On The Idle Time Model In Computer Networks

Saulius Minkevičius
VU Institute of Mathematics and Informatics
Akademijos 4, 08663 Vilnius, Lithuania
and Vilnius University, Naugarduko 24, 03225
Vilnius, Lithuania
e-mail: minkevicius.saulius@gmail.com

Edvinas Greičius
Vilnius University, Naugarduko 24, 03225 Vilnius,
Lithuania
e-mail: edvinas.greicius@gmail.com

Abstract – **An open queueing network model in light traffic and heavy traffic has been developed. The probability limit theorem for the idle time process of customers has been presented in light traffic in open queueing networks. Also, we present an application of the theorem - an idle time model from the computer network practice.**

Keywords* – *mathematical models of technical systems; performance evaluation; reliability theory; queueing theory; open queueing network; light traffic; heavy traffic.

I Introduction and statement of the problem

The modern queueing theory is one of the most powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. In this paper, we analyze queueing systems, arising in the network theory and communications theory (called an open queueing network). In this paper, probability limit theorems are considered by investigating the values of a virtual time of a customer and an idle time of a customer in an open queueing network.

One of the main directions of research in the theory of queues corresponds with the asymptotic analysis of formulas or equations, describing the distribution of this and another probabilistic characteristic of a queue. For such a development of the analysis, there must be formulas or equations and besides that an unlimited convergence of the queue to the critical own point. This way, the first results about limited behaviour of single channel queues in heavy traffic are achieved (see [6, 7]). In the single phase case, where intervals of times between the arrival of customers are independent identically distributed random variables and there is one single device, working independently of the output in heavy traffic, is fully investigated in well known papers [2, 3]. Probability limit theorems for a virtual waiting time of a customer in a single queue are proved under various conditions of heavy traffic (see [8]). Probability limit theorems for a virtual waiting time of a customer in heavy traffic and an idle time of a customer in light traffic are closely connected. Also probability limit theorems for the waiting time of a customer and the queue length of customers in a multiphase queue are proved under various conditions of heavy traffic (see [10]).

So, in the sequel, we present a probability limit theorem in conditions of light traffic for another important probabilistic characteristic of an open queueing network (idle time of a customer). Note that there are only some works designed for the investigation of the idle time of a customer in a single-server queue (see surveys [16, 18] and paper [17]). Also, note that the research of the idle time of a customer in more general systems than the classical system $GI/G/1$ (multiserver queue, multiphase queue, open queueing network, etc.) has just started (see again [16, 17, 18]).

The idle function of computer networks shows in which part of time the computer network is not busy (idle). So in this paper, we present the probability limit theorem for the idle time of a customer in light traffic in the queueing network. The service discipline is "first come, first served" (FCFS). We consider the open queueing network with the FCFS service discipline at each station and general distributions of interarrival and service time. The queueing network studied has k single server stations, each of which has an associated infinite capacity waiting room. Each station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed either to another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables. The

 ISBN: 978-1-61208-432-9

basic components of the queueing network are arrival processes, service processes, and routing processes.

We begin with a probability space (Ω, B, P) on which these processes are defined. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, ..., k$; defined on the probability space. The random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have a support in $\{0, 1, 2, ..., k\}$.

We define $\mu_j = \left(E\left[S_n^{(j)}\right]\right)^{-1}$, $\sigma_j = D\left(S_n^{(j)}\right)$ and $\lambda_j = \left(E\left[z_n^{(j)}\right]\right)^{-1}$, $a_j = D\left(z_n^{(j)}\right)$, $j = 1, 2, ..., k$; with all of these terms assumed finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right)$, $i, j = 1, 2, ..., k$. The $k \times k$ matrix $P = (p_{ij})$ is assumed to have a spectral radius strictly smaller than a unit. The matrix P is called a routing matrix. In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer, served at the station i, is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network.

At first, let us define $I_j(t)$ as the idle time of a customer at the jth station of the queueing network in time t (time t at which an open queueing network is not busy (idle) serving customers at the jth station of the queueing network), $\beta_j = 1 - \dfrac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij}}{\mu_j}$, $\hat{\sigma}_j^2 = \sum_{i=1}^{k} p_{ij}^2 \cdot \mu_i \cdot \left(\sigma_j + \left(\dfrac{\mu_i}{\mu_j}\right)^2 \cdot \sigma_i\right) + \lambda_j \cdot \left(\sigma_j + \left(\dfrac{\lambda_j}{\mu_j}\right)^2 \cdot a_j\right)$, $j = 1, 2, \cdots, k$ and $t > 0$.

We suppose that the following conditions are fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} < \mu_j, \ j = 1, 2, \ldots, k. \tag{1}$$

In addition, we assume throughout that

$$\max_{1 \leq j \leq k} \sup_{n \geq 1} E\left\{\left(z_n^{(j)}\right)^{2+\gamma}\right\} < \infty \ \text{ for some } \ \gamma > 0, \tag{2}$$

$$\max_{1 \leq j \leq k} \sup_{n \geq 1} E\left\{\left(S_n^{(j)}\right)^{2+\gamma}\right\} < \infty \ \text{ for some } \ \gamma > 0. \tag{3}$$

Conditions (2) and (3) imply the Lindeberg condition for the respective sequences.

One of the results of the paper is the following probability limit theorem for the idle time of a customer in an open queueing network (the proof can be found in [12]).

Theorem 1. *If conditions (1) - (3) are fulfilled, then*

$$\lim_{n \to \infty} P\left(\frac{I_j(nt) - \beta_j \cdot n \cdot t}{\hat{\sigma}_j \cdot \sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2 \cdot t/2) dy,$$

$0 \leq t \leq 1$ *and* $j = 1, 2, \ldots, k$.

II Idle time function of a computer network

Now we present a technical example from the computer network practice. Assume that queues of customers requests arrive at the computer v_j at the rate λ_j per hour during business hours, $j = 1, 2, \ldots, k$. These queues are served at the rate μ_j per hour by the computer v_j, $j = 1, 2, \ldots, k$. After service by the computer v_j, with probability p_j (usually $p_j \geq 0.9$), they leave the network and with probability p_{ji}, $i \neq j$, $1 \leq i \leq k$ (usually $0 < p_{ji} \leq 0.1$) arrive at the computer v_i, $i = 1, 2, \ldots, k$. Also, we assume the computer v_j to be idle, when the idle time of the waiting for service computer is less than k_j, $j = 1, 2, \ldots, k$.

In this section, we prove the following theorem on the idle time function of the computer network (probability of idle time in a computer network).A computer network is idle when it is not busy.

Theorem 2. *If* $t \geq \max_{1 \leq j \leq k} \dfrac{k_j}{\hat{\beta}_j}$ *and conditions (1) - (3) are fulfilled, all computers in the network are idle.*

Therefore, using Theorem 1, we get

$$\lim_{n \to \infty} P\left(\frac{I_j(n) - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2/2) dy, \ j = 1, 2, \ldots, k. \tag{4}$$

Let us investigate a computer network which consists of the elements (computers) v_j, $j = 1, 2, \ldots, k$.

Denote

$$X_j = \begin{cases} 1, & \text{if the element } v_j \text{ is idle} \\ 0, & \text{if the element } v_j \text{ is not idle,} \end{cases}$$

$j = 1, 2, \ldots, k$.

 ISBN: 978-1-61208-432-9

Note that $\{X_j = 1\} = \{I_j(t) < k_j\}, \quad j = 1, 2, \ldots, k$.

Assume the structural function of the system of elements is connected with scheme 1 from k (see, for example, [2]) as follows:

$$\phi(X_1, X_2, \ldots, X_k) = \begin{cases} 1, & \sum_{i=1}^k X_i \geq 1 \\ 0, & \sum_{i=1}^k X_i < 1. \end{cases}$$

Suppose $y = \sum_{i=2}^k X_i$. Let us estimate the idle function of the system (computer network) using the formula of the full conditional probability

$$h(X_1, X_2, \ldots, X_k) = E\phi(X_1, X_2, \ldots, X_k) =$$

$$P(\phi(X_1, X_2, \ldots, X_k) = 1) = P(\sum_{i=1}^k X_i \geq 1) =$$

$$P(X_1 + y \geq 1) = P(X_1 + y \geq 1 | y = 1) \cdot P(y = 1) +$$

$$P(X_1 + y \geq 1 | y = 0) \cdot P(y = 0) = P(X_1 \geq 0) \cdot P(y = 1) +$$

$$P(X_1 \geq 1) \cdot P(y = 0) \leq P(y = 1) + P(X_1 \geq 1)$$

$$= P(y = 1) + P(X_1 = 1) \leq P(y \geq 1) + P(X_1 = 1)$$

$$= P(\sum_{i=2}^k X_i \geq 1) + P(X_1 = 1) \leq \cdots \leq$$

$$\leq \sum_{i=1}^k P(X_i = 1) = \sum_{i=1}^k P(I_i(t) \leq k_i).$$

Thus,

$$0 \leq h(X_1, X_2, \ldots, X_k) \leq \sum_{i=1}^k P(I_i(t) \leq k_i). \tag{5}$$

Applying Theorem 1 (with $t = 1$), we obtain that

$$0 \leq \lim_{t \to \infty} P(I_j(t) < k_j) = \lim_{n \to \infty} P(I_j(n) < k_j) =$$

$$\lim_{t \to \infty} P\left(\frac{I_j(n) - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}} < \frac{k_j - \beta_j \cdot n}{\hat{\sigma}_j \cdot \sqrt{n}}\right) =$$

$$\int_{-\infty}^{-\infty} \exp(-y^2/2) dy = 0. \tag{6}$$

Then (see (6)),

$$\lim_{t \to \infty} P\left(I_j(t) < k_j\right) = 0, \; j = 1, 2, \ldots, k. \tag{7}$$

So, $h(X_1, X_2, \ldots, X_k) = 0$. (see (5) and (7)). The proof of the theorem is completed.

Further, we suppose that the following alternative conditions are fulfilled:

$$\lambda_j + \sum_{i=1}^k \mu_i \cdot p_{ij} > \mu_j, \; j = 1, 2, \ldots, k. \tag{8}$$

Let us define $V_j(t)$ as a virtual waiting time of a customer at the jth station of the queueing network in time t, $j = 1, 2, \ldots, k$. Note that this condition guarantees that, with probability one, there exists a virtual waiting time of a customer and this virtual waiting time of a customer when is constantly growing. One of the results of the paper is the following theorem on the probability limit for the virtual waiting time of a customer in an open queueing network.

Theorem 3. *If conditions (8) are fulfilled, then*

$$\lim_{n \to \infty} P\left(\frac{V_j(nt) - \beta_j \cdot n \cdot t}{\hat{\sigma}_j \cdot \sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2/2) dy,$$

$0 \leq t \leq 1$ *and* $j = 1, 2, \ldots, k$.

Proof. This theorem is proved under conditions $\lambda_j > \mu_j, \; j = 1, 2, \ldots, k$ (see, for example, [12]). Applying the methods of [15], it can be proved that this theorem is true under more general conditions(8). The proof of the theorem is complete.

Applying Theorem 3, we get the following result.

Theorem 4. *If* $t \geq \max_{1 \leq j \leq k} \frac{k_j}{\hat{\beta}_j}$ *and conditions (1), (2), and (8) are fulfilled, the computer network becomes unreliable (all computers fail). So, in that case, all the computers are busy.*

Proof. The proof is similar to that of Theorem 2. The proof of the theorem is complete.

Finally, we find the exact expression for $h(X_1, X_2, \ldots, X_k), \; t > 0$. Next, we prove the following theorem on this probability.

Theorem 5 $h(X_1, X_2, \ldots, X_k)$ *is equal to* $\exp(-\sum_{j=1}^k P(I_j(t) < k_j))$.

Proof. First denote $\lambda_j, \; j = 1, 2, \ldots, k$ as intensivities of structural elements, that form a complex stochastic system. Then the probability of stopping this system is equal to $e^{-\sum_{j=1}^k \lambda_j}$.

However,

$$\lambda_j = MX_j = P(X_j = 1) = P(I_j(t) < k_j), \; j = 1, 2, \ldots, k. \tag{9}$$

Applying (8), we obtain that $h(X_1, X_2, \ldots, X_k)$ is equal to

$$e^{-\sum_{j=1}^k \lambda_j} = e^{-\sum_{j=1}^k P(V_j(t) < k_j)}.$$

 ISBN: 978-1-61208-432-9

The proof is complete.

As a result, using Theorem 5, it is possible to estimate the idle time of a complex computer network.

III Concluding remarks and future research

1.1. Conditions (1)-(3) mean that the number of jobs arriving at the node of the network is greater than the number of service jobs at the same node of the network. It is clear from this note, that the length of jobs in the node of the network is constantly growing with probability one.
1.2. Conditions (1), (2), and (8) mean that the total number of jobs arriving at the node of the network is less than the number of service jobs at the same node of the network.It is clear from this note, that the length of jobs in the node of the network is constantly decreasing with probability one.
2. Now all the cases of traffic in computer networks are investigated – light traffic (see Theorem 1), average traffic (see Theorem 3) and heavy traffic (see Theorem 2). The investigation of all these cases is new.
3. If the conditions of the Theorem 1 and Theorem 4 are fulfilled (i. e., conditions (1) and (8) are satisfied), the network is either idle or busy. Conditions (1) and (8) are fundamental, - the behaviour of the whole network and its evolution is not clear, if conditions (1) and (8) are not satisfied. Therefore, this fact is the object of further research and discussion.
4. The theorems of this paper are proved for a class of open queueing networks in light traffic with the service principle FCFS, endless waiting time of customers at the each node of the queueing system, and the intervals between the arrival of customers at the open queueing networks are independent identically distributed random variables. However, similar theorems can be applied to a wider class of open queueing networks in the light traffic: when the arrival and service of customers in a queue are distributed in groups, also, when interarrival times of customers at the open queueing network are weakly dependent random variables, etc.

IV Acknowledgement

This research was partly supported by the National Complex Programme "Theoretical and engineering aspects of e-service technology creation and application in high-performing calculation platforms".

References

[1] P. Billingsley, *Convergence of probability measures*, Wiley, New York (1968).

[2] A. Borovkov, *Stochastic processes in queueing theory*, Nauka, Moscow (1972) (in Russian).

[3] A. Borovkov, *Asymptotic methods in theory of queues*, Nauka, Moscow (1980) (in Russian).

[4] D. Iglehart, Multiple channel queues in heavy traffic. IV. Law of the iterated logarithm, Zeitschrift für Wahrscheinlicht-Keitstheorie und Verwandte Gebiete, 17, 168-180 (1971).

[5] D. Iglehart, Weak convergence in queueing theory, *Advances in Applied Probability*, 5, 570-594 (1973).

[6] J. Kingman, On queues in heavy traffic, J. R. Statist. Soc., 24, 383-392 (1962a).

[7] J. Kingman, The single server queue in heavy traffic, Proc. Camb. Phil. Soc., 57, 902-904(1962b).

[8] E. Kyprianou, The virtual waiting time of the $GI/G/1$ queue in heavy traffic, Advances in Applied Probability, 3, 249-268 (1971)

[9] S. Minkevičius, Weak convergence in multiphase queues, Lietuvos Matematikos Rinkinys, 26, 717-722 (1986) (in Russian).

[10] S. Minkevičius, Transient phenomena in multiphase queues, Lietuvos Matematikos Rinkinys, 31. 136-145 (1991).

[11] S. Minkevičius, On the law of the iterated logarithm in multiphase queues, Lietuvos Matematikos Rinkinys, 35, 360-366 (1995).

[12] S. Minkevičius, The probability limit theorem for the idle time of a customer in open queueing networks, Informatica (accepted, 2015).

[13] J.J. Morder, S. E. Elmaghraby (eds.). *Handbook of operational research models and applications*, Van Nostrand Reinhold, New York (1978).

[14] M. Reiman, Open queueing networks in heavy traffic, Mathematics of Operations Research, 9, 441-459 (1984).

[15] L. Sakalauskas, S. Minkevičius, On the law of the iterated logarithm in open queueing networks, European Journal of Operational Research, 120, 632-640 (2000).

[16] L. Takacs, Occupation time problems in the theory of queues. Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, Heidelberg, New York, 98, 91-131 (1974).

 ISBN: 978-1-61208-432-9

[17] W. Whitt, Weak convergence theorems for priority queues: preemptive-resume discipline, Journal of Applied Probability, 8, 74=94 (1971).

[18] W. Whitt, Heavy traffic limit theorems for queues: a survey, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, New York, 98, 307-350 (1974).