Impulse Response and Generating Functions of *sinc*^{N} FIR Filters

J. Le Bihan

Laboratoire RESO
Ecole Nationale d'Ingénieurs de Brest (ENIB)
CS 73862, 29238 BREST cedex 3, France
E-mail: jean.le_bihan@enib.fr

Abstract – ***Sinc***^{N} **finite impulse response (FIR) filters are built as a cascade of *N sinc* filters, each of length *M*. They are used for digital signal processing applications, in various areas including telecommunications. Generating functions for the *sinc*^{N} FIR filters transfer functions are given. It is shown that z-Transform techniques can offer an efficient method to derive straightforward recurrence relations for fast computation of the impulse response. Moreover, a simple expression, valid for all the filter coefficients, is also obtained. It is new and general, compared to previously published formula.**

Keywords – digital filter ; impulse response ; coefficient computation ; z-Transform.

I. INTRODUCTION

Different structures for digital filters have been proposed for applications as decimation, interpolation or noise suppression in Sigma-Delta A/D conversion [1, 2]. As suitable filters for high speed operation, cascaded-integrator-comb (CIC) filters [1] and *sinc*^{N} impulse finite response (FIR) filters [3, 4] are among the well known solutions. The use of *sinc*^{N} filters constituted of a cascade of *N sinc* filters, each of length *M*, requires the calculation of all of the impulse response coefficients. So adequate recursions or expressions for rapid calculation of these coefficients have been the subject of numerous investigations, as in [5-7]. Interesting computation aids have already been published: a closed form expression for the first *M* coefficients has been given in [6], while a recurrence formula has been presented in [7].

It has been proved that using z-Transform techniques can be an efficient method to derive digital filter coefficients [8, 9]. Here, employing these techniques, new results are obtained. Simple effective recurrence relations are derived for the computation of the coefficients of *sinc*^{N} FIR filters. In addition, a simple expression is deduced, which is valid not only for the first coefficients of the impulse response but for all of them. In addition, generating functions for the transfer functions of *sinc*^{N} FIR filters are presented.

In this paper, the general form of the transfer function of the *sinc*^{N} FIR filters is given in section II, as well as its useful basic properties. In section III, it is shown how recurrence relations for the filter coefficients can be derived using z-Transform techniques applied to the expression of the transfer function considered as a z-transform. Similarly, the way to obtain an expression for the filter coefficients, new and general compared to formula previously published by other authors, is described in section IV. Finally, in section V, forms of generating functions for the transfer functions of *sinc*^{N} FIR filters are expressed.

II. TRANSFER FUNCTION OF *sinc*^{N} FILTERS

The transfer function $H^{M,N}(z)$ of *sinc*^{N} filters can be written as

$$H^{M,N}(z) = \left(\frac{1}{M} \frac{1-z^{-M}}{1-z^{-1}} \right)^N \tag{1}$$

or

$$H^{M,N}(z) = M^{-N} \left(1 + z^{-1} + z^{-2} + \ldots + z^{-(M-1)} \right)^N \tag{2}$$

to which corresponds the following magnitude response $\left| H^{M,N}\left(e^{j\omega}\right) \right| = \left| \frac{\sin(M\omega/2)}{M\sin(\omega/2)} \right|^N$ (Fig. 1).

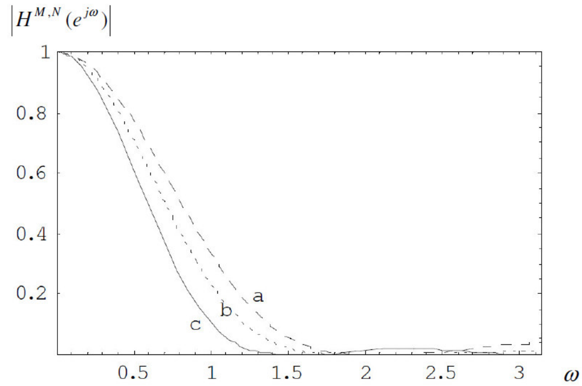

Fig. 1. Magnitude responses of $H^{3,3}(e^{j\omega})$ (a), $H^{3,4}(e^{j\omega})$ (b) and $H^{4,3}(e^{j\omega})$ (c), versus normalized angular frequency ω.

For simplicity, we will consider the scaled transfer function $G^{M,N}(z) = M^N H^{M,N}(z)$

$$G^{M,N}(z) = \left(\frac{1-z^{-M}}{1-z^{-1}} \right)^N \tag{3}$$

$G^{M,N}(z)$ is a polynomial, with degree *N(M-1),* of the form

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$$G^{M,N}(z) = \sum_{j=0}^{N(M-1)} g_j^{M,N} z^{-j} \qquad (4)$$

where $g_j^{M,N}$ represent the coefficients of the impulse response of the $sinc^N$ filters. $h_j^{M,N}$ and $g_j^{M,N}$ are related by $h_j^{M,N} = M^{-N} g_j^{M,N}$. Note that $g_j^{M,N} = 0\ (\forall j < 0)$, $g_0^{M,N} = 1\ (\forall M, \forall N)$, $G^{M,0}(z) = 1\ (\forall M)$, $G^{1,N}(z) = 1\ (\forall N)$ and $g_{N(M-1)-j}^{M,N} = g_j^{M,N}$.

In the following, we will propose simple recurrence relations useful for the computation of $sinc^N$ FIR filter coefficients, and especially a new general expression of these coefficients.

III. RECURRENCE RELATIONS FOR THE FILTER COEFFICIENTS

In this section, several relations are derived, according to different criteria such as simplicity, low order, varying index. Each of these independent relations can be useful, depending on the way chosen for computation, and implemented separately.

A. Calculation of $g_j^{M,N}$ from 3 coefficients with the same values M and N, and lower indexes j-1, j-M and j-(M+1)

Differentiating $\ln G^{M,N}(z)$ using (3) and multiplying by $G^{M,N}(z)$ yields

$$\frac{dG^{M,N}(z)}{dz} = N\left(\frac{Mz^{-(M+1)}}{1-z^{-M}} - \frac{z^{-2}}{1-z^{-1}}\right)G^{M,N}(z) \qquad (5)$$

which can be written

$$\begin{aligned}&\left(1 - z^{-1} - z^{-M} + z^{-(M+1)}\right)\left(-z\frac{dG^{M,N}(z)}{dz}\right) = \\ &N\left(z^{-1} - Mz^{-M} + (M-1)z^{-(M+1)}\right)G^{M,N}(z)\end{aligned} \qquad (6)$$

By using basic z-Transform techniques, (6) leads immediately to

$$\begin{aligned}g_j^{M,N} &= \frac{1}{j}(N-1+j)g_{j-1}^{M,N} - \left[M(N+1)-j\right]g_{j-M}^{M,N} \\ &+\left[M(N+1)-(N-1)-j\right]g_{j-(M+1)}^{M,N}\end{aligned} \qquad (7)$$

which can also be written

$$\begin{aligned}g_j^{M,N} &= g_{j-1}^{M,N} + g_{j-M}^{M,N} - g_{j-(M+1)}^{M,N} \\ &+\frac{1}{j}\Big[(N-1)\left(g_{j-1}^{M,N} - g_{j-(M+1)}^{M,N}\right) \\ &-M(N+1)\left(g_{j-M}^{M,N} - g_{j-(M+1)}^{M,N}\right)\Big]\end{aligned} \qquad (8)$$

An advantage of (8) is the fact that M and N keep constant values. Relative drawbacks are that the number of terms is not the lowest possible and that shifts in j indexes increase with M and may be large. This recurrence relation is the same as (11) in [7], but here it has been straightforwardly derived using z-Transform techniques.

For example, for $M=8$, $N=4$ and $j=9$, the relation gives:

$$g_9^{8,4} = g_8^{8,4} + g_1^{8,4} - g_0^{8,4} + \frac{1}{9}[3(g_8^{8,4} - g_0^{8,4}) - 40(g_1^{8,4} - g_0^{8,4})]$$

i.e., $g_9^{8,4} = 161 + 4 - 1 + \frac{1}{9}\left[3(161-1) - 40(4-1)\right] = 204$

B. Calculation of $g_j^{M,N}$ from 2 coefficients with the same value M, and indexes (j-1, N) and (j, N-1)

Differentiating $G^{M,N}(z)$ using (3) yields

$$\frac{dG^{M,N}(z)}{dz} = N\left[\frac{Mz^{-(M+1)}}{1-z^{-1}} - \frac{z^{-2}(1-z^{-M})}{(1-z^{-1})^2}\right]\left(\frac{1-z^{-M}}{1-z^{-1}}\right)^{N-1} \qquad (9)$$

which can be written

$$\begin{aligned}&(1-z^{-1})\left(-z\frac{dG^{M,N}(z)}{dz}\right) = \\ &N\left\{-MG^{M,N-1}(z) + \left[M-(M-1)z^{-1}\right]G^{M,N}(z)\right\}\end{aligned} \qquad (10)$$

By using basic z-Transform techniques, it can be immediately deduced

$$g_j^{M,N} = \frac{1}{MN-j}\left[(MN-j+1-N)g_{j-1}^{M,N} + MNg_j^{M,N-1}\right] \qquad (11)$$

Advantages of (11) are a very low number of terms, the fact that M keeps a constant value and that shifts in N and j indexes equal only one and occur separately. This new and simple relation allows a fast recursive calculation of the impulse response coefficients. Therefrom, computation is quite easy, even permitting to fill up the start of the table of filter coefficients simply by hand.

For example, for $M=8$, $N=4$ and $j=9$, the relation gives:

$$g_9^{8,4} = \frac{1}{23}\left(20g_8^{8,4} + 32g_9^{8,3}\right)$$

i.e., $g_9^{8,4} = \frac{1}{23}\left(20\times161 + 32\times46\right) = 204$

C. Calculation of $g_j^{M,N}$ from 3 coefficients with the same value M, and indexes (j-1, N) and (j, N-1) and (j-M, N-1)

Using (3), $G^{M,N}(z)$ can be linked to $G^{M,N-1}(z)$ as follows

$$G^{M,N}(z) = \frac{1-z^{-M}}{1-z^{-1}}G^{M,N-1}(z) \qquad (12)$$

By using basic z-Transform techniques, it can be immediately deduced from (12)

$$g_j^{M,N} = g_{j-1}^{M,N} + g_j^{M,N-1} - g_{j-M}^{M,N-1} \qquad (13)$$

Advantages of (13) are a low number of terms, with no multiplying factors, the fact that M keeps a constant value and that shifts in N equal only one. Relative drawbacks are that the number of terms is not the lowest possible and that shifts in j index increase with M and may be large. This simple relation allows a fast recursive calculation of the impulse response coefficients. Computation is quite easy, even permitting to fill up the start of the table of filter coefficients simply by hand.

For example, for $M=8$, $N=4$ and $j=9$, the relation gives:

$$g_9^{8,4} = g_8^{8,4} + g_9^{8,3} - g_1^{8,3}$$

i.e., $g_9^{8,4} = 161 + 46 - 3 = 204$

D. Calculation of $g_j^{M,N}$ from N+1 coefficients with the same lower value M-1, and indexes (j-k, k), k=0..N

Using (3), $G^{M,N}(z)$ can be expressed as follows

$$G^{M,N}(z) = \left[1 + z^{-1}\frac{1-z^{-(M-1)}}{1-z^{-1}}\right]^N \tag{14}$$

which can be written

$$G^{M,N}(z) = \sum_{k=0}^{N}\binom{N}{k} z^{-k} G^{M-1,k}(z) \tag{15}$$

By using basic z-Transform techniques, (15) yields

$$g_j^{M,N} = \sum_{k=0}^{N}\binom{N}{k} g_{j-k}^{M-1,k} \tag{16}$$

An advantage of (16) is the fact that *M-1* keeps constant in the right side of the equation. Drawbacks are that the number of terms, the number of sets of coefficients involved in the relation, as well as shifts in *j* index increase with *N*.

For example, for *M=8*, *N=4* and *j=9*, $g_j^{M,N}$ can be easily computed using the values of the coefficients $g_{9-k}^{7,k}$, weighted by the binomial coefficients $\binom{4}{k}$:

$$g_9^{8,4} = \sum_{k=0}^{4}\binom{4}{k} g_{9-k}^{7,k} = g_9^{7,0} + 4g_8^{7,1} + 6g_7^{7,2} + 4g_6^{7,3} + g_5^{7,4}$$

i.e., $g_{10}^{8,4} = 0 + 4\times 0 + 6\times 6 + 4\times 28 + 56 = 204$

TABLE I

Values of $g_j^{M,N}$ *for N = 1,…,4 and M = 1,…,8*

	M	*N(M-1)*	M^N	$g_0^{M,N}$	$g_1^{M,N}$	$g_2^{M,N}$	$g_3^{M,N}$	$g_4^{M,N}$	$g_5^{M,N}$	$g_6^{M,N}$	$g_7^{M,N}$	$g_8^{M,N}$	$g_9^{M,N}$	$g_{10}^{M,N}$	$g_{11}^{M,N}$	$g_{12}^{M,N}$	$g_{13}^{M,N}$	$g_{14}^{M,N}$
N=1	*1*	*0*	*1*	1														
	2	*1*	*2*	1														
	3	*2*	*3*	1	1													
	4	*3*	*4*	1	1													
	5	*4*	*5*	1	1	1												
	6	*5*	*6*	1	1	1												
	7	*6*	*7*	1	1	1	1											
	8	*7*	*8*	1	1	1	1											
N=2	*1*	*0*	*1*	1														
	2	*2*	*4*	1	2													
	3	*4*	*9*	1	2	3												
	4	*6*	*16*	1	2	3	4											
	5	*8*	*25*	1	2	3	4	5										
	6	*10*	*36*	1	2	3	4	5	6									
	7	*12*	*49*	1	2	3	4	5	6	7								
	8	*14*	*64*	1	2	3	4	5	6	7	8							
N=3	*1*	*0*	*1*	1														
	2	*3*	*8*	1	3													
	3	*6*	*27*	1	3	6	7											
	4	*9*	*64*	1	3	6	10	12										
	5	*12*	*125*	1	3	6	10	15	18	19								
	6	*15*	*216*	1	3	6	10	15	21	25	27							
	7	*18*	*343*	1	3	6	10	15	21	28	33	36	37					
	8	*21*	*512*	1	3	6	10	15	21	28	36	42	46	48				
N=4	*1*	*0*	*1*	1														
	2	*4*	*16*	1	4	6												
	3	*8*	*81*	1	4	10	16	19										
	4	*12*	*256*	1	4	10	20	31	40	44								
	5	*16*	*625*	1	4	10	20	35	52	68	80	85						
	6	*20*	*1296*	1	4	10	20	35	56	80	104	125	140	146				
	7	*24*	*2401*	1	4	10	20	35	56	84	116	149	180	206	224	231		
	8	*28*	*4096*	1	4	10	20	35	56	84	120	161	204	246	284	315	336	344

In this section, using z-Transform techniques, efficient recursive formulae for computing the impulse response coefficients $g_j^{M,N}$ of $sinc^N$ filters have been derived, especially (11) and (13). The values of the coefficients $g_j^{M,N}$, for $M \le 8$ and $N \le 4$, are given in Table I. For symmetry reasons, only the coefficients for $j = 0..INT\left[\frac{N(M-1)}{2}\right]$ are shown.

Futher investigations should be undertaken in order to evaluate precisely the number of operations and the computation time when using such or such recurrence relation or a combination of them, and, in a way, the more efficient strategy.

IV. GENERAL EXPRESSION FOR THE FILTER COEFFICIENTS

Let us write $G^{M,N}(z)$ under the form

$$G^{M,N}(z) = (1 - z^{-M})^N Y^N(z) \tag{17}$$

with $Y^N(z)$ defined as follows

$$Y^N(z) = \left(\frac{1}{1 - z^{-1}}\right)^N \tag{18}$$

A. Explicit expression for the coefficients y_j^N of $Y^N(z)$

Differentiating $\ln Y^N(z)$ and multiplying by $(1 - z^{-1})\left[-zY^N(z)\right]$ gives

$$(1 - z^{-1})\left(-z\frac{dY^N(z)}{dz}\right) = Nz^{-1}Y^N(z) \tag{19}$$

Basic z-Transform techniques allow to obtain

$$y_j^N = \frac{N - 1 + j}{j} y_{j-1}^N \tag{20}$$

This relation permits easy recursive computation of y_j^N from y_{j-1}^N. Moreover, (20) leads to the following simple explicit relation for the coefficients y_j^N

$$y_j^N = \binom{N - 1 + j}{j} \tag{21}$$

B. General expression for the coefficients $g_j^{M,N}$ of $G^{M,N}(z)$

Developing $(1 - z^{-M})^N$ in (17) leads to the following relation between $G^{M,N}(z)$ and $Y^N(z)$

$$G^{M,N}(z) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} z^{-kM} Y^N(z) \tag{22}$$

Then using the z-Transform translation property, the coefficients $g_j^{M,N}$ can be written

$$g_j^{M,N} = \sum_{k=0}^{[j/M]} (-1)^k \binom{N}{k} \binom{N - 1 + j - kM}{j - kM} \tag{23}$$

with *j=0..N(M-1)*, and where *[j/M]* denotes the integer quotient of *j* and *M*.

It is worth noting that this new expression of the coefficients $g_j^{M,N}$ applies for all values of *M*, *N* and *j*. This simple formula allows to calculate any coefficient numerically. It offers the obvious advantage, compared to previous results, to be general and valid for all the coefficients of the impulse response of the $sinc^N$ filters. In fact, in [6] (cf. (19)) and in [7] (cf. (12) and (13)), explicit expressions are given only for the very first coefficients of the impulse response of the $sinc^N$ filters. Note that here (23) leads immediately to these explicit expressions for *j=0..(M-1)* and for *j=M*, with *k=0* and *k=1* respectively.

For example, for *M=8*, *N=4* and *j=9 ([j/M]=1),* $g_j^{M,N}$ can be easily computed using the products of binomial coefficients $\binom{N}{k}\binom{N-1+j-kM}{j-kM}$, i.e., $\binom{4}{k}\binom{12-8k}{9-8k}$, multiplied by $(-1)^k$:

$$g_9^{8,4} = \sum_{k=0}^{1} (-1)^k \binom{4}{k}\binom{12-8k}{9-8k} = \binom{12}{9} - 4\binom{4}{1},$$

i.e., $g_9^{8,4} = 220 - 4 \times 4 = 204$

In this section, a simple general expression (23) has been obtained for rapid computation of any of the coefficients of $sinc^N$ filters.

As a summary, useful recurrence relations and formula (8), (11), (13), (16) and (23) have been derived for computing the $sinc^N$ FIR filter coefficients $h_j^{M,N} = M^{-N} g_j^{M,N}$.

V. GENERATING FUNCTIONS FOR THE TRANSFER FUNCTIONS OF $sinc^N$ FIR FILTERS

Let us consider (12) $G^{M,N}(z) = \frac{1 - z^{-M}}{1 - z^{-1}} G^{M,N-1}(z)$, with $G^{M,0}(z) = 1$.

A. Ordinary generating function

Multiplying $G^{M,N}(z)$ by x^N and summing with *N* varying from 0 to infinity leads to the following ordinary generating function

$$\Gamma_o^M(z,x) = \sum_{N=0}^{\infty} G^{M,N}(z) x^N = \frac{1}{1 - \frac{1 - z^{-M}}{1 - z^{-1}} x} \tag{24}$$

Conversely, developing $\Gamma_o^M(z,x)$ into series expansion generates the expression of the filter transfer functions $G^{M,N}(z)$ as the coefficients of x^N ($N = 0,..,\infty$).

B. Exponential generating function

Multiplying $G^{M,N}(z)$ by $\frac{x^N}{N!}$ and summing with *N* varying from 0 to infinity leads to the following exponential generating function

$$\Gamma_e^M(z,x) = \sum_{N=0}^{\infty} G^{M,N}(z) \frac{x^N}{N!} = e^{\frac{1 - z^{-M}}{1 - z^{-1}} x} \tag{25}$$

Conversely, developing $\Gamma_e^M(z,x)$ into series expansion generates the expression of the filter transfer functions $G^{M,N}(z)$ as the coefficients of $\frac{x^N}{N!}$ ($N = 0,..,\infty$).

VI. CONCLUSION

Sinc^N filters, constituted of a cascade of *N sinc* filters, each of length *M*, are useful for digital signal processing applications, in various domains including telecommunications. In this paper, z-Transform has been used as an efficient tool for deriving various formulae allowing to compute the impulse response coefficients of *sinc*^{N} FIR filters.

In particular, straightforward recursive relations have been demonstrated. Further investigations should be made to evaluate the computation time and the more efficient strategy when exploiting these relations separately or in combination.

Moreover, a simple general expression – new compared to previous published formula – has been given, valid for all the coefficients, whatever their rank j and the values of *M* and *N*.

In addition, generating functions for the *sinc*^{N} FIR filters transfer functions have been given.

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