Zero-Sum Games with Distributionally Robust Chance Constraints

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***Abstract*—We consider a two-player zero-sum game with random linear chance constraints whose distributions are known to belong to moments based uncertainty sets. We show that a Saddle Point Equilibrium problem is equivalent to a primal-dual pair of Second-Order Cone Programs. The game with chance constraints can be used in various applications, e.g., risk constraints in portfolio optimization, resource constraints in stochastic shortest path problem, renewable energy aggregators in the local market.**

***Keywords*—Distributionally robust chance constraints, Zero-sum game, Saddle point equilibrium, Second-order cone program.**

I. Introduction

A two-player zero-sum game is defined using a single payoff function, where one player plays the role of maximizer and another player plays the role of minimizer. More commonly, a zero-sum game is introduced with a payoff matrix, where the rows and the columns are the actions of player 1 and player 2, respectively. A Saddle Point Equilibrium (SPE) is the solution concept to study the zero-sum games and it exists in the mixed strategies [1]. Dantzig and later Adler showed the equivalence between linear programming problems and two-player zero-sum games [2] [3]. Charnes [4] generalized the zero-sum game considered in [1] by introducing linear inequality constraints on the mixed strategies of both the players and called it a constrained zero-sum game. An SPE of a constrained zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs [4]. Singh and Lisser [5] considered a stochastic version of constrained zero-sum game considered by Charnes [4], where the mixed strategies of each player are restricted by random linear inequality constraints, which are modelled using chance constraints. When the random constraint vectors follow a multivariate elliptically symmetric distribution, the zero-sum game problem is equivalent to a primal-dual pair of Second-Order Cone Programs (SOCPs) [5].

Nash equilibrium is the generalization of SPE and it is used as a solution concept for the general-sum games [6] [7]. Under certain conditions on payoff functions and strategy sets, there always exists a Nash equilibrium [8]. The general-sum games under uncertainties are considered in the literature [9]–[13], which capture both risk neutral and risk averse situations.

In this paper, we consider a more general two player zero-sum game as compared to [5]. Unlike in [5], the strategy set of each player is defined by a compact polyhedral set, which is further restricted by some random linear inequalities and the information on the distribution of the random constraint vectors is not exactly known. We consider two different uncertainty sets based on the partial information on the mean vectors and covariance matrices of the random constraint vectors. We show that, there exists an SPE of the game Z_α and an SPE problem is equivalent to a primal-dual pair of SOCPs.

The rest of the paper is organized as follows. The definition of a distributionally robust zero-sum game is given in Section II. Section III presents the reformulation of distributionally robust chance constraints as second order cone constraints under two different uncertainty sets. Section IV outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game.

II. The model

We consider a two player zero-sum game, where each player has continuous strategy set. Let $C^1 \in \mathbb{R}^{K_1 \times m}$, $C^2 \in \mathbb{R}^{K_2 \times n}$, $d^1 \in \mathbb{R}^{K_1}$ and $d^2 \in \mathbb{R}^{K_2}$. We consider $X = \{x \in \mathbb{R}^m \mid C^1 x = d^1, x \geq 0\}$ and $Y = \{y \in \mathbb{R}^n \mid C^2 y = d^2, y \geq 0\}$ as the strategy sets of player 1 and player 2, respectively. We assume that X and Y are compact sets. Let $u : X \times Y \to \mathbb{R}$ be a payoff function associated to the zero-sum game and we assume that player 1 (resp. player 2) is interested in maximizing (resp. minimizing) $u(x, y)$ for a fixed strategy y (resp. x) of player 2 (resp. player 1). For a given strategy pair $(x, y) \in X \times Y$, the payoff function $u(x, y)$ is given by

$$u(x, y) = x^T G y + g^T x + h^T y, \qquad (1)$$

where $G \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^n$. The first term of (1) results from the interaction between both the players whereas the second and third term represents the individual impact of

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player 1 and player 2 on the game, respectively. The strategy sets are often restricted by random linear constraints, which are modelled using chance constraints. The chance constraint based strategy sets appear in many practical problems, e.g., risk constraints in portfolio optimization [14]. In this paper, we consider the case, where the strategies of player 1 satisfy the following random linear constraints,

$$(a_k^1)^T x \leq b_k^1,\ k = 1, 2, \ldots, p, \tag{2}$$

whilst the strategies of player 2 satisfy the following random linear constraints

$$(a_l^2)^T y \geq b_l^2,\ l = 1, 2, \ldots, q. \tag{3}$$

Let $\mathcal{I}_1 = \{1, 2, \ldots, p\}$ and $\mathcal{I}_2 = \{1, 2, \ldots, q\}$ be the index sets for the constraints of player 1 and player 2, respectively. For each $k \in \mathcal{I}_1$ and $l \in \mathcal{I}_2$, the vectors a_k^1 and a_l^2 are random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the case, where the only information we have about the distributions of a_k^1 and a_l^2 is that they belong to some uncertainty sets $\mathcal{D}_k^1$ and $\mathcal{D}_l^2$, respectively. The uncertainty sets $\mathcal{D}_k^1$ and $\mathcal{D}_l^2$, are constructed based on the partially available information on the distributions of a_k^1 and a_l^2, respectively. Using the worst case approach, the random linear constraints (2) and (3) can be formulated as distributionally robust chance constraints given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\left((a_k^1)^T x \leq b_k^1\right) \geq \alpha_k^1,\ \forall\, k \in \mathcal{I}_1, \tag{4}$$

and

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\left((-a_l^2)^T y \leq -b_l^2\right) \geq \alpha_l^2,\ \forall\, l \in \mathcal{I}_2, \tag{5}$$

where α_k^1 and α_l^2 are the confidence levels of player 1 and player 2 for kth and lth constraints, respectively. Therefore, for a given $\alpha^1 = (\alpha_k^1)_{k \in \mathcal{I}_1}$ and $\alpha^2 = (\alpha_l^2)_{l \in \mathcal{I}_2}$, the feasible strategy sets of player 1 and player 2 are given by

$$S^1_{\alpha^1} = \Big\{x \in X \mid \inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\{(a_k^1)^T x \leq b_k^1\} \geq \alpha_k^1,\ \forall\, k \in \mathcal{I}_1\Big\}, \tag{6}$$

and

$$S^2_{\alpha^2} = \Big\{y \in Y \mid \inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\{(-a_l^2)^T y \leq -b_l^2\} \geq \alpha_l^2,\ \forall\, l \in \mathcal{I}_2\Big\}. \tag{7}$$

We call the zero-sum game with the strategy set $S^1_{\alpha^1}$ for player 1 and the strategy set $S^2_{\alpha^2}$ for player 2 as a distributionally robust zero-sum game. We denote this game by Z_α. A strategy pair $(x^*, y^*) \in S^1_{\alpha^1} \times S^2_{\alpha^2}$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0,1]^p \times [0,1]^q$, if

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y),\ \forall\, x \in S^1_{\alpha^1},\ y \in S^2_{\alpha^2}.$$

III. Reformulation of distributionally robust chance constraints

We consider two different uncertainty sets based on the partial information about the mean vectors and covariance matrices of the random constraint vectors a_k^i, $i = 1, 2$, $k \in \mathcal{I}_i$. For each uncertainty set, distributionally robust chance constraints (4) and (5) are reformulated as second-order cone constraints.

A. Moments Based Uncertainty Sets

For each player i, $i = 1, 2$, we consider the case, where the mean vector and covariance matrix of the random vector a_k^i for all $k \in \mathcal{I}_i$ are known to belong to polytopes $U_{\mu_k^i}$ and $U_{\Sigma_k^i}$, respectively. We consider polytopes $U_{\mu_k^i} = Conv(\mu_{k1}^i, \mu_{k2}^i, \ldots, \mu_{kM}^i)$ and $U_{\Sigma_k^i} = Conv(\Sigma_{k1}^i, \Sigma_{k2}^i, \ldots, \Sigma_{kM}^i)$, where $\Sigma_{kj}^i \succ 0$, $j = 1, 2, \ldots M$; $Conv$ denotes the convex hull and $\Sigma_{kj}^i \succ 0$ implies that Σ_{kj}^i is a positive definite matrix. For each $i = 1, 2$, and $k \in \mathcal{I}_i$, the uncertainty set for the distribution of a_k^i is defined by

$$\mathcal{D}_k^i\left(\mu_k^i, \Sigma_k^i\right) = \left\{F_k^i \;\middle|\; \begin{array}{c} E_{F_k^i}\left[a_k^i\right] \in U_{\mu_k^i} \\ COV_{F_k^i}[a_k^i] \in U_{\Sigma_k^i} \end{array}\right\}, \tag{8}$$

where $E_{F_k^i}$ and $COV_{F_k^i}$ are expectation and covariance operator under probability distribution F_k^i, respectively. The uncertainty set (8) is considered in [15]. As for the second uncertainty set, we consider the case, where the mean vector of a_k^i lies in an ellipsoid of size $\gamma_{k1}^i \geq 0$ centered at μ_k^i and the covariance matrix of a_k^i lies in a positive semidefinite cone defined with a linear matrix inequality. It is defined by

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \;\middle|\; \begin{array}{l} \left(\mathbb{E}_{F_k^i}[a_k^i] - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \\ \times \left(\mathbb{E}_{F_k^i}[a_k^i] - \mu_k^i\right) \leq \gamma_{k1}^i, \\ \\ COV_{F_k^i}[a_k^i] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right\}, \tag{9}$$

where $\Sigma_k^i \succ 0$ and $\gamma_{k2}^i > 0$; for the given matrices B_1 and B_2, $B_1 \preceq B_2$ implies that $B_2 - B_1$ is a positive semidefinite matrix. The uncertainty set (9) is considered in [16].

B. Second-order cone constraint reformulation

We show that the distributionally robust chance constraints (4) and (5) are equivalent to second-order cone constraints for the uncertainty sets defined by (8) and (9).

Lemma III.1. *For each $i = 1, 2$, and $k \in \mathcal{I}_i$, let the distribution F_k^i of random vector a_k^i, lies in uncertainty set $\mathcal{D}_k^i\left(\mu_k^i, \Sigma_k^i\right)$ defined by* (8)*. Then, the constraints* (4) *and* (5) *are equivalent to* (10) *and* (11)*, respectively, given by*

$$\begin{aligned} &(\mu_{kj}^1)^T x + \sqrt{\frac{\alpha_k^1}{1 - \alpha_k^1}}\,||(\Sigma_{kj}^1)^{\frac{1}{2}} x|| \leq b_k^1, \\ &\forall\ j = 1, 2, \ldots, M,\ k \in \mathcal{I}_1, \end{aligned} \tag{10}$$

$$\begin{aligned} &-(\mu_{kj}^2)^T y + \sqrt{\frac{\alpha_k^2}{1 - \alpha_k^2}}\,||(\Sigma_{kj}^2)^{\frac{1}{2}} y|| \leq -b_k^2, \\ &\forall\ j = 1, 2, \ldots, M,\ k \in \mathcal{I}_2. \end{aligned} \tag{11}$$

Proof. Based on the structure of uncertainty set (8), (4) can be written as

$$\inf_{(\mu, \Sigma) \in \mathcal{U}_k^1} \inf_{F_k^1 \in \mathcal{D}(\mu, \Sigma)} \mathbb{P}\big\{(a_k^1)^T x \leq b_k^1\big\} \geq \alpha_k^1,$$

where

$$\mathcal{D}(\mu,\Sigma)=\left\{F_k^1 \left|\mathbb{E}_{F_k^1}[a_k^1]=\mu, COV_{F_k^1}[a_k^1]=\Sigma\right.\right\},$$

and

$$\mathcal{U}_k^1=\left\{(\mu,\Sigma)\ \left|\mu\in U_{\mu_k^1},\Sigma\in U_{\Sigma_k^1}\right.\right\}.$$

The bound of one-sided Chebyshev inequality can be achieved by a two-point distribution given by equation (2) of [17]. Therefore, we have

$$\inf_{F_k^1\in\mathcal{D}(\mu,\Sigma)}\mathbb{P}\big\{(a_k^1)^T x\le b_k^1\big\}=\begin{cases}1-\frac{1}{1+\frac{(\mu^T x-b_k^1)^2}{(x^T\Sigma x)}},\\ \text{if}\ \ \mu^T x\le b_k^1,\\ \\ 0,\ \ \text{otherwise.}\end{cases}$$

For the case $\mu^T x > b_k^1$,

$$\inf_{F_k^1\in\mathcal{D}(\mu,\Sigma)}\mathbb{P}\big\{a_k^1 x\le b_k^1\big\}=0,$$

which makes constraint (4) infeasible for any $\alpha_1 > 0$. Therefore, for $x\in S_{\alpha_1}^1$ the condition $\mu^T x\le b_k^1$ always holds and the constraint (4) is equivalent to

$$\inf_{(\mu,\Sigma)\in\mathcal{U}_k^1} 1-\frac{1}{1+(\mu^T x-b_k^1)^2/(x^T\Sigma x)}\ge\alpha_k^1,$$

which can be reformulated as

$$\frac{\min\limits_{\mu\in U_{\mu_k^1}}\left(b_k^1-\mu^T x\right)}{\max\limits_{\Sigma\in U_{\Sigma_k^1}}\sqrt{x^T\Sigma x}}\ge\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}. \tag{12}$$

The above inequality (12) can be reformulated as (10). Similarly, we can show that (5) is equivalent to (11). □ □

Lemma III.2. *For each $i=1,2$, and $k\in\mathcal{I}_i$, let the distribution F_k^i of random vector a_k^i, lies in the uncertainty set $\mathcal{D}_k^i\left(\mu_k^i,\Sigma_k^i\right)$ defined by* (9)*. Then, the constraints* (4) *and* (5) *are equivalent to* (13) *and* (14)*, respectively, given by*

$$(\mu_k^1)^T x+\left(\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}\sqrt{\gamma_{k2}^1}+\sqrt{\gamma_{k1}^1}\right)\left\|\left(\Sigma_k^1\right)^{\frac{1}{2}}x\right\|\le b_k^1,$$
$$\forall\ k\in\mathcal{I}_1, \tag{13}$$

$$-(\mu_k^2)^T y+\left(\sqrt{\frac{\alpha_k^2}{1-\alpha_k^2}}\sqrt{\gamma_{k2}^2}+\sqrt{\gamma_{k1}^2}\right)\left\|\left(\Sigma_k^2\right)^{\frac{1}{2}}y\right\|\le -b_k^2,$$
$$\forall\ k\in\mathcal{I}_2. \tag{14}$$

Proof. Based on the structure of the uncertainty set (9), the constraint (4) can be written as

$$\inf_{(\mu,\Sigma)\in\tilde{\mathcal{U}}_k^1}\ \inf_{F_k^1\in\mathcal{D}(\mu,\Sigma)}\mathbb{P}\big\{a_k^1 x\le b_k^1\big\}\ge\alpha_k^1,$$

where

$$\mathcal{D}(\mu,\Sigma)=\left\{F_k^1 \left|\mathbb{E}_{F_k^1}[a_k^1]=\mu, COV_{F_k^1}[a_k^1]=\Sigma\right.\right\}$$

and

$$\tilde{\mathcal{U}}_k^1=\left\{(\mu,\Sigma)\left|\ \begin{array}{l}\left(\mu-\mu_k^1\right)^\top\left(\Sigma_k^1\right)^{-1}\left(\mu-\mu_k^1\right)\le\gamma_{k1}^1,\\ \Sigma\preceq\gamma_{k2}^1\Sigma_k^1.\end{array}\right.\right\}.$$

Using the similar arguments as in the Lemma III.1, the constraint (4) is equivalent to

$$\frac{b_k^1+v_1(x)}{\sqrt{v_2(x)}}\ge\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}, \tag{15}$$

where

$$v_1(x)=\begin{cases}\min\limits_{\mu}-\mu^T x\\ \text{s.t.}\ \ \left(\mu-\mu_k^1\right)^\top\left(\Sigma_k^1\right)^{-1}\left(\mu-\mu_k^1\right)\le\gamma_{k1}^1,\end{cases}$$
$$v_2(x)=\begin{cases}\max\limits_{\Sigma}\ x^T\Sigma x\\ \text{s.t.}\ \Sigma\preceq\gamma_{k2}^1\Sigma_k^1.\end{cases} \tag{16}$$

Let $\beta\ge 0$ be a Lagrange multiplier associated with the constraint of optimization problem (16). By applying the KKT conditions, the optimal solution of (16) is given by $\mu=\mu_k^1+\frac{\sqrt{\gamma_{k1}^1}\Sigma_k^1 x}{\sqrt{x^T\Sigma_k^1 x}}$ and the associated Lagrange multiplier is given by $\beta=\sqrt{\frac{x^T\Sigma_k^1 x}{4\gamma_{k1}^1}}$. Therefore, the corresponding optimal value $v_1(x)=-(\mu_k^1)^T x-\sqrt{\gamma_{k1}^1}\sqrt{x^T\Sigma_k^1 x}$. Since, $u^T\Sigma u\le u^T\gamma_{k2}^1\Sigma_k^1 u$ for any $u\in\mathbb{R}^n$, then, $v_2(x)=\gamma_{k2}^1 x^T\Sigma_k^1 x$. Therefore, using (15), (4) is equivalent to (13). Similarly, we can show that (5) is equivalent to (14). □ □

The reformulation of feasible strategy sets (6) and (7) for uncertainty sets (8) and (9) can be written as

$$S_{\alpha^1}^1=\Big\{x\in X\mid(\mu_{kj}^1)^T x+\kappa_{\alpha_k^1}||(\Sigma_{kj}^1)^{\frac{1}{2}}x||\le b_k^1,$$
$$\forall\ j=1,2,\ldots,M,\ k\in\mathcal{I}_1\Big\}, \tag{17}$$

and

$$S_{\alpha^2}^2=\Big\{y\in Y\mid -(\mu_{lj}^2)^T y+\kappa_{\alpha_l^2}||(\Sigma_{lj}^2)^{\frac{1}{2}}y||\le -b_l^2,$$
$$\forall\ j=1,2,\ldots,M,\ l\in\mathcal{I}_2.\Big\}. \tag{18}$$

For each $i=1,2$, $k\in\mathcal{I}_i$, if $\kappa_{\alpha_k^i}=\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$, (17) and (18) represent the reformulations of (6) and (7) under uncertainty set defined by (8), respectively. For each $i=1,2$, $k\in\mathcal{I}_i$, if $\kappa_{\alpha_k^i}=\left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}\sqrt{\gamma_{k2}^i}+\sqrt{\gamma_{k1}^i}\right)$, and $M=1$, (17) and (18) represent the reformulations of (6) and (7) under uncertainty set defined by (9), respectively.

We assume that the strategy sets (17) and (18) satisfy the strict feasibility condition given by Assumption III.3.

Assumption III.3.
1) *There exists an $x\in S_{\alpha^1}^1$ such that the inequality constraints of $S_{\alpha^1}^1$ defined by* (17) *are strictly satisfied.*
2) *There exists an $y\in S_{\alpha^2}^2$ such that the inequality constraints of $S_{\alpha^2}^2$ defined by* (18) *are strictly satisfied.*

The conditions given in Assumption III.3 are Slater's condition, which are sufficient for strong duality in a convex optimization problem. We use these conditions in order to derive equivalent SOCPs for the zero-sum game Z_α.

IV. Existence and characterization of Saddle Point Equilibrium

In this section, we show that there exists an SPE of the game Z_α if the distributions of the random constraint vectors of both the players belong to the uncertainty sets defined in Section III-A. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game Z_α.

Theorem IV.1. *Consider the game Z_α, where the distributions of the random constraint vectors a_k^i, $k \in \mathcal{I}_i$, $i = 1, 2$, belong to the uncertainty sets described in Section III-A. Then, there exists an SPE of the game for all $\alpha \in (0,1)^p \times (0,1)^q$.*

Proof. Let $\alpha \in (0,1)^p \times (0,1)^q$. For uncertainty sets described in Section III-A, it follows from Lemma III.1 and Lemma III.2 that the strategy sets $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are given by (17) and (18), respectively. It is easy to see that $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are convex and compact sets. The function $u(x,y)$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem [1]. $\square$

A. Equivalent Primal-Dual Pair of Second-Order Cone Programs

From the minimax theorem [1], (x^*, y^*) is an SPE for the game Z_α if and only if

$$x^* \in \arg\max_{x \in S^1_{\alpha^1}} \min_{y \in S^2_{\alpha^2}} u(x,y), \tag{19}$$

$$y^* \in \arg\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x,y). \tag{20}$$

We start with problem $\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x,y)$. The inner optimization problem $\max_{x \in S^1_{\alpha^1}} u(x,y)$ can be equivalently written as

$$\begin{aligned}
&\max_{x, t^1_{k,j}} x^T G y + g^T x + h^T y \\
&\text{s.t.} \\
&(i)\ -x^T \mu^1_{k,j} - \kappa_{\alpha^1_k} \left\| t^1_{k,j} \right\| + b^1_k \geq 0, \\
&\forall\ j = 1, 2 \ldots, M,\ k \in \mathcal{I}_1, \\
&(ii)\ t^1_{k,j} - \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} x = 0,\ \forall\ j = 1, 2 \ldots, M,\ k \in \mathcal{I}_1, \\
&(iii)\ C^1 x = d^1,\ x_r \geq 0,\ \forall\ r = 1, 2, \ldots, m.
\end{aligned} \tag{21}$$

Let $\lambda^1 = \left(\lambda^1_{k,j}\right)_{j=1, k \in \mathcal{I}_1}^M \in \mathbb{R}^{Mp}$, $\delta^1_{k,j} \in \mathbb{R}^m$ for all $k \in \mathcal{I}_1$, $j = 1, 2 \ldots M$, and $\nu^1 \in \mathbb{R}^{K_1}$ be the Lagrange multipliers of constraints $(i), (ii)$, and equality constraints given in (iii) of (21), respectively. Then, the Lagrangian dual problem of the SOCP (21) can be written as

$$\begin{aligned}
&\min_{\lambda_1 \geq 0,\ \delta^1_{k,j},\ \nu^1} \max_{x \geq 0,\ t^1_{k,j}} \Big\{ x^T G y + g^T x + h^T y \\
&+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \Big[\lambda^1_{k,j} \big(-x^T \mu^1_{k,j} - \kappa_{\alpha^1_k} \left\| t^1_{k,j} \right\| + b^1_k \big) \\
&+ (\delta^1_{k,j})^T \Big(t^1_{k,j} - \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} x \Big) \Big] + (\nu^1)^T (d^1 - C^1 x) \Big\} \\
&= \min_{\lambda_1 \geq 0, \delta^1_{k,j}, \nu^1} \Big[\max_{x \geq 0} x^T \Big(G y \\
&- \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \big(\lambda^1_{k,j} \mu^1_{k,j} + \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} \delta^1_{k,j} \big) - (C^1)^T \nu^1 + g \Big) \\
&+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \max_{t^1_{k,j}} \Big((\delta^1_{k,j})^T t^1_{k,j} - \kappa_{\alpha^1_k} \lambda^1_{k,j} \left\| t^1_{k,j} \right\| \Big) + h^T y \\
&+ (\nu^1)^T d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \lambda^1_{k,j} b^1_k \Big].
\end{aligned}$$

The inner maximization problems in the above Lagrangian dual problem will be unbounded unless we have the following dual constraints

$$\begin{aligned}
&G y - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \big(\lambda^1_{k,j} \mu^1_{k,j} + \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} \delta^1_{k,j} \big) \\
&- (C^1)^T \nu^1 + g \leq 0, \\
&||\delta^1_{k,j}|| \leq \kappa_{\alpha^1_k} \lambda^1_{k,j}, \forall\ k \in \mathcal{I}_1, j = 1, 2 \ldots, M.
\end{aligned}$$

Under Assumption III.3, the Lagrangian dual problem of (21) has zero duality gap [18]. Therefore, the problem $\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x,y)$ is equivalent to the following SOCP

$$\begin{aligned}
&\min_{y,\ \nu^1,\ \delta^1_{k,j},\ \lambda^1_{k,j}} h^T y + (\nu^1)^T d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \lambda^1_{k,j} b^1_k \\
&\text{s.t.} \\
&(i)\ G y - \sum_{k \in \mathcal{I}_1} \sum_{j=1}^M \big(\lambda^1_{k,j} \mu^1_{k,j} + \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} \delta^1_{k,j} \big) \\
&- (C^1)^T \nu^1 + g \leq 0, \\
&(ii)\ -(\mu^2_{lj})^T y + \kappa_{\alpha^2_l} ||(\Sigma^2_{lj})^{\frac{1}{2}} y|| \leq -b^2_l, \\
&\forall\ j = 1, 2, \ldots, M,\ l \in \mathcal{I}_2, \\
&(iii)\ ||\delta^1_{k,j}|| \leq \kappa_{\alpha^1_k} \lambda^1_{k,j},\ \lambda^1_{k,j} \geq 0, \\
&\forall\ k \in \mathcal{I}_1,\ j = 1, 2 \ldots, M, \\
&(iv)\ C^2 y = d^2,\ y_s \geq 0,\ \forall\ s = 1, 2, \ldots, n.
\end{aligned} \tag{22}$$

Similarly, problem $\max_{x\in S^1_{\alpha^1}} \min_{y\in S^2_{\alpha^2}} u(x,y)$ is equivalent to the following SOCP

$$\max_{x,\ \nu^2,\ \delta^2_{l,j},\ \lambda^2_{l,j}} \; g^Tx + (\nu^2)^T d^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{M} \lambda^2_{l,j} b^2_l$$

$$\begin{aligned}
&\text{s.t. } (i)\ G^Tx - \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{M}\left(\lambda^2_{l,j}\mu^2_{l,j} + \left(\Sigma^2_{l,j}\right)^{\frac{1}{2}}\delta^2_{l,j}\right)\\
&-(C^2)^T\nu^2 + h \geq 0,\\
&(ii)\ (\mu^1_{kj})^Tx + \kappa_{\alpha^1_k}||(\Sigma^1_{kj})^{\frac{1}{2}}x|| \leq b^1_k,\\
&\forall\ j = 1,2,\ldots,M,\ k\in\mathcal{I}_1,\\
&(iii)\ ||\delta^2_{l,j}|| \leq \kappa_{\alpha^2_l}\lambda^2_{l,j},\ \lambda^2_{l,j}\geq 0,\ \forall\ \ l\in\mathcal{I}_2,\ j=1,2,\ldots M\\
&(iv)\ C^1x = d^1,\ x_r\geq 0,\ \forall\ r=1,2,\ldots,m.
\end{aligned} \tag{23}$$

It follows from the duality theory of SOCPs that (22) and (23) form a primal-dual pair [18].

Remark IV.2. *For each $i = 1,2$, and $k\in\mathcal{I}_i$, if $\kappa_{\alpha^i_k} = \sqrt{\frac{\alpha^i_k}{1-\alpha^i_k}}$,* (22) *and* (23) *represent the primal-dual pair of SOCPs for the uncertainty sets defined by* (8)*. For each $i = 1,2$, and $k\in\mathcal{I}_i$, if $\kappa_{\alpha^i_k} = \left(\sqrt{\frac{\alpha^i_k}{1-\alpha^i_k}}\sqrt{\gamma^i_{k2}} + \sqrt{\gamma^i_{k1}}\right)$ and $M = 1$,* (22) *and* (23) *represent the primal-dual pair of SOCPs for the uncertainty set defined by* (9)*.*

Next, we show that the equivalence between the optimal solutions of (22)-(23) and an SPE of the game Z_α.

Theorem IV.3. *Consider the zero-sum game Z_α, where the feasible strategy sets of player 1 and player 2 are given by* (17) *and* (18)*, respectively. Let Assumption III.3 holds. Then, for a given $\alpha\in(0,1)^p\times(0,1)^q$, (x^*,y^*) is an SPE of the game Z_α if and only if there exists $(\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and $(\nu^{2*},(\delta^{2*}_{l,j})_{l,j},\lambda^{2*})$ such that $(y^*,\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and $(x^*,\nu^{2*},(\delta^{2*}_{l,j})_{l,j},\lambda^{2*})$ are optimal solutions of* (22) *and* (23)*, respectively.*

Proof. Let (x^*,y^*) be an SPE of the game Z_α. Then, x^* and y^* are the solutions of (19) and (20), respectively. Therefore, there exists $(\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and $(\nu^{2*},(\delta^{2*}_{l,j})_{l,j},\lambda^{2*})$ such that $(y^*,\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and $(x^*,\nu^{2*},(\delta^{2*}_{l,j})_{l,j},\lambda^{2*})$ are optimal solutions of (22) and (23) respectively.

Let $(y^*,\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and $(x^*,\nu^{2*},(\delta^{2*}_{l,j})_{l,j},\lambda^{2*})$ be optimal solutions of (22) and (23), respectively. Under Assumption III.3, (22) and (23) are strictly feasible. Therefore, strong duality holds for primal-dual pair (22)-(23). Then, we have

$$\begin{aligned}
&g^Tx^* + (\nu^{2*})^Td^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{M}\lambda^{2*}_{l,j}b^2_l = h^Ty^*\\
&+(\nu^{1*})^Td^1 + \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{M}\lambda^{1*}_{k,j}b^1_k.
\end{aligned} \tag{24}$$

Consider the constraint (i) of (22) at optimal solution $(y^*,\nu^{1*},(\delta^{1*}_{k,j})_{k,j},\lambda^{1*})$ and multiply it by x^T, where $x\in S^1_{\alpha_1}$. Then, by using Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&x^TGy^* + g^Tx + h^Ty^* \leq h^Ty^* + (\nu^{1*})^Td^1\\
&+\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{M}\lambda^{1*}_{k,j}b^1_k,\ \forall\ x\in S^1_{\alpha_1}.
\end{aligned} \tag{25}$$

Similarly, we have

$$\begin{aligned}
&x^{*T}Gy + g^Tx^* + h^Ty \geq g^Tx^*\\
&+(\nu^{2*})^Td^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{M}\lambda^{2*}_{l,j}b^2_l,\ \forall\ y\in S^2_{\alpha_2}.
\end{aligned} \tag{26}$$

Take $x = x^*$ and $y = y^*$ in (25) and (26), then from (24), we get

$$\begin{aligned}
u(x^*,y^*) &= h^Ty^* + (\nu^{1*})^Td^1 + \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{M}\lambda^{1*}_{k,j}b^1_k\\
&= g^Tx^* + (\nu^{2*})^Td^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{M}\lambda^{2*}_{l,j}b^2_l.
\end{aligned} \tag{27}$$

It follows from (25), (26), and (27) that (x^*,y^*) is an SPE of the game Z_α. □ □

V. Conclusion

We show the existence of a mixed strategy SPE for a two-player distributionally robust zero-sum chance-constrained game under three different uncertainty sets based on first two moments. Under Slater's condition, the Saddle Point Equilibria of the game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. The Saddle Point Equilibria of zero-sum games can be computed efficiently because SOCPs are polynomial time solvable. The uncertainty sets considered in the paper have positive semidefinite cone structure, which leads to the reformulation of distributionally robust chance constraints as second order cone constraints. Moreover, these reformulations play a major role in deriving the equivalent primal-dual pair of SOCPs. The tractable reformulation of the zero-sum game problem with different payoff structure, as well as the uncertainty sets other than the ones considered in the paper could be an interesting area for the future research.

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