On the Distribution of the Queue Size in a Packet Buffer

Blazej Adamczyk and Andrzej Chydzinski
Department of Computer Networks and Systems
Silesian University of Technology
Gliwice, Poland
email: {blazej.adamczyk, andrzej.chydzinski}@polsl.pl

***Abstract*—In this paper, we address the performance of the packet queueing mechanism at the output interface of a router. Specifically, we derive transient and stationary distribution of the queue size, which are fundamental performance characteristics of every queueing mechanism. The results are obtained directly in the time domain, without previous application of the Laplace transform, which is usually the case in the transient analysis. The model can be used for both passive buffer management (tail drop) and active buffer management, in which the dropping probability is a function of the queue size. It can be used also for traffic types of different statistical properties. Theoretical results are illustrated with numerical examples.**

***Keywords*-packet buffer; queue size distribution; passive buffer management; active buffer management.**

I. INTRODUCTION

The buffers for packets, found in networking equipment, serve the purpose of temporarily storing sudden surges of packets caused by unpredictable peaks of network traffic. The buffers are integral components of packet networks. In their absence, the utilization of physical links between nodes would experience a notable decrease.

The size of the queue in a buffer is an important performance parameter. This is due to the fact that the time a packet spends in the buffer extends its total trip time to destination. Hence, several mathematical models have been proposed for calculations of the queue size, see, e.g., the classic monograph [1]. The classic models however, like the well-known M/M/1/N or M/G/1/N, constitute rather rough approximation of reality. They do not reflect important properties of real traffic and/or the packet service process at the router. For instance, they allow neither for modelling of an arbitrary packet interarrival time distribution nor its autocorrelated structure. Moreover, the classic analysis was focused on stationary characteristics mostly, without the full transient solution of the model.

Therefore, a few more complex queueing models were proposed and solved, with focus on advanced traffic modelling, e.g., [2]–[4], and the transient analysis, e.g., [5]–[7].

Recently, the situation got more complicated when active buffer management was advocated by the Internet Engineering Task Force (IETF) [8][9]. Most packet buffers exploited nowadays are governed by the passive, tail-drop algorithm [10] [11], i.e. incoming packets are dropped when the buffer is full. In active buffer management, the packets are dropped before the buffer gets full, with probability evolving in time. Many such schemes have been proposed for router's buffers, see, e.g., [12]–[19]. The most straightforward of these algorithms are based on the concept that the dropping probability should be a function of the queue size, see, e.g., [20]–[25].

This created a new research problem, i.e. finding the queue size distribution in a buffer with active management. The problem has been already solved to a large extent for algorithms exploiting the queue-size based dropping, namely, the solutions for simple traffic models were given in, e.g., [26]–[32], while for advanced traffic models in [33]–[38]. Moreover, works [28][29][32]–[38] encompass the transient analysis of various characteristics.

Now, in the transient analysis, the typical approach is based on moving equations to the Laplace transform domain [29] [33][38], solving them in this domain, and returning to the time domain with the help of the transform inversion algorithm. This method was successful in solving many queueing problems, but it can be viewed as rather complicated.

In this paper, we derive the transient queue size distribution directly in the time domain, without the help of the Laplace transform. Moreover, we give the formulas for the stationary distribution, and for the average queue size, both in the transient and the stationary case.

The considered model encompasses the active buffer management, in which the dropping probability is a function of the queue size. However, by a proper parameterization, it can be also used for calculations in the passive management case. The packet arrival stream is modeled by the Markovian arrival process [39], which is very robust as it can approximate with high accuracy any interarrival time distribution, with any autocorrelation. The function for packet dropping probabilities is general in the model and can assume any form.

The only cost we have to pay for these direct derivations is that the service time distribution (which is proportional to the packet size distribution) is approximated by the exponential distribution. Fortunately, this should not constitute a big problem in practice for the following reason. Packet sizes are strictly limited by the maximum transmission unit (MTU), e.g., in the 40-1500 bytes range. Therefore, if only the traffic is not dominated by small packets, then the coefficient of variation of the packet size is less than 1. Consider, for instance, the following pessimistic scenario: 50% of packets are of size 40 bytes, 50% of size 1500 bytes. The average packet size is in this case 770 bytes, while the coefficient of variation is 0.948. The analogous coefficient of variation for the exponential distribution is 1. Therefore, we can expect the queue sizes obtained using the exponential service time distribution, to be not far from the real ones, in this pessimistic

 ISBN: 978-1-68558-147-3

scenario. In reality, the coefficient of variation of the packet size is usually significantly less than 1, often less than 0.5. Therefore, the discrepancy between the queue sizes obtained using the exponential approximation should be on the side of pessimistic overestimation.

The rest of the paper is structured as follows. In Section II, the details of the queueing model are presented. In Section III, the actual analysis of the queue size distribution, its average value and the standard deviation, are carried out. The section is divided into two parts, devoted to the steady-state analysis, and the transient analysis, respectively. In Section IV, numerical examples are presented. They include stationary and transient results, with full distributions accompanied by average values and standard deviations. Moreover, the convergence to the stationary state is demonstrated. The final conclusions are gathered in Section V.

II. The model

The buffer is modeled herein by the single-server queueing system of finite capacity, namely, the packets arrive to the buffer according to the arrival process, which will be defined below. In the buffer, they form a queue in the arrival order, in a First-In-First-Out (FIFO) manner. The packets are served and removed from the head of the queue by the transmission process. Packet transmission time is random and exponentially distributed with parameter μ. The capacity of the system (buffer) is K packets, which includes the one being transmitted, if applicable. If upon a packet arrival the buffer is full, the arriving packet is dropped. Moreover, every arriving packet can be dropped even if the buffer is not full. This happens with probability $d(n)$, where n is the queue size upon arrival of this packet. Function $d(n)$ can have any form if only it meets $0 \leq d(n) \leq 1$ for every n.

The packet arrival proces is modeled by the Markovian arrival process [39]. This process has an internal modulating Markov chain with m states, which can modulate the actual interarrival times to have a complicated form of the distribution of interarrival times and autocorrelation.

In practice, the Markovian arrival process is parameterized by two $m \times m$ matrices, D_0 and D_1. Diagonal elements of D_1 cover arrivals of packets without switching the modulating state, while off-diagonal elements cover arrivals of packets accompanied by switching the modulating state. Off-diagonal entries of D_0 cover switching the modulating state without arrivals.

More on the properties and detailed characteristics of the Markovian arrival process can be found in [39].

In what follows, by $X(t)$ we will denote the queue size (in packets) at the time t, including the one being transmitted, if applicable. By $J(t)$ we will denote the modulating state at the time t, i.e. the state of the internal Markov chain. The space of possible states is $\{1, \ldots, m\}$.

III. Queue size analysis

First of all, we can notice that the two-dimensional process, $(X(t), J(t))$ constitutes a continuous-time Markov chain in the considered model.

Indeed, at any particular moment in time, t, the evolution of the arrival process after t depends only on $J(t)$, i.e. the modulating state at t. It does not depend on the values of the modulating state before t.

If there is an ongoing service at t, the remaining service time is exponentially distributed with parameter μ, which is a consequence of the memoryless property of the exponential distribution. In other words, the distribution of the remaining service time counting from t does not depend on already passed service.

Finally, future dropping of packets, counting from t, depends only on the current queue size, $X(t)$.

Summarizing, the evolution of the system counting from t depends only on the current queue size, $X(t)$, and the current modulating state, $J(t)$, which makes the process $(X(t), J(t))$ to be a continuous-time Markov chain.

Let Q be the rate matrix of this two-dimensional Markov chain. Obviously, Q must be of size $(K+1)m \times (K+1)m$, because it covers simultaneously changes of the queue size, with possible values $0, \ldots, K$ and changes of the modulating state, with possible values $1, \ldots, m$.

We will derive now the matrix Q.

Firstly, note that when the system is empty, the change of the modulating state without changing the queue size can happen either when there is no packet arrival, which is covered by D_0, or when an arriving packet gets dropped immediately, which is covered by $d(0)D_1$. Therefore, the two possibilities together are covered by the matrix:

$$D_0 + d(0)D_1. \tag{1}$$

When the queue size is $i > 0$, the change of the modulating state without changing the queue size can happen either when there is no packet arrival, which is covered by D_0, or when a packet is dropped instantly, which is covered by $d(i)D_1$. Moreover, we have to exclude the service completion events, happening with intensity μ. Therefore, the three possibilities together are covered by the matrix:

$$D_0 + d(i)D_1 - \mu I, \tag{2}$$

where I is the $m \times m$ identity matrix.

A successful packet arrival event, with perhaps a change of the modulating state, is covered by the matrix:

$$(1 - d(i))D_1. \tag{3}$$

In this case, the queue size increases by 1.

Then, any non-zero queue size can be decreased by 1 at any time by the service process. This can happen only without a change of the modulating state and is covered by the matrix:

$$\mu I. \tag{4}$$

In the considered model, instant changes of the queue size by more than 1 are impossible, neither up nor down. Therefore, such events have probability 0.

Summarizing these considerations, we obtain matrix Q in the following form:

$$Q = [Q_{ij}]_{i,j=0,\ldots,K}, \tag{5}$$

$$Q_{ij} = \begin{cases} D_0 + d(0)D_1, & \text{if } \ i=0, j=0, \\ D_0 + d(i)D_1 - \mu I, & \text{if } \ i=j, i>0, \\ (1-d(i))D_1, & \text{if } \ i=j-1, \\ \mu I, & \text{if } \ i=j+1, \\ \mathrm{O}, & \text{if } \ i>j+1, \\ \mathrm{O}, & \text{if } \ i<j-1, \end{cases} \tag{6}$$

where each entry of Q is an $m \times m$ submatrix, giving its total size $(K+1)m \times (K+1)m$. O in (6) denotes the zero matrix.

It is also useful to present Q in a graphical form. Namely, from (6) we have:

$$Q = \begin{pmatrix} U & Z_0 & \mathrm{O} & \mathrm{O} & \cdots & \mathrm{O} \\ \mu I & Y_1 & Z_1 & \mathrm{O} & \cdots & \mathrm{O} \\ \mathrm{O} & \mu I & Y_2 & Z_3 & \cdots & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mu I & Y_3 & \ddots & \mathrm{O} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathrm{O} & \mathrm{O} & \mathrm{O} & \cdots & \mu I & Y_K \end{pmatrix}, \tag{7}$$

with

$$U = D_0 + d(0)D_1, \tag{8}$$

$$Y_i = D_0 + d(i)D_1 - \mu I, \quad i \geq 1, \tag{9}$$

$$Z_i = (1-d(i))D_1, \quad i \geq 0. \tag{10}$$

A. Stationary solution

Having rate matrix Q, we can obtain the stationary distribution of the queue size and the modulating state:

$$q_{nj} = \lim_{t\to\infty} \mathbb{P}(X(t)=n, J(t)=j|X(0)=k, J(0)=i), \tag{11}$$

using the system of linear equations:

$$qQ = [0,\ldots,0], \qquad \sum_{n=0}^{K}\sum_{j=1}^{m} q_{nj} = 1, \tag{12}$$

where

$$q = [q_{01},\ldots,q_{0m},q_{11},\ldots,q_{1m},\ldots,q_{K1},\ldots,q_{Km}]. \tag{13}$$

It is known that system (13) has a unique solution, if the Markov chain is finite and aperiodic, as in our case [1]. It is also known that this solution does not depend on initial conditions, $X(0)=k$ and $J(0)=i$.

Having computed vector q, we can obtain easily the distribution of the queue size in the stationary state. Defining:

$$p_n = \lim_{t\to\infty} \mathbb{P}(X(t)=n|X(0)=k, J(t)=i), \tag{14}$$

we have

$$p_n = \sum_{j=1}^{m} q_{nj}. \tag{15}$$

The average queue size in the stationary state, A, equals:

$$A = \sum_{n=0}^{K} n \sum_{j=1}^{m} q_{nj}, \tag{16}$$

while the standard deviation of the queue size in the stationary state, S, is:

$$S = \sqrt{\sum_{n=0}^{K} n^2 \sum_{j=1}^{m} q_{nj} - A^2}. \tag{17}$$

Finally, the stationary state distribution of the modulating state can be also obtained. Namely, we have:

$$r_j = \lim_{t\to\infty} \mathbb{P}(J(t)=j|X(0)=k, J(t)=i) = \sum_{n=0}^{K} q_{nj}. \tag{18}$$

B. Transient solution

Having rate matrix Q, we can obtain also the distribution of the queue size and the modulating state at any time. Defining:

$$q_{kinj}(t) = \mathbb{P}(X(t)=n, J(t)=j|X(0)=k, J(t)=i), \tag{19}$$

we have:

$$q_{kinj}(t) = \left[e^{Qt}\right]_{(k\cdot m+i, n\cdot m+j)}, \tag{20}$$

where e^{Qt} is the matrix exponential, while

$$\left[M\right]_{(a,b)}$$

denotes the (a,b) entry of matrix M.

From $q_{kinj}(t)$, we can obtain the distribution of the queue size at the time t, i.e.:

$$p_{kin}(t) = \mathbb{P}(X(t)=n|X(0)=k, J(t)=i). \tag{21}$$

Namely, we have

$$p_{kin}(t) = \sum_{j=1}^{m} q_{kinj}(t). \tag{22}$$

Finally, defining $A_{ki}(t)$ to be the average queue size at the time t, given that $X(0)=k$ and $J(t)=i$, we obtain:

$$A_{ki}(t) = \sum_{n=0}^{K} n \sum_{j=1}^{m} q_{kinj}(t). \tag{23}$$

The standard deviation of the average queue size at the time t, given that $X(0)=k$ and $J(t)=i$, is equal to:

$$S_{ki}(t) = \sqrt{\sum_{n=0}^{K} n^2 \sum_{j=1}^{m} q_{kinj}(t) - (A_{ki}(t))^2}. \tag{24}$$

C. Special case - passive buffer management

It is easy to see that the presented model can be applied to the passive buffer management as well. If we set:

$$d(n) = \begin{cases} 0, & \text{for} \quad n < K, \\ 1, & \text{for} \quad n \geq K, \end{cases} \tag{25}$$

than the resulting model is equivalent to the classic, tail-drop buffer management. All the presented derivations and formulas remain valid, because in the definition of the model given in Section II, function $d(n)$ has an arbitrary form.

 ISBN: 978-1-68558-147-3

D. Special cases - arrival processes of various types

The complex Markovian arrival process assumed in the definition of the model in Section II can be simplified when needed. For instance:

- setting $m = 1$ and $D_0 = -\lambda$, $D_1 = \lambda$, we get the Poisson process;
- setting $D_0 = T$ and $D_1 = -T\mathbf{1}\alpha$, we get the renewal process, in which the interarrival time distribution is of phase type (a phase-type distribution can approximate any distribution with arbitrary accuracy);
- setting $D_1 = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$ we get the Markov-modulated Poisson process, which is a simple and popular model of autocorrelated traffic.

Again, all the presented derivations and formulas remain valid in the each case listed above.

IV. Numerical examples

In the examples, the following parameterization of the Markovian arrival process is used:

$$D_0 = \begin{bmatrix} -0.4395723602 & 0.03517495366 & 0.01134675916 \\ 0.04652171271 & -0.6814139155 & 0.04652171271 \\ 0.01248143512 & 0.01248143523 & -2.4156995522 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.21503476203 & 0.09914223374 & 0.07887365167 \\ 0.04725831044 & 0.48250711000 & 0.05860506971 \\ 0.13318251133 & 0.04765638867 & 2.20989778185 \end{bmatrix}.$$

These matrices describe a moderately autocorrelated stream, with the 1-lag autocorrelation of 0.188 and the rate of 1.1. Moreover, it is assumed that the packet transmission rate is 1. Thus, the queue is slightly overloaded, with $\rho = 1.1$. To reduce this overload, the following active buffer management is used in the system (see Figure 1):

$$d(n) = \begin{cases} 0, & \text{for} \quad n < 5, \\ 0.0002(n-5)^3, & \text{for} \quad 5 \le n < 20, \\ 1, & \text{for} \quad n \ge 20, \end{cases} \qquad (26)$$

which is a third-degree polynomial, suggested in [22]. The buffer size is $K = 20$.

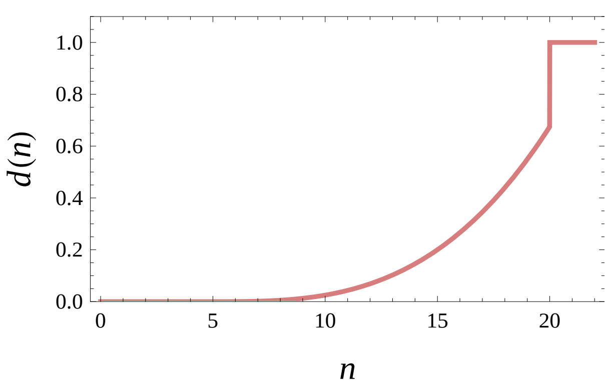

Figure 1. Function $d(n)$ used in numerical examples.

In Table I, the average queue size and its standard deviation are presented at different moments in time. (The initial queue size was 0, while the initial modulating state was 1 in this calculations). The results for t up to 1000 were obtained from formulas (23) and (24), while the results for $t = \infty$ were obtained from (16) and (17), respectively.

In Figures 2 and 3, we can see the evolution of the average queue size and its standard deviation in time for the initial queue size of 10 and the initial modulating state of 1.

As we can see in Table I and Figure 2, the full convergence of the average value to the stationary state takes about 120s, while the standard deviation stabilized much quicker than that. It is interesting that in the case of the initial queue of 10 packets (i.e. 50% of the buffer), the average queue size is not monotonic (see Figure 2). It is decreasing during the first 10s interval, then begins to increase until the stationary state is

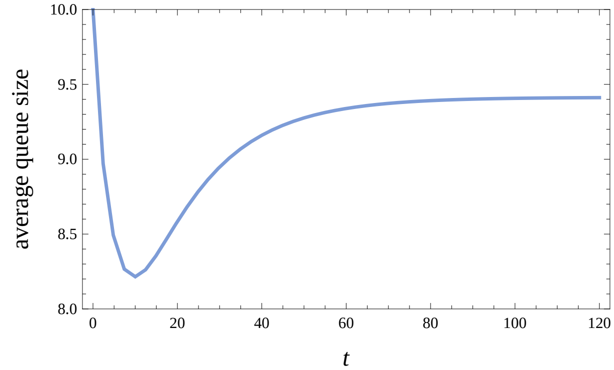

Figure 2. Average queue size in time for $X(0) = 10$ and $J(0) = 1$.

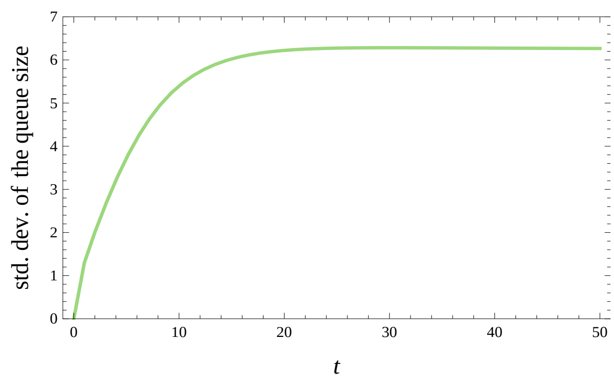

Figure 3. Std. dev. of the queue size in time for $X(0) = 10$ and $J(0) = 1$.

TABLE I
Average queue size and its standard deviation in time for $X(0) = 0$ and $J(0) = 1$.

	average queue size at t	std. dev. of the queue size at t
$t = 0.1$	0.038	0.201
$t = 1.0$	0.346	0.734
$t = 10$	3.630	4.253
$t = 100$	9.389	6.254
$t = 1000$	9.413	6.251
$t = \infty$	9.413	6.251

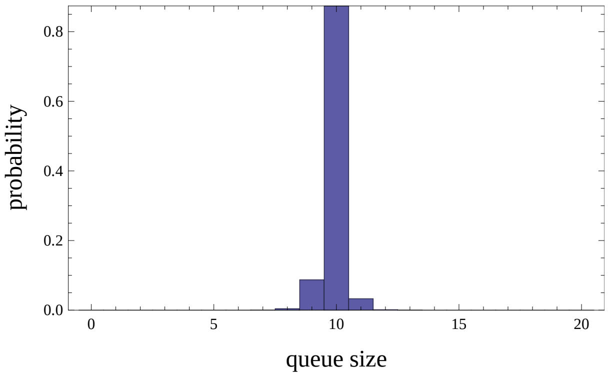

Figure 4. Distribution of the queue size at $t = 0.1$. Initial conditions: $X(0) = 10$ and $J(0) = 1$.

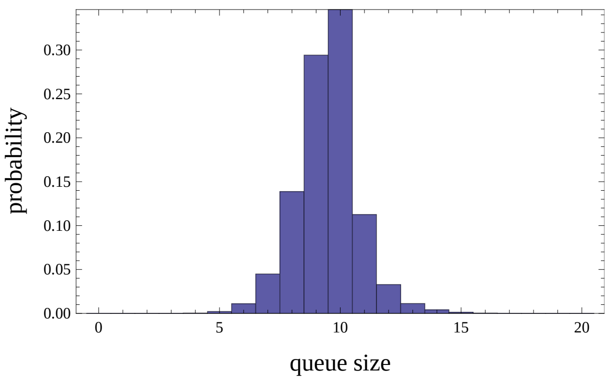

Figure 5. Distribution of the queue size at $t = 1.0$. Initial conditions: $X(0) = 10$ and $J(0) = 1$.

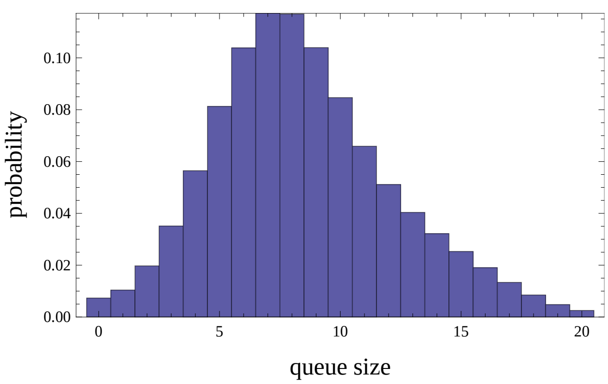

Figure 6. Distribution of the queue size at $t = 5.0$. Initial conditions: $X(0) = 10$ and $J(0) = 1$.

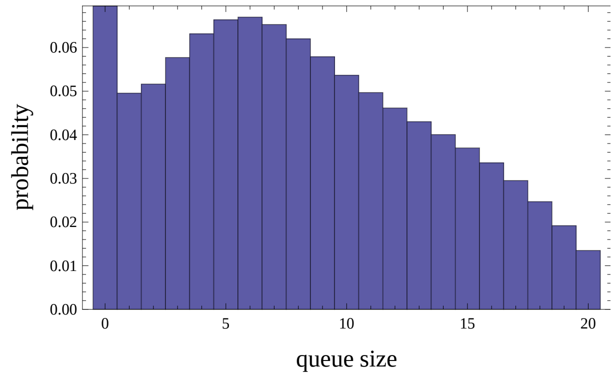

Figure 7. Distribution of the queue size at $t = 10$. Initial conditions: $X(0) = 10$ and $J(0) = 1$.

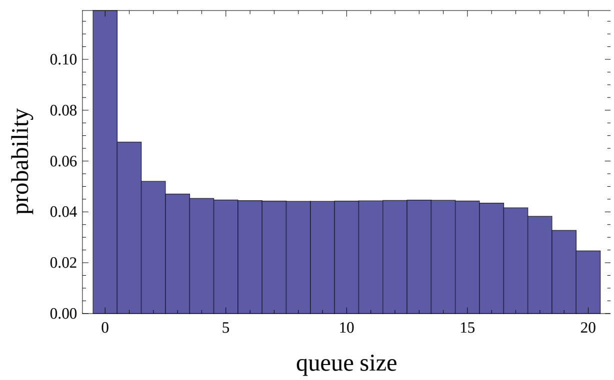

Figure 8. Distribution of the queue size at $t = 20$. Initial conditions: $X(0) = 10$ and $J(0) = 1$.

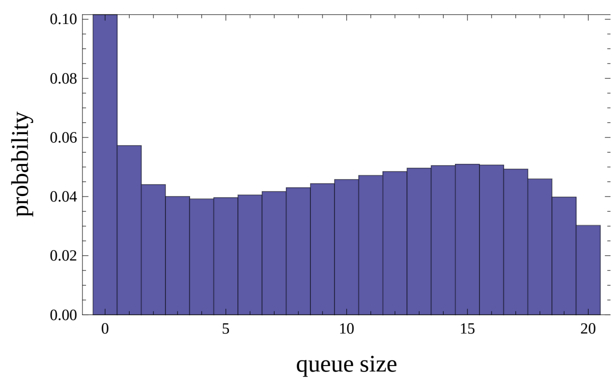

Figure 9. Stationary distribution of the queue size ($t = \infty$).

reached.

Now, in Figures 4–8, the full distribution of the queue size is depicted at different times, while in Figure 9 the stationary distribution is shown. Figures 4–8 were obtained by means of formulas (22) and (20), while Figure 9 by means of (12) and (15).

In these figures, we can track the evolution of the shape of the queue size distribution towards the stationary distribution. Namely, very early, the probability mass is concentrated around the initial queue size, equal 10 (see Figures 4 and 5). Then, it becomes more spread out – see Figure 6. After some more time, a peak at 0 occurs (Figure 7). At $t = 20$, the distribution starts to resemble quite well the stationary distribution – compare Figures 8 and 9.

V. Conclusions

In this paper, we derived transient and stationary distribution of the queue size in a packet buffer, as well as its average value and standard deviation. The derivations were carried out in the time domain, without previous application of the Laplace transform. The analyzed model can be used for both passive buffer management (tail drop) and active buffer management, in which the dropping probability is a function of the queue size. It can be used for many traffic types, of different statistical properties, including the Poisson process, phase-type renewal process, Markov-modulated Poisson process, and others. Theoretical results are illustrated with numerical examples.

The only significant simplification of the model was approximation of the service time by the exponential distribution. This should not constitute a big problem in practice, because the real coefficient of variation of the packet size is usually smaller than the coefficient of variation of the exponential distribution. Therefore, the error caused by this approximation should be on the side of pessimistic overestimation of the queue size.

Acknowledgement

This research was funded by National Science Centre, Poland, grant number 2020/39/B/ST6/00224.

References

[1] L. Kleinrock, "Queueing Systems: Theory", John Wiley and Sons New York, 1975.

[2] D. M. Lucantoni, "New results on the single server queue with a batch Markovian arrival process", Commun. Stat., Stochastic Models, vol. 7, no. 1, pp. 1–46, 1991.

[3] W. Fischer and K. Meier-Hellstern, "The Markov-modulated Poisson process (MMPP) cookbook", Performance Evaluation, vol. 18, no. 2, pp. 149–171, 1992.

[4] A. N. Dudin, A. A. Shaban, and V. I. Klimenok, "Analysis of a queue in the BMAP/G/1/N system", International Journal of Simulation: Systems, Science and Technology, vol. 6, no. 1–2, pp. 13–23, 2005.

[5] T. Hofkens, K. Spaey, and C. Blondia, "Transient Analysis of the D-BMAP/G/1 Queue with an Application to the Dimensioning of a Playout Buffer for VBR Video", Proceedings of Networking'04. Lecture Notes in Computer Science, vol. 3042, pp. 1338–1343, 2004.

[6] A. Chydzinski, "Transient Analysis of the MMPP/G/1/K Queue", Telecommunication Systems, vol. 32, no. 4, pp. 247–262, 2006.

[7] A. Chydzinski, "On the Queue Length Distribution in BMAP Systems", Journal of Communications Software and Systems, vol. 2, no. 4, pp. 275–282, 2006.

[8] Internet Engineering Task Force, "Request for Comments 7567: ETF Recommendations Regarding Active Queue Management" F. Baker and G. Fairhurst, Eds.; 2015.

[9] Internet Engineering Task Force, "Request for Comments 7928: Characterization Guidelines for Active Queue Management (AQM)", N. Kuhn, F. Natarajan, and N. Khademi, Eds.; 2016.

[10] G. Appenzeller, I. Keslassy, and N. McKeown, "Sizing router buffers", ACM SIGCOMM Computer Communication Review, vol. 34, pp. 281–292, 2004.

[11] B. Spang, S. Arslan, and N. McKeown, "Updating the theory of buffer sizing", Performance Evaluation, vol. 151,102232, 2021.

[12] K. Nichols and Van Jacobson, "Controlling Queue Delay", Queue, vol. 55, no. 7, pp. 42–50, 2012.

[13] R. Pan, P. Natarajan, C. Piglione, and M. S. Prabhu, "PIE: A lightweight control scheme to address the bufferbloat problem", IEEE International Conference on High Performance Switching and Routing, pp. 148–155, 2013.

[14] G. Kahe and A. H. Jahangir, "A self-tuning controller for queuing delay regulation in TCP/AQM networks", Telecommunication Systems, vol. 71, no. 2, pp. 215–229, 2019.

[15] S. B. Alaoui, E. H. Tissir, and N. Chaibi, "Analysis and design of robust guaranteed cost Active Queue Management", Computer Communications, vol. 159, pp. 124–1321, 2020.

[16] H. Wang, "Trade-off queuing delay and link utilization for solving bufferbloat", ICT Express, vol. 6, no. 4, pp. 269–272, 2020.

[17] R. Hotchi, "Active queue management supporting TCP flows using disturbance observer and Smith predictor", IEEE Access, vol. 8, pp. 173401–173413, 2020.

[18] C. Wang, X. Chen, J. Cao, and J. Qiu, "Neural Network-Based Distributed Adaptive Pre-Assigned Finite-Time Consensus of Multiple TCP/AQM Networks", IEEE Transactions on Circuits and Systems, vol. 68, no. 1, pp. 387–395, 2021.

[19] J. Shen, Y. Jing, and T. Ren, "Adaptive finite time congestion tracking control for TCP/AQM system with input saturation", International Journal of Systems Science, vol. 53, no. 2, pp. 253–264, 2022.

[20] S. Floyd and V. Jacobson, "Random early detection gateways for congestion avoidance", IEEE/ACM Transactions on Networking, vol. 1, no. 4, pp. 397–413, 1993.

[21] K. Zhou, K. L. Yeung, and V. Li, "Nonlinear RED: asimple yet efficient active queue management scheme", Computer Networks, vol. 50, pp. 3784–3784, 2006.

[22] J. Domanska, D. Augustyn, and A. Domanski, "The choice of optimal 3-rd order polynomial packet dropping function for NLRED in the presence of self-similar traffic", Bulletin of the Polish Academy of Sciences. Technical Sciences, vol. 60, no. 4, pp. 779–786, 2012.

[23] A. Gimenez, M. A. Murcia, J. M. Amigo, O. Martinez-Bonastre, and J. Valero, "New RED-Type TCP-AQM Algorithms Based on Beta Distribution Drop Functions", Applied Sciences, vol. 12, no. 21, 2022.

[24] C. Feng, L. Huang, C. Xu, and Y. Chang, "Congestion Control Scheme Performance Analysis Based on Nonlinear RED" IEEE Systems Journal, vol. 11, no. 4, pp. 2247–2254, 2017.

[25] S. Patel and Karmeshu, "A New Modified Dropping Function for Congested AQM Networks", Wireless Personal Communications, vol. 104, no. 1, pp. 37–55, 2019.

[26] T. Bonald, M. May, and J.-C. Bolot, "Analytic evaluation of RED performance", Proceedings of INFOCOM, pp. 1415–1424, 2000.

[27] A. Chydzinski and L. Chrost, "Analysis of AQM queues with queue-size based packet dropping", International Journal of Applied Mathematics and Computer Science, vol. 21, no. 3, pp. 567-–577, 2011.

[28] W. M. Kempa, "A direct approach to transient queue-size distribution in a finite-buffer queue with AQM", Applied Mathematics & Information Sciences, vol. 7, no. 3, pp. 909-915, 2013.

[29] W. M. Kempa, "Time-dependent queue-size distribution in the finite GI/M/1 model with AQM-type dropping", Acta Electrotech. Et Inform., vol. 13, pp. 85–90. 2013.

[30] A. Chydzinski, M. Barczyk, and D. Samociuk, "The Single-Server Queue with the Dropping Function and Infinite Buffer", Mathematical Problems in Engineering, vol. 2018, ID 3260428, pp. 1–12, 2018.

[31] A. Chydzinski, "Queues with the dropping function and non-Poisson arrivals", IEEE Access, vol. 8, no. 1, pp. 39819–39829, 2020.

[32] A. Chydzinski, "Non-Stationary Characteristics of AQM Based on the Queue Length", Sensors, vol. 23, no. 1, 485, 2023.

[33] P. Mrozowski and A. Chydzinski, "Queues with dropping functions and autocorrelated arrivals", Methodology and Computing in Applied Probability, vol. 20, no. 1, pp. 97–115, 2018.

[34] A. Chydzinski and B. Adamczyk, "Queues with the dropping function and general service time", PLoS ONE, vol. 14, no. 7, e0219444, pp.1-21, 2019.

[35] A. Chydzinski, "Waiting Time in a General Active Queue Management Scheme", IEEE Access, vol. 11, pp. 66535-66543, 2023.

[36] A. Chydzinski, "Output Stream from the AQM Queue with BMAP Arrivals", Journal of Sensor and Actuator Networks, vol. 13(1), no. 4, pp. 1-18, 2024.

[37] A. Chydzinski, "Throughput of the Queue With Probabilistic Rejections", IEEE Access, vol. 11, pp. 138141-138150, 2023.

[38] A. Chydzinski, "Loss Process at an AQM Buffer", Journal of Sensor and Actuator Networks, vol. 12(4), no. 55, pp. 1-17, 2023.

[39] A. N. Dudin, V. I. Klimenok, and V. M. Vishnevsky, "The Theory of Queuing Systems with Correlated Flows", Springer Cham, 2020.

 ISBN: 978-1-68558-147-3