Half-Band FIR Filters for Signal Compression

Pavel Zahradnik and Boris Šimák
Department of Telecommunication Engineering
Czech Technical University in Prague
Prague, Czech Republic
zahradni, simak@fel.cvut.cz

Miroslav Vlček
Department of Applied Mathematics
Czech Technical University in Prague
Prague, Czech Republic
vlcek@fd.cvut.cz

***Abstract*—An efficient design of equiripple half-band FIR filters for signal compression is presented. Solution of the approximation problem in terms of generating function and zero phase transfer function for the equiripple half-band FIR filter is shown. The equiripple half-band FIR filters are optimal in the Chebyshev sense. The closed form solution provides an efficient computation of the impulse response of the filter. One example is included.**

***Keywords*-FIR filter; half-band filter; equiripple approximation; wavelet compression;**

I. Introduction

Half-band (HB) filters are basic building blocks in wavelet analysis [1], signal compression and in multirate signal processing [2]. The only available method for designing equiripple (ER) HB finite impulse response (FIR) filters is based on the numerical McClellan - Parks program [3]. It is usually combined with a clever "Trick" [4]. Besides this, some design methods are available for almost ER HB FIR filters, e.g. [5],[6]. No general non-numerical design of an ER HB FIR filter was found in references. In our paper we are primarily concerned with the ER approximation of HB FIR filters and with the related non-numerical design procedure suitable for practical design of ER HB FIR filters. We present the generating function and the zero phase transfer function of the ER HB FIR filter. These functions give an insight into the nature of this approximation problem. Our design procedure is based on the Chebyshev polynomials of the second kind. Based on the differential equation for the Chebyshev polynomials of the second kind, we have derived formulas for an effective evaluation of the coefficients of the impulse response. We present an approximating degree equation which is useful in practical filter design. The advantage of the proposed approach over the numerical design procedures relies on the fact that the coefficients of the impulse response are evaluated by formulas. Hence the speed of the design is deterministic.

II. Impulse Response, Transfer Function and Zero Phase Transfer Function

A HB filter is specified by the minimal passband frequency $\omega_p T$ (or maximal stopband frequency $\omega_s T$) and by the minimal attenuation in the stopband a_s [dB] (or maximal attenuation in the passband a_p [dB]). The antisymmetric behavior of its frequency response implies the relations $\omega_s T = \pi - \omega_p T$ and $10^{0.05 a_p} + 10^{0.05 a_s} = 1$. The goal in the filter design is to get the minimum filter length N satisfying the filter specification and to evaluate the coefficients of the impulse response of the filter. We assume the impulse response $h(k)$ with odd length $N = 2(2n+1)+1$ coefficients and with even symmetry $h(k) = h(N-1-k)$. The impulse response of the HB FIR filter with the length $N = 2(2n+1)+1$ contains $2n$ zero coefficients as follows

$$\begin{aligned} h(2n+1) &= a(0) = 0.5 \\ 2h(2n+1 \pm 2k) &= a(2k) = 0 \ , \ k = 1 \dots n \\ 2h(2n+1 \pm (2k+1)) &= a(2k+1) \ , \ k = 0 \dots n \ . \end{aligned} \tag{1}$$

The transfer function of the HB FIR filter is

$$H(z) = z^{-(2n+1)} \left[\frac{1}{2} + \sum_{k=0}^{n} a(2k+1)\, T_{2k+1}(w) \right] \tag{2}$$

where $T_m(w)$ is Chebyshev polynomials of the first kind. The frequency response $H(e^{j\omega T})$ of the HB FIR filter is

$$H(e^{j\omega T}) = e^{-j(2n+1)\omega T}\ Q(\cos \omega T) \tag{3}$$

where $Q(w)$ is a polynomial in the variable $w = (z + z^{-1})/2$ which on the unit circle reduces to a real valued zero phase transfer function $Q(w)$ of the real argument $w = \cos(\omega T)$.

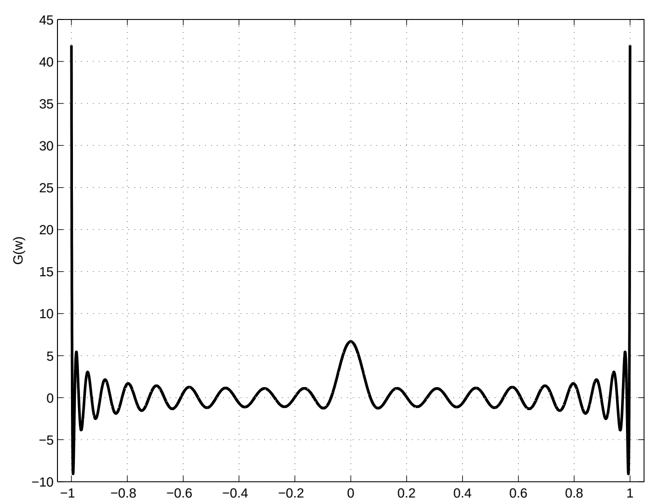

Fig. 1. Generating polynomial $G(w)$ for $n = 20$, $\kappa' = 0.03922835$, $A = 1.08532371$ and $B = 0.95360863$.

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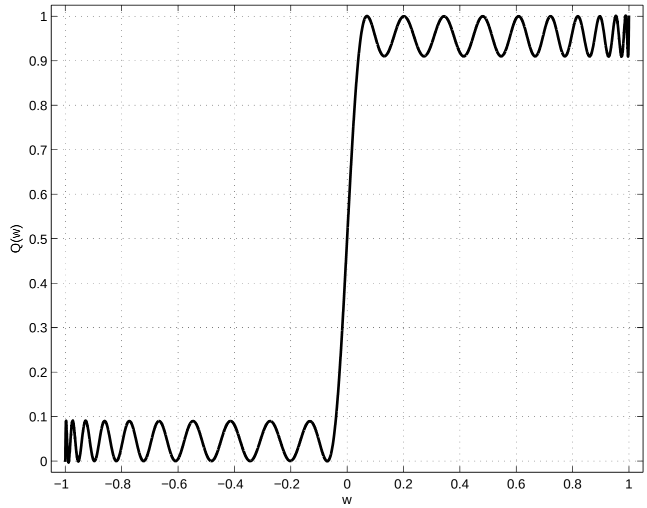

Fig. 2. Zero phase transfer function $Q(w)$ for $n = 20$, $\kappa' = 0.03922835$, $A = 1.08532371$, $B = 0.95360863$ and $\mathcal{N} = 0.55091994$.

III. Generating Polynomial and Zero Phase Transfer Function of an ER HB FIR Filter

A straightforward theory for the generating polynomial of an ER HB FIR filter is currently unavailable. The generating polynomial of an ER HB FIR filter is related to the generating polynomial of the almost ER HB FIR filter presented in [5]. Based on our experiments conducted in [5], we have found that the generating polynomial $G(w)$ (Fig. 1) of the ER HB FIR filter is obtained by weighting of Chebyshev polynomials in the generating polynomial of the almost ER HB FIR filter, namely

$$G(w) = AU_n\left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right) + BU_{n-1}\left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right) \tag{4}$$

where $U_n(x)$ and $U_{n-1}(x)$ are Chebyshev polynomials of the second kind and A, B, κ' are real numbers. The zero phase transfer function $Q(w)$ (Fig. 2) is related to the generating polynomial

$$Q(w) = \frac{1}{2} + \frac{1}{\mathcal{N}}\int G(w)dw \tag{5}$$

where the norming factor $\mathcal{N}$ is given by (17). The generating polynomial $G(w)$ and the zero phase transfer function $Q(w)$ show the nature of the approximation of an ER HB FIR filter.

IV. Differential Equation and Impulse Response of an ER HB FIR Filter

The Chebyshev polynomial of the second kind $U_x(w)$ fulfils the differential equation

$$(1-x^2)\frac{d^2U_n(x)}{dx^2} - 3x\frac{dU_n(x)}{dx} + n(n+2)U_n(x) = 0 \ . \tag{6}$$

Using substitution

$$x = \left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right) \tag{7}$$

we get the differential equation (6) in the form

$$\begin{aligned} w(w^2-\kappa'^2)&\left[(1-w^2)\frac{d^2U_n(w)}{dw^2} - 3w\frac{dU_n(w)}{dw}\right] \\ &+ \left[\kappa'^2(1-w^2) + 2w^2(1-w^2)\right]\frac{dU_n(w)}{dw} \\ &+ 4w^3n(n+2)U_n(w) = 0 \ . \end{aligned} \tag{8}$$

Based on the differential equation (8), we have derived the non-numerical procedure for the evaluation of the impulse response $h_n(k)$ corresponding to polynomial $\mathcal{U}_n(w)$

$$\mathcal{U}_n(w) = \int U_n\left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right)dw \ . \tag{9}$$

This procedure is summarized in Tab. I. The impulse response $h(k)$ of the ER HB FIR filter is

$$h(k) = \frac{1}{2} + \frac{A}{\mathcal{N}}h_n(k) + \frac{B}{\mathcal{N}}h_{n-1}(k) \ . \tag{10}$$

The non-numerical evaluation of the impulse response $h(k)$ is essential in the practical filter design.

V. Degree of an ER HB FIR Filter

The exact degree formula is not available. In the practical filter design, the degree n can be obtained with excellent accuracy from the specified minimal passband frequency ω_pT and from the minimal attenuation in the stopband a_s [dB] using the approximating degree formula

$$n \doteq \frac{a_s[dB] - 18.18840664\,\omega_pT + 33.64775300}{18.54155181\,\omega_pT - 29.13196871} \ . \tag{11}$$

The exact relation between the minimal attenuation in the stopband a_s [dB], the minimal passband frequency ω_pT and the degree n were obtained experimentally.

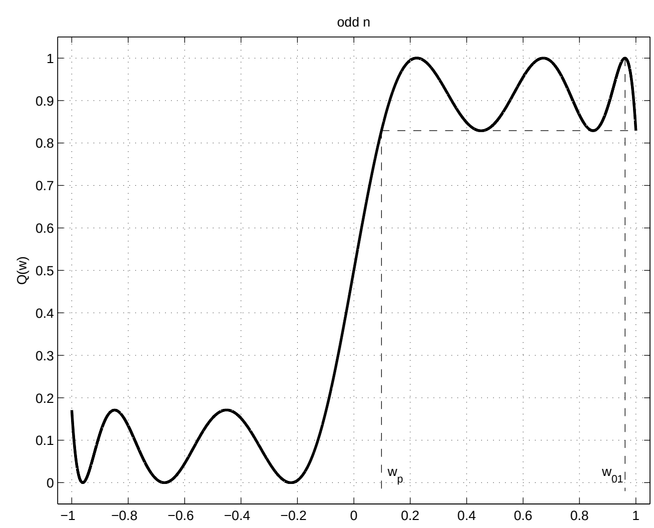

Fig. 3. $Q(w)$ for odd n.

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TABLE I
ALGORITHM FOR THE EVALUATION OF THE COEFFICIENTS $h_n(k)$.

given — n (integer value), κ' (real value)

initialization

$$\alpha(2n) = \frac{1}{(1-\kappa'^2)^n}$$

$$\alpha(2n-2) = -(2n\kappa'^2 + 1)\,\alpha(2n)$$

$$\alpha(2n-4) = -\frac{4n+1+(n-1)(2n-1)\kappa'^2}{2n}\,\alpha(2n-2) - \frac{(2n+1)(n+1)\kappa'^2}{2n}\,\alpha(2n)$$

body

(*for* $k = n$ *down to* 3)

$$\alpha(2k-6) =$$

$$\Big\{ -\left[3(n(n+2)-k(k-2)) + 2k - 3 + 2(k-2)(2k-3)\kappa'^2\right]\alpha(2k-4)$$
$$-\left[3(n(n+2)-(k-1)(k+1)) + 2(2k-1) + 2k(2k-1)\kappa'^2\right]\alpha(2k-2)$$
$$-\left[n(n+2)-(k-1)(k+1)\right]\alpha(2k) \quad \Big\} \;/\; \left[n(n+2)-(k-3)(k-1)\right]$$

(*end loop on* k)

integration

(*for* $k = 0$ *to* n) $\quad a(2k+1) = \dfrac{\alpha(2k)}{2k+1}$ $\quad$ (*end loop on* k)

impulse response $h_n(k)$

$$h_n(2n+1) = 0$$

(*for* $k = 0$ *to* n) $\quad h_n(2n+1 \pm (2k+1)) = \dfrac{a(2k+1)}{2}$ $\quad$ (*end loop on* k)

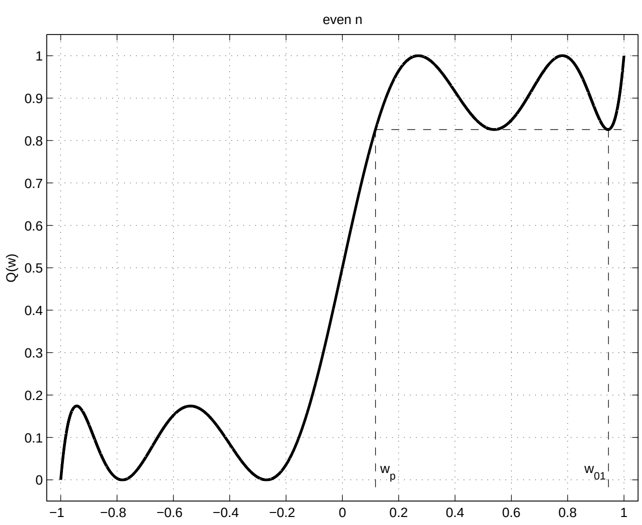

Fig. 4. $Q(w)$ for even n.

VI. SECONDARY VALUES OF THE ER HB FIR FILTER

The secondary real values κ', A and B can be obtained from the specified passband frequency $\omega_p T$ and from the degree n of the generating polynomial. In practical filter design, the approximating formulas

$$\kappa' = \frac{n\omega_p T - 1.57111377\,n + 0.00665857}{-1.01927560\,n + 0.37221484} \tag{12}$$

$$A = \left(0.01525753\,n + 0.03682344 + \frac{9.24760314}{n}\right)\kappa'$$
$$+ 1.01701407 + \frac{0.73512298}{n} \tag{13}$$

and

$$B = \left(0.00233667\,n - 1.35418408 + \frac{5.75145813}{n}\right)\kappa'$$
$$+ 1.02999650 - \frac{0.72759508}{n} \tag{14}$$

obtained experimentally provide excellent accuracy. Further, the exact values κ', A and B can be obtained numerically (e.g. using the Matlab function $fminsearch$) by satisfying the equality (see Fig. 3 - 4)

$$Q(w_p) = \begin{cases} Q(1) & \text{if } n \text{ is odd} \\ Q(w_{01}) & \text{if } n \text{ is even .} \end{cases} \tag{15}$$

The value

$$w_{01} = \sqrt{\kappa'^2 + (1-\kappa'^2)\cos^2\frac{\pi}{2n+1}} \tag{16}$$

was introduced in [6]. Relation (15) guarantees the equiriple behaviour of the generating polynomial $Q(w)$.

VII. DESIGN OF THE ER HB FIR FILTER

The design procedure is as follows:

- Specify the ER HB FIR filter by the minimal passband frequency $\omega_p T$ and by the minimal attenuation in the stopband a_s [dB].
- Calculate the integer degree n of the generating polynomial (11).
- Calculate the real values κ' (12), A (13) and B (14).
- Evaluate the partial impulse responses $h_n(k)$ and $h_{n-1}(k)$ (Tab. I).

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- Evaluate the final impulse response $h(k)$ (10) where the real norming factor $\mathcal{N}$ is

$$\mathcal{N} = \begin{cases} 2\,[\,A\mathcal{U}_n(1) + \mathrm{B}\mathcal{U}_{n-1}(1)\,] & \text{if n is even} \\ 2\,[\,A\mathcal{U}_n(w_{01}) + \mathrm{B}\mathcal{U}_{n-1}(w_{01})\,] & \text{if n is odd .} \end{cases} \tag{17}$$

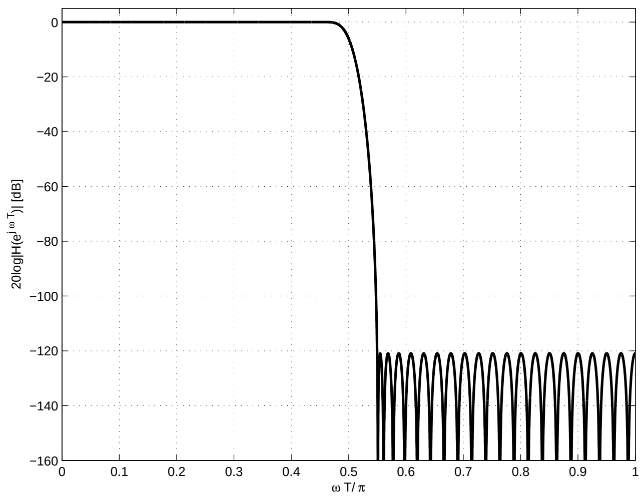

Fig. 5. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

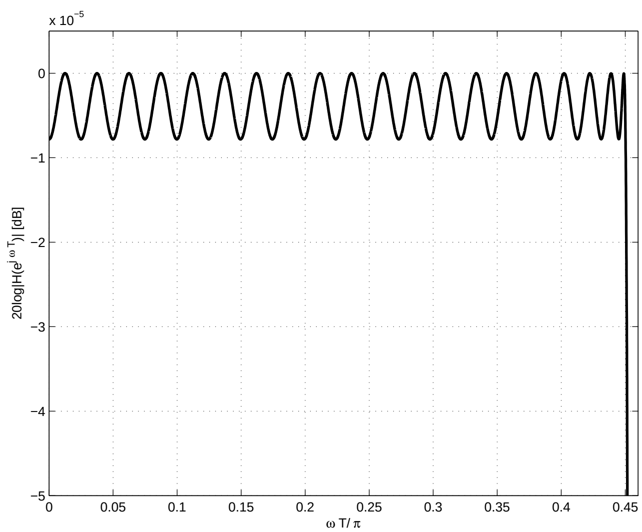

Fig. 6. Passband of the filter.

VIII. Example

Design the ER HB FIR filter specified by the minimal passband frequency $\omega_p T = 0.45\pi$ and by the minimal attenuation in the stopband $a_s = -120$ dB.
We get $n = 38.3856 \to 39$ (11), $\kappa' = 0.15571103$ (12), $A = 1.17117396$ (13), $B = 0.83763199$ (14) and $\mathcal{N} = -2747.96038544$ (17). The impulse response $h(k)$ (Tab. II) with the length $N = 159$ coefficients is evaluated using Tab. I and eq. (10). The actual values $\omega_{p\ act} T = 0.4502\pi$ and $a_{act} = -120.91$ dB satisfy the filter specification. The amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 5. The detailed view of its passband is shown in Fig. 6. For the specified values $\omega_p T = 0.45\pi$ and $N = 159$, the comparative numerical design based on the "Trick" [3] combined with the Remez algorithm using the Matlab function $firpm$ results in the slightly unsatisfactory minimal passband frequency $\omega_{p\ act} T = 0.44922001\pi < 0.45\pi$ and consequently in a slightly better minimal attenuation in the stopband $a_{s\ act} = -123.29066608$ [dB].

TABLE II
Coefficients of the Impulse Response.

k	$h(k)$	k	$h(k)$
0 , 158	-0.00000070	42 , 116	0.00231877
2 , 156	0.00000158	44 , 114	-0.00283354
4 , 154	-0.00000331	46 , 112	0.00344038
6 , 152	0.00000622	48 , 110	-0.00415347
8 , 150	-0.00001087	50 , 108	0.00498985
10 , 148	0.00001799	52 , 106	-0.00597048
12 , 146	-0.00002852	54 , 104	0.00712193
14 , 144	0.00004363	56 , 102	-0.00847897
16 , 142	-0.00006481	58 , 100	0.01008867
18 , 140	0.00009384	60 , 98	-0.01201717
20 , 138	-0.00013287	62 , 96	0.01436125
22 , 136	0.00018446	64 , 94	-0.01726924
24 , 134	-0.00025161	66 , 92	0.02098117
26 , 132	0.00033779	68 , 90	-0.02591284
28 , 130	-0.00044697	70 , 88	0.03285186
30 , 128	0.00058370	72 , 86	-0.04348979
32 , 126	-0.00075311	74 , 84	0.06223123
34 , 124	0.00096097	76 , 82	-0.10523903
36 , 122	-0.00121375	78 , 80	0.31802058
38 , 120	0.00151871	79	0.50000000
40 , 118	-0.00188398		

IX. Conclusion

This paper has presented the equiripple approximation of halfband FIR filters. The generating polynomial and the zero phase transfer function illustrate the nature of the approximation problem. The coefficients of the impulse response are straightforwardly evaluated from the filter specification.

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