = 1.00067186$ (13) and $n_{DC} = 394.8084 \rightarrow n = 395$ (14). Using the recursive algorithm (Tab. III) we get the impulse response $h_{DC}(k)$ of the DC-pass FIR filter. Its length is 791 coefficients. The amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the DC-pass FIR filter is shown in Fig. 5. The final impulse

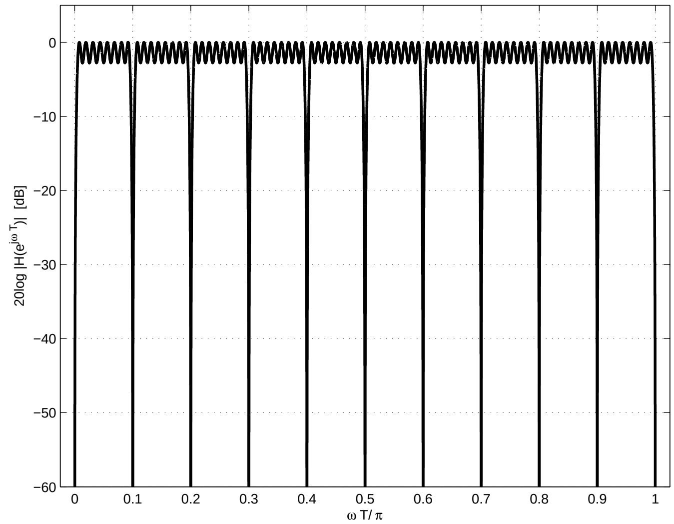

Fig. 4. Amplitude frequency response of the comb FIR filter of type 2.

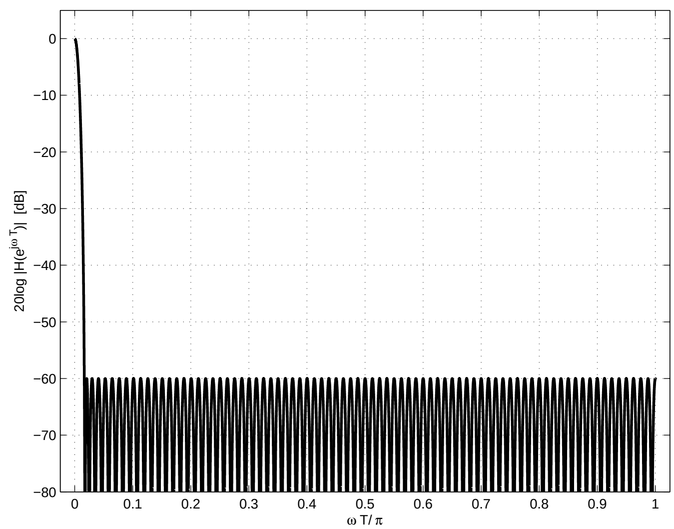

Fig. 5. Amplitude frequency response of the DC-pass FIR filter.

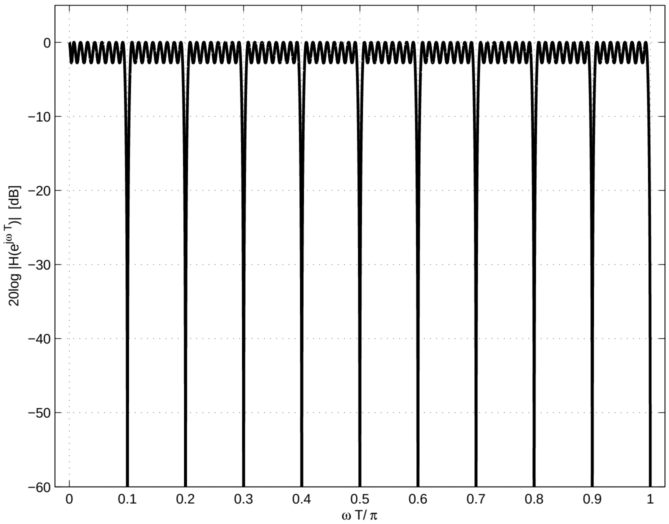

Fig. 6. Amplitude frequency response of the comb5 FIR filter.

response $h(k)$ of the length 791 coefficients of the comb5 FIR filter is calculated using (15). The amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the comb5 FIR filter is shown in Fig. 6.

VII. Conclusion

In this paper a fifth type of the comb FIR filter was introduced and its design procedure was presented. The coefficients of the impulse response of the filter are evaluated based on the filter specification. Independent control can be exercised over the number of notch bands, the width of the notch bands and attenuations in both pass bands and stop bands.

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