Upper bounds and optimal solutions for a Deterministic and Stochastic linear Bilevel Problem

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***Abstract*—In this paper, we compute upper bounds and optimal solutions for a deterministic linear bilevel programming problem and then, for a stochastic version of this problem. The latter is formulated while adding probabilistic knapsack constraints in the upper level problem of the initial deterministic model. The upper bounds are computed using a Lagrangian iterative minmax algorithm and linear programming relaxations. To this purpose, we first transform both problems into the so called Global Linear Complementarity problems. We then, use these models to derive equivalent mixed integer programming formulations. This allows comparing the iterative minmax algorithm and the linear programming upper bounds with the optimal solution of the problem for the deterministic and stochastic instances as well. Our numerical results show tight near optimal bounds for both, the stochastic and deterministic linear programming relaxations and larger gaps for the iterative minmax algorithm.**

***Keywords*-Linear bilevel programming; stochastic programming; mixed integer programming.**

I. Introduction

In mathematical programming, the bilevel programming problem (BPP) is a hierarchical optimization problem. It consists in optimizing an objective function subject to a constrained set in which another optimization problem is embedded. The first level optimization problem (upper-level problem) is known as the leader's problem while the lower-level is known as the follower's problem. Formally, it can be written as follows

$$\begin{aligned} \min_{\{x\in X,y\}} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \leq 0 \\ & \min_{\{y\}} f(x,y) \\ & \text{s.t.} \quad g(x,y) \leq 0 \end{aligned}$$

where $x \in R^{n_1}$, $y \in R^{n_2}$, $F : R^{n_1} \times R^{n_2} \rightarrow R$ and $f : R^{n_1} \times R^{n_2} \rightarrow R$ are the decision variables and the objective valued functions for the upper and lower level problems, respectively. Similarly, the functions $G : R^{n_1} \times R^{n_2} \rightarrow R^{m_1}$ and $g : R^{n_1} \times R^{n_2} \rightarrow R^{m_2}$ denote upper and lower level constraints. Bilevel programming is commonly used to model situations in which two or more decision makers control part of the variables within a particular decision process [1]. The main goal is thus, to find an optimal point such that the leader and the follower minimizes their respective objective valued functions $F(x,y)$, $f(x,y)$ subject to their respective linking constraints $G(x,y)$ and $g(x,y)$. Notice that either the leader (or the follower) might also have their own particular constraints such as the set X in the above leader problem. Applications concerning BPP include transportation, networks design, management and planning among others (for different domains of applications see for instance [6]).

It has been shown that BPPs are strongly NP-hard even for the simplest case in which all the involved functions are affine [8]. Hereafter, we only consider the case in which all the above functions $F(x,y), f(x,y), G(x,y), g(x,y)$ are linear. Besides, if a particular constrained set exists in the leader or in the follower problem, we assume that it is a polyhedral affine space.

Stochastic programming (SP), on the other side, is an optimization technique which deals with the uncertainty of the input parameters of a mathematical program [16]. The underlying idea of SP is that the input parameters can be modeled as random variables to which the theory of probabilities can be applied. The probability distributions governing the data are usually assumed to be known in advance or that they can be estimated. The probability space is also usually assumed to be discrete and as such, one can consider finite sets of scenarios for the input parameters. There are two well known scenario based approaches in SP. The first one is known as the recourse model approach [5], [7] while the second one is known as probabilistic constrained approach [7]. The literature related to SP has grown considerably in last decades. A general survey can be found for instance in [14] and the reader is also referred

 ISBN: 978-1-61208-202-8

to [3], [9], [15] or to a more recent book in [16] for a deeper comprehension.

In this paper, we consider the probabilistic knapsack constrained approach proposed in [7] when embedded into the upper level problem. Under this approach, it is imposed a threshold risk on the probability of occurrence for some (or all) of the constraints within a particular mathematical model. This means that some of the constraints should be satisfied, at least for a given percentage, while the rest of them are discarded.

The paper is organized as follows. In Section II, we provide a brief state of the art concerning joint aspects of bilevel and stochastic programming. Then, in Section III, we state the linear bilevel programming problem (LBPP) and briefly explain the probabilistic constrained approach considered. In Section IV, we derive the Global Linear Complementarity problem (GLCP) and also explain how the iterative minmax (IMM) algorithm works in order to compute the upper bounds. In Section V, we derive from the GLCPs, mixed integer and linear programming formulations (Resp. MIP and LP) according to [1]. Numerical results are given for the LBPP and for the stochastic LBPP (SLBPP) in Section VI. Finally, in Section VII we give the main conclusions of the paper.

II. Related Work

Although there exist many application domains in which bilevel programming can be suitably applied, joint stochastic and bilevel programming aspects have not yet widely been explored so far. Some preliminary works are the following [2], [4], [11]–[13], [17].

In [11], Luh et al. study a deterministic pricing problem and propose a stochastic counterpart for it by assuming that the inducible region is subject to uncertainty. Here, the inducible region is defined as the feasible set of the follower problem induced by the decision of the leader problem. Next, Patriksson et al. also incorporates uncertainty in the input data of hierarchical mathematical Programming problems [13]. In both papers [11], [13], the authors discuss theoretical aspects such as necessary and sufficient conditions for optimality, existence of solutions, convexity, and propose algorithms to deal with the problem at hand. Subsequently, Christiansen et al. [4], consider a stochastic bilevel programming problem which corresponds to an application in structural optimization where again, theoretical aspects such as existence of optimal solutions, Lipschitz continuity and differentiability aspects are discussed. More recently, applications concerning telecommunication network problems have been studied in [2], [17]. Therein, the analysis covers both theoretical and also practically oriented issues. In particular, special attention is given to different formulations of one and two stage stochastic bilevel programming problems where necessary optimality conditions for each of these problem instances are stated. Additionally, in [17], it is also proposed an algorithm which uses a stochastic quasi-gradient method to solve the problem.

Finally in [12], Özaltin et al. consider a stochastic bilevel knapsack problem with uncertain right-hand sides, and derive necessary and sufficient conditions for the existence of an optimal solution. In particular, they provide an equivalent two stage stochastic formulation when the leader problem take only integer values for the decision variables, although at the cost of having binary decision variables in the follower problem. Branching based algorithms are proposed to solve large scale instances of the problem.

In this paper, we focus more on computational numerical experiments rather than on theoretical aspects. Hence, we proceed as follows. We first compute upper bounds and optimal solutions for a generic linear bilevel programming problem (LBPP). We then, extend this generic LBPP by introducing knapsack probabilistic constraints in the upper level problem [7]. Hence, we compute upper bounds and optimal solutions for this stochastic LBPP (SLBPP) as well. The upper bounds are computed using a Lagrangian iterative minmax (IMM) algorithm proposed in [10] and also using linear programming (LP) relaxations we formulate from the so called Global Linear Complementarity Problem (GLCP) according to [1]. In [10], Kosuch et al. neither provide optimal solutions for deterministic or stochastic problems nor calculate gaps to measure IMM efficiency. Furthermore, even when Audet et al. propose links to derive an equivalent MIP formulation from a linear bilevel programming problem [1], they do not provide numerical comparisons to measure the tightness of its LP relaxation. Therefore, this paper can be seen as an extension of the works presented in [10] and [1] in the sense that now, we do provide optimal solutions and upper bounds for the IMM and for the LP relaxations as well as numerical comparisons between them, for deterministic and stochastic instances. In particular, we compute the optimal solutions using the MIP equivalent formulations [1].

III. Problem Formulation

In this section, we first present the generic LBPP under study. Then, we extend this generic model by adding knapsack probabilistic constraints in the upper level problem according to [7]. Since the probabilistic constrained approach introduces binary variables in the problem, we then obtain a mixed integer linear bilevel programming problem (MILBP) which we transform back into a LBPP [1]. We consider the following LBPP:

$$\text{LBP1:} \quad \max_{\{x\}} \quad c_1^T x + d_1^T y \tag{1}$$

$$\text{s.t.} \quad A^1 x + B^1 y \leq b^1 \tag{2}$$

$$0 \leq x \leq \mathbf{1}_{n_1} \tag{3}$$

$$y \in \arg\max_{\{y\}} \{c_2^T x + d_2^T y\} \tag{4}$$

$$\text{s.t.} \quad A^2 x + B^2 y \leq b^2 \tag{5}$$

$$0 \leq y \leq \mathbf{1}_{n_2} \tag{6}$$

where $x \in R^{n_1}$ and $y \in R^{n_2}$ are decision variables. Vectors $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ are vectors of size n_1 and n_2 with entries equal to one. Matrices A^1, B^1, A^2, B^2 and vectors $c_1, c_2, d_1, d_2, b_1 \in R^{m_1}, b_2 \in R^{m_2}$ are input real matrices/vectors defined accordingly. In LBP1, (1)-(3) correspond to the leader's problem while (4)-(6) represent the follower's problem. Knapsack probabilistic constraints can be added to the upper-level problem of LBP1 as follows. Let $w = w(\omega) \in R_+^{n_1}$ and $S = S(\omega) \in R_+$ be two random variables distributed according to a discrete probability distribution Ω. We consider the following knapsack probabilistic constraints in the upper level problem

$$P\left\{w^T(\omega)x \leq S(\omega)\right\} \geq (1-\alpha) \tag{7}$$

where α represents the risk we take while not satisfying some of the constraints. Since Ω is discrete, one may suppose that $w = w(\omega)$ and $S = S(\omega)$ are concentrated in a finite set of scenarios such as $w(\omega) = \{w_1, .., w_K\}$ and $S(\omega) = \{s_1, ..s_K\}$, respectively with probability vector $p^T = (p_1, .., p_K)$ for all k such that $\sum_{k=1}^{K} p_k = 1$ and $p_k \geq 0$. According to [7], constraints in (7) can be transformed into the following pair of deterministic constraints

$$w_k^T x \leq s_k + M_k z_k \quad k = 1:K \tag{8}$$
$$p^T z \leq \alpha \tag{9}$$

where vector $z^T = (z_1, .., z_K,)$ is composed of binary variables. This means, if $z_k = 0$ then the constraint is included, otherwise it is not activated. M_k for each $k = 1 : K$ is defined as

$$M_k = \sum_{i=1}^{n_1} w_k^i - s_k$$

where w_k^i denotes the ith component of vector w_k. Putting it altogether yields the following deterministic mixed integer linear bilevel program

$$\begin{aligned}
\text{MILBP1:} \quad \max_{\{x,z\}} \quad & c_1^T x + d_1^T y \\
\text{s.t.} \quad & A^1 x + B^1 y \leq b^1 \\
& 0 \leq x \leq \mathbf{1}_{n_1} \\
& w_k^T x \leq s_k + M_k z_k \quad k = 1:K \\
& p^T z \leq \alpha \\
& z_k \in \{0,1\}^K \\
& y \in \arg\max_{\{y\}} \{c_2^T x + d_2^T y\} \\
\text{s.t.} \quad & \quad A^2 x + B^2 y \leq b^2 \\
& \quad 0 \leq y \leq \mathbf{1}_{n_2}
\end{aligned}$$

Although MILBP1 contains binary variables, it can be converted back into an equivalent continuous LBPP [1] as follows

$$\begin{aligned}
\text{LBP2:} \quad \max_{\{x,z\}} \quad & c_1^T x + d_1^T y \\
\text{s.t.} \quad & A^1 x + B^1 y \leq b^1 \\
& 0 \leq x \leq \mathbf{1}_{n_1} \\
& w_k^T x \leq s_k + M_k z_k \quad k = 1:K \\
& p^T z \leq \alpha \\
& 0 \leq z_k \leq 1, \quad \forall k \\
& v = 0_K \\
& (y, v) \in \arg\max_{\{y,v\}} \{c_2^T x + d_2^T y + \mathbf{1}_K^T v\} \\
\text{s.t.} \quad & \quad A^2 x + B^2 y \leq b^2 \\
& \quad 0 \leq y \leq \mathbf{1}_{n_2} \\
& \quad v \leq z \qquad (10) \\
& \quad v \leq \mathbf{1}_K - z \qquad (11)
\end{aligned}$$

In LPB2, we denote by $\mathbf{1}_K$ and 0_K, the vector of all ones and the vector of all zeros of dimension K. As explained in [1], the transformation from MILBP1 into LBP2 can be done by performing the following steps. First the binary variables $z \in \{0,1\}^K$ for each $k = 1 : K$ in the upper level problem should be relaxed inside the interval [0,1]. In parallel, a new continuous variable vector $v = 0_K$ should be placed in the leader's problem imposing that all its entries be equal to zero. In fact, vector v is introduced in the follower's problem when adding the term $\mathbf{1}_K^T v$ in its objective function together with the new constraints (10)-(11). The term added in the objective function together with the latter constraints will enforce all the entries in vector z to be either equal to zero or one. We then, have derived an equivalent LBPP formulation for MILBP1. Notice that v is a variable vector in the follower's problem while vector z is a variable vector in the leader's problem.

In the next section, we derive the so called Global Linear Complementarity Counterparts for LBP1 and LBP2. Subsequently, we briefly present and explain the Lagrangian iterative minmax algorithm proposed in [10].

IV. THE GLCP AND IMM ALGORITHM

In this section, we explain all the necessary transformation steps until reaching the GLCP counterparts for LBP1 and LBP2. Then, we present IMM algorithm and describe how it works in order to compute the upper bounds. Finally, we derive from the GLCP problems equivalent MIP formulations according to [1] together with their LP relaxations.

A. The Global Linear Complementarity Problem

The GLCP is a single level quadratic optimization problem. The main idea of deriving the GLCP consists of replacing the original follower's problem with its initial constraints, dual constraints and complementary slackness conditions. The decision variables of GLCP are thus: the leader, the follower and the follower's dual variables as well. In order to derive a GLCP model for LBP1, we first write the dual of the follower problem as follows

$$\text{LBPD1:} \quad \min_{\{\lambda,\mu\}} \quad \lambda^T(b^2 - A^2x) + \mathbf{1}_K^T\mu \tag{12}$$
$$\text{s.t.} \quad (B^2)^T\lambda + I_{n_2}\mu \geq d_2 \tag{13}$$
$$\lambda \geq 0, \mu \geq 0 \tag{14}$$

where λ and μ are Lagrangian multipliers vectors of appropriate size. I_{n_2} represents the identity matrix of order n_2. Now, we add the complementary slackness conditions we construct by using LBP1 and LBPD1 together with its respective dual constraints (13)-(14). We may obtain the so called GLCP counterpart for LBP1 as follows

$$\text{LBPG1:} \quad \max_{\{x,y,\lambda,\mu\}} \quad c_1^Tx + d_1^Ty$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$
$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$(B^2)^T\lambda + I_{n_2}\mu \geq d_2$$
$$\lambda \geq 0, \mu \geq 0$$
$$(b^2 - A^2x - B^2y)^T\lambda = 0 \tag{15}$$
$$(\mathbf{1}_{n_2} - I_{n_2}y)^T\mu = 0 \tag{16}$$
$$((B^2)^T\lambda + I_{n_2}\mu - d_2)^Ty = 0 \tag{17}$$

where (15)-(17) are the complementary slacknes conditions. To derive the GLCP counterpart for LBP2, we proceed similarly as for LBP1. In this case, the dual formulation for the follower problem can be written as

$$\text{LBPD2:} \quad \min_{\{\lambda_1,\mu_1,\mu_2,\mu_3\}} \quad \lambda_1^T(b^2 - A^2x) + \mu_1^Tz +$$
$$+\mu_2^T(\mathbf{1}_K - z) + \mu_3^T\mathbf{1}_{n_2} \tag{18}$$
$$\text{s.t.} \quad (B^2)^T\lambda_1 + I_{n_2}\mu_3 \geq d_2 \tag{19}$$
$$I_K\mu_1 + I_K\mu_2 \geq \mathbf{1}_K \tag{20}$$
$$\lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0 \tag{21}$$

where λ_1, μ_1, μ_2 and μ_3 are Lagrangian multiplier vectors respectively. Subsequently, the GLCP in this case reads

$$\text{LBPG2:} \quad \max_{\{x,y,z,\mu_1,\mu_2,\mu_3,\lambda_1\}} \quad c_1^Tx + d_1^Ty$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$
$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$w_k^Tx \leq s_k + M_kz_k \quad k = 1:K$$
$$p^Tz \leq \alpha, \quad 0 \leq z_k \leq 1 \quad \forall k = 1:K$$
$$(B^2)^T\lambda_1 + I_{n_2}\mu_3 \geq d_2$$
$$I_K\mu_1 + I_K\mu_2 \geq \mathbf{1}_K$$
$$\lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$$
$$\lambda_1^T(b^2 - A^2x - B^2y) = 0 \tag{22}$$
$$\mu_1^Tz = 0 \tag{23}$$
$$\mu_2^T(\mathbf{1}_K - z) = 0 \tag{24}$$
$$\mu_3^T(\mathbf{1}_{n_2} - y) = 0 \tag{25}$$
$$y^T((B^2)^T\lambda_1 + I_{n_2}\mu_3 - d_2) = 0 \tag{26}$$

In LBPG2, the last constraints (22)-(26) are due to the complementary slackness condition.

In the next subsection, we briefly illustrate how IMM algorithm works when solving a minmax relaxation derived from the GLCP [10].

B. The IMM Algorithm

To show how the IMM algorithm works, we take for illustration purposes, the GLCP we have already derived from the previous subsection denoted by LBPG2. Notice that this model is a quadratic optimization problem since their complementary constraints (22)-(26) are quadratic, and thus it is hard to solve directly. The first step of IMM consists in relaxing these quadratic constraints into the following Lagrangian function

$$\mathcal{L}(x, y, z, \lambda_1, \mu_1, \mu_2, \mu_3) =$$
$$= c_1^Tx + d_1^Ty +$$
$$+\lambda_1^T(b^2 - A^2x - B^2y) +$$
$$+\mu_1^Tz + \mu_2^T(\mathbf{1}_K - z) +$$
$$+\mu_3^T(\mathbf{1}_{n_2} - z) +$$
$$+y^T((B^2)^T\lambda_1 + I_{n_2}\mu_3 - d_2) \tag{27}$$

This allows writing a minmax relaxation for LBPG2 as follows

$$\text{LGN2:} \quad \min_{\{\mu_1,\mu_2,\mu_3,\lambda_1\}} \max_{\{x,y,z\}} \quad \mathcal{L}(x, y, z, \lambda_1, \mu_1, \mu_2, \mu_3)$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$
$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$w_k^Tx \leq s_k + M_kz_k \quad k = 1:K$$
$$p^Tz \leq \alpha, \quad 0 \leq z_k \leq 1 \quad \forall k = 1:K$$
$$(B^2)^T\lambda_1 + I_{n_2}\mu_3 \geq d_2$$
$$I_K\mu_1 + I_K\mu_2 \geq \mathbf{1}_K$$
$$\lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$$

The second step of IMM consists of decomposing LGN into two linear programming subproblems: LGNs and LGNd as

$$\text{LGNs:} \quad \max_{\{x,y,z,\varphi\}} \quad \varphi$$
$$\varphi \leq \mathcal{L}(x, y, z, \lambda_1^q, \mu_1^q, \mu_2^q, \mu_3^q),$$
$$\forall q = 0, 1, ..., N - 1 \tag{28}$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$

 ISBN: 978-1-61208-202-8

$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$w_k^T x \leq s_k + M_k z_k \quad k = 1 : K$$
$$p^T z \leq \alpha, \quad 0 \leq z_k \leq 1 \quad \forall k = 1 : K$$

and

$$\text{LGNd:} \quad \min_{\{\mu_1,\mu_2,\mu_3,\lambda_1,\beta\}} \beta$$
$$\beta \geq \mathcal{L}(x^q, y^q, z^q, \lambda_1, \mu_1, \mu_2, \mu_3),$$
$$\forall q = 1, ..., N \tag{29}$$
$$\text{s.t.} \quad (B^2)^T \lambda_1 + I_{n_2}\mu_3 \geq d_2$$
$$I_K\mu_1 + I_K\mu_2 \geq \mathbf{1}_K$$
$$\lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$$

where φ and β are defined as free real variables. Finally, the third step of the algorithm consists in solving iteratively both LGNs and LGNd. At iteration q, the auxiliary constraint (28) (resp. (29)) is added to LGNs (resp. LGNd) in order to enforce the convergence of their optimal solution values towards the optimal solution value of LGN. The iteration process stops when either $\beta - \varphi < \delta$ or $(\beta - \varphi)/\beta < \varepsilon$ for small $\delta > 0$ and $\varepsilon > 0$. The convergence of IMM is proven in [10]. Notice that even when IMM does not converge to a stationary point, it provides, at least, an upper bound for the GLCP. Hereafter, we denote by LGN1 and LGN2 the minmax relaxations we formulate starting from LBP1 and LBP2 respectively. In this paper, we compute upper bounds for LGN1 and LGN2 using IMM algorithm. Afterward, we compare these upper bounds with LP relaxations we derived from equivalent MIP formulations according to [1].

V. MIP and LP Formulations

In this subsection, we present for each GLCP problems (LBPG1 and LBPG2 respectively) an equivalent MIP formulation. The method basically consists of replacing each quadratic constraint of the GLCP by two linear constraints that include a new binary variable. According to [1], a MIP formulation for LBPG1 can be written as follows

$$\text{MIP1:} \quad \max_{\{x,y,\lambda,\mu,\nu^1,\nu^2,\nu^3\}} c_1^T x + d_1^T y$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$
$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$(B^2)^T\lambda + I_{n_2}\mu \geq d_2$$
$$\lambda \geq 0, \mu \geq 0$$
$$b^2 - A^2x - B^2y + L\nu^1 \leq L\mathbf{1}_{m_2} \tag{30}$$
$$\lambda \leq L\nu^1, \quad \nu^1 \in \{0,1\}^{m_2} \tag{31}$$
$$\mathbf{1}_{n_2} - I_{n_2}y + L\nu^2 \leq L\mathbf{1}_{n_2} \tag{32}$$
$$\mu \leq L\nu^2, \quad \nu^2 \in \{0,1\}^{n_2} \tag{33}$$
$$(B^2)^T\lambda + I_{n_2}\mu - d_2 + L\nu^3 \leq L\mathbf{1}_{n_2} \tag{34}$$
$$y \leq L\nu^3, \quad \nu^3 \in \{0,1\}^{n_2} \tag{35}$$

In this model, constraints in (30)-(31),(32)-(33),(34)-(35) are equivalent to constraints (15),(16),(17) in LBPG1 respectively. These constraints force at least one of the terms within each product term to be equal to zero. To this end, a large constant L is needed [1]. Similarly, we can derive a MIP formulation for LBPG2 as follows

$$\text{MIP2:} \quad \max_{\{x,y,z,\mu_1,\mu_2,\mu_3,\lambda_1,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5\}} c_1^T x + d_1^T y$$
$$\text{s.t.} \quad A^1x + B^1y \leq b^1$$
$$0 \leq x \leq \mathbf{1}_{n_1}$$
$$A^2x + B^2y \leq b^2$$
$$0 \leq y \leq \mathbf{1}_{n_2}$$
$$w_k^T x \leq s_k + M_k z_k \quad k = 1 : K$$
$$p^T z \leq \alpha, \quad 0 \leq z_k \leq 1 \quad \forall k = 1 : K$$
$$(B^2)^T\lambda_1 + I_{n_2}\mu_3 \geq d_2$$
$$I_K\mu_1 + I_K\mu_2 \geq \mathbf{1}_K$$
$$\lambda_1 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0$$
$$b^2 - A^2x - B^2y + L\theta_1 \leq L\mathbf{1}_{m_2} \tag{36}$$
$$\lambda \leq L\theta_1, \quad \theta_1 \in \{0,1\}^{m_2} \tag{37}$$
$$z + L\theta_2 \leq L\mathbf{1}_K \tag{38}$$
$$\mu_1 \leq L\theta_2, \quad \theta_2 \in \{0,1\}^K \tag{39}$$
$$\mathbf{1}_K - z + L\theta_3 \leq L\mathbf{1}_K \tag{40}$$
$$\mu_2 \leq L\theta_3, \theta_3 \in \{0,1\}^K \tag{41}$$
$$\mathbf{1}_{n_2} - y + L\theta_4 \leq L\mathbf{1}_{n_2} \tag{42}$$
$$\mu_3 \leq L\theta_4, \quad \theta_4 \in \{0,1\}^{n_2} \tag{43}$$
$$(B^2)^T\lambda_1 + I_{n_2}\mu_3 - d_2 + L\theta_5 \leq L\mathbf{1}_{n_2} \tag{44}$$
$$y \leq L\theta_5, \quad \theta_5 \in \{0,1\}^{n_2} \tag{45}$$

Analogously, in this model constraints (36)-(45) replace constraints (22)-(26) in LPBG2. We denote by LP1 and LP2 the corresponding linear programming relaxations derived from MIP1 and MIP2, respectively.

VI. Numerical Results

In this section, we present numerical results for MIP1, MIP2, LP1, LP2, LGN1 and LGN2. The input data is generated as follows. The entries in matrices A^1, A^2, B^1, B^2 are filled with random values uniformly picked from [-1,1] except for the last row which is uniformly filled with values in [0,1]. The entries of b^1, b^2 are generated in the following way:

$$b_i^1 = \sum_{j=1}^{n_1} A_{ij}^1 + \sum_{j=1}^{n_2} B_{ij}^1 + \rho_i^1, \quad i = \{1,..,m_1\} \tag{46}$$
$$b_i^2 = \sum_{j=1}^{n_1} A_{ij}^2 + \sum_{j=1}^{n_2} B_{ij}^2 + \rho_i^2, \quad i = \{1,..,m_2\} \tag{47}$$

Table I
UPPER BOUNDS AND OPTIMAL SOLUTIONS FOR THE DETERMINISTIC PROBLEM (LBPP)

#	Instance Size				MIP1	LGN1		Time LGN1	# LPs	LP1		Time LP1	Gaps	
	m_1	m_2	n_1	n_2		Ubs	Std			Ubs	Std		LGN1	LP1
1	25	25	50	50	318.6297	400.5155	55.2680	0.4578	19.9000	324.0572	53.2167	0.1703	20.5718	1.5381
2	25	25	50	100	570.9695	754.9484	45.3639	0.5563	15.7000	579.6330	45.8222	0.1641	24.2832	1.3892
3	25	25	50	150	800.2270	1078.6	38.0506	0.7156	14	806.3612	39.1128	0.1844	25.7922	0.7522
4	25	25	50	250	1319.1	1758.4	69.2061	1.1703	12.4000	1324	55.1661	0.2797	24.9855	0.3751
5	25	25	100	50	534.4669	616.3301	46.8627	0.9031	30.7000	541.6181	49.7854	0.2375	13.3941	1.3551
6	25	25	100	100	823.3725	993.2053	43.7868	0.9297	22.1000	830.9123	50.2394	0.2422	17.1145	0.8795
7	25	25	100	150	1061	1323.7	75.8933	1.0781	18.3000	1062.8	94.5218	0.2562	19.9749	0.1677
8	25	25	100	250	1501	1975.4	80.0849	1.5844	15.7000	1512.4	85.2606	0.2719	24.0231	0.7090
9	25	25	150	50	796.3497	879.2877	71.7902	1.1422	31.2000	799.2495	71.0114	0.2391	9.4376	0.3403
10	25	25	150	100	1050	1232	47.2699	1.4094	27.6000	1057.9	56.2905	0.2391	14.8390	0.7229
11	25	25	150	150	1288.4	1567.1	66.5875	1.4109	20.6000	1303.8	61.2962	0.2609	17.8121	1.1917
12	25	25	150	250	1763.7	2213.5	96.8940	2.3250	20.1000	1768.8	83.8630	0.3000	20.3168	0.2782
13	25	25	250	50	1348.7	1436.5	68.2226	1.4578	29.6000	1349.4	59.5212	0.2656	6.0902	0.0493
14	25	25	250	100	1541.7	1736.5	61.1188	2.0219	31.1000	1552	55.1324	0.2797	11.2055	0.6480
15	25	25	250	150	1777	2047.9	59.2596	1.9344	23.5000	1782.3	44.7907	0.2781	13.2086	0.2986
16	25	25	250	250	2292.5	2723	48.3565	2.5031	20.1000	2297.4	61.8148	0.3234	15.8171	0.2108
17	50	50	50	50	181.8254	256.2218	48.6907	0.8703	17.6000	184.6658	45.4258	0.2516	29.3389	1.4090
18	50	50	50	100	399.8696	570.6325	83.5419	2.3438	24	404.9089	63.2648	0.2562	29.8847	1.2420
19	50	50	50	250	1116.7	1581.4	60.9783	2.9844	14.8000	1117.8	57.5520	0.3422	29.3974	0.1003
20	50	50	50	500	2413.4	3281.7	68.6972	3.3344	10.6000	2415.2	56.4318	0.4969	26.4553	0.0747
21	50	50	100	50	338.3935	401.1113	80.9225	2.6719	34.6000	338.8181	68.0222	0.2562	15.5971	0.1357
22	50	50	100	100	639.0975	804.6371	58.2480	5.3641	39	642.3280	62.9545	0.2813	20.7142	0.5395
23	50	50	100	250	1403.2	1836.7	54.4123	4.9594	21.6000	1408.8	76.2232	0.3516	23.6332	0.4281
24	50	50	100	500	2595.3	3471.6	89.8953	5.4188	14.6000	2596.1	102.6473	0.4656	25.2606	0.0311
25	50	50	250	50	1146.4	1223.7	79.9139	6.2484	54.6000	1156.5	75.4866	0.3266	6.2685	0.8414
26	50	50	250	100	1374.8	1544.5	66.4153	7.4703	50.8000	1381.5	64.8566	0.3563	10.9772	0.4713
27	50	50	250	250	2136.1	2551.2	96.8040	10.4750	37	2137.3	69.5131	0.4203	16.2371	0.0592
28	50	50	250	500	3282.1	4180.2	79.3646	11.9203	23.7000	3287	73.9953	0.5359	21.4839	0.1534
29	50	50	500	50	2392.7	2472.2	94.0571	12.0469	60.4000	2394.9	88.9616	0.4484	3.2086	0.0864
30	50	50	500	100	2586.4	2750	43.8231	11.7516	52	2590.4	47.6176	0.4609	5.9523	0.1548
31	50	50	500	250	3386.5	3828.9	63.0136	16.3172	45.7000	3390.8	57.1894	0.5453	11.5474	0.1240
32	50	50	500	500	4574	5499.7	92.8812	18.2703	29.9000	4574.8	82.5642	0.6703	16.8290	0.0173
33	100	100	150	150	764.0421	999.6294	89.3524	32.7984	49.5000	767.1260	91.5415	0.4359	23.7201	0.3816
34	100	100	150	200	1010.3	1330.6	56.0125	38.4875	45.4000	1010.5	56.4980	0.4531	24.0978	0.0212
35	100	100	150	300	1483.4	2006.4	61.0033	33.0938	43.8000	1485.2	42.4945	0.5594	26.0603	0.1240
36	100	100	150	500	2518.9	3400.1	73.6681	39.9469	30.4000	2520.9	92.7737	0.7281	25.9167	0.0807
37	100	100	200	150	1107.4	1358.6	102.1064	46.0844	60.4000	1108.5	77.3447	0.4437	18.4364	0.1010
38	100	100	200	200	1362.7	1703.6	161.0181	49.2656	51.6000	1363.7	137.4408	0.4719	20.0013	0.0709
39	100	100	200	300	1774.1	2296.3	52.6927	38.4375	49.2000	1776	44.1224	0.5906	22.7256	0.1049
40	100	100	200	500	2742	3602.8	55.1760	45.6688	33.4000	2744.2	41.0465	0.7656	23.8907	0.0809
41	100	100	300	150	1495.9	1758.3	102.0865	65.8469	84.6000	1497.8	97.7465	0.5406	14.9401	0.1207
42	100	100	300	200	1782.6	2157.1	56.6420	45.8000	70.4000	1783.5	66.7431	0.5938	17.3630	0.0469
43	100	100	300	300	2196.5	2718.8	30.9680	50.4688	58.2000	2197.8	31.8063	0.7000	19.2058	0.0609
44	100	100	300	500	3259.4	4111.9	67.2305	73.8406	47	3261	91.4262	0.8625	20.7402	0.0461
45	100	100	500	150	2525	2768.5	140.0770	61.5875	89.4000	2525.2	130.6626	0.7438	8.7985	0.0105
46	100	100	500	200	2782.7	3146.5	73.4873	53.1187	67.6000	2786.3	105.9573	0.7813	11.5874	0.1255
47	100	100	500	300	3246.4	3765.5	128.5285	80.2375	76	3249.1	120.3366	0.8500	13.7950	0.0848
48	100	100	500	500	4159.1	5026.3	71.2547	90.6281	51.6000	4160.3	61.3897	1	17.2430	0.0282

where ρ_i^1 and ρ_i^2 for each i, are random numbers picked from the interval $[0, 2]$. This procedure ensures that the inducible region generated by the upper level and lower level constraints be non-empty and bounded. Each input data vector w_k, for each probabilistic constraint in LBP2, is chosen uniformly distributed from [0,1] while s_k are picked from the interval $[\frac{1}{2}\widetilde{W}_k, \widetilde{W}_k]$. Here, $\widetilde{W}_k$ is computed as $\widetilde{W}_k = w_k^T \mathbf{1}_{n_1}$ for $k = 1 : K$. Finally, vectors c_1, c_2, d_1, d_2 are randomly chosen from $(0, 10]$ and $\alpha = 0.05$. Again, this procedure guarantees boundedness for the feasible region of the bilevel instances, although it does not guarantee non-emptiness anymore [10].

Without loss of generality we set the large value L needed for the resolution of the MIP and LP formulations be equal to $L = 10^5$. The IMM algorithm as well as the MIP and LP formulations are implemented using Matlab 7.8 and Cplex 12.2. The simulations are run in a 2100 MHz computer with 2 Gb Ram under windows XP.

Table I shows numerical results for MIP1, LGN1 and LP1 while table II shows the same information for MIP2, LGN2 and LP2, respectively. These numerical results correspond to averages computed over 50 sample runs for each instance, except for the instances 33 to 48 in tables I and II. For these instances, we only compute the average over 10 runs since solving the MIP models become prohibitive for larger instances. The two tables provide similar information. In table I, columns 2 to 5 give the instance sizes. Column 6 provides the optimal solution of MIP1. Columns 7 and 8 give the upper bounds and the standard deviation obtained while using IMM to solve LGN1. Columns 9 and 10 give the cpu time in seconds and the number of LPs IMM needs to converge. Similarly, columns 11 to 13 provide the upper bounds we obtain with the LP1 relaxation, its standard deviation and the cpu time in seconds. Finally, relative gaps are given in columns 14 and 15 for LGN1 and LP1, respectively. The gaps are computed as $\left(\frac{Ubs-MIP1}{Ubs}\right) \cdot 100$ in each case.

Table II provides exactly the same information for MIP2, LP2 and LGN2. The only difference now, is that the second column gives the number of scenarios $k = \{1, .., K\}$ we add in the leader's problem. From the numerical results, we mainly observe in table I, that the gaps decrease with

 ISBN: 978-1-61208-202-8

Table II
UPPER BOUNDS AND OPTIMAL SOLUTIONS FOR THE STOCHASTIC PROBLEM (SLBPP)

#	Instance Size					MIP2	LGN2		Time LGN2	# LPs	LP2		Time LP2	Gaps	
	K	m_1	m_2	n_1	n_2		Ubs	Std			Ubs	Std		LGN2	LP2
1	25	25	25	100	100	784.3203	989.2347	33.8779	1.3563	21.4000	820.4212	34.9967	0.2172	20.6631	4.3371
2		25	25	100	250	1513.4	2019.9	43.3322	1.7859	14	1564.5	69.4705	0.2281	25.0775	3.2478
3		25	25	250	100	1433.4	1742.2	48.9163	3.5781	35.5000	1583	45.0512	0.2406	17.7093	9.4391
4		25	25	250	250	2082.1	2700.1	57.2095	4.0766	23.4000	2267.3	76.3264	0.3516	22.8960	8.1650
5	50	25	25	100	100	760	979.1	42.86	1.7828	23.6999	795.01	49.1429	0.2	22.3878	4.3532
6		25	25	100	250	1487.31	1990.43	53.6890	1.9218	14	1555.41	51.2573	0.2250	25.2638	4.3767
7		25	25	250	100	1399.5	1707	56.8898	3.6875	29.9000	1553.6	53.2098	0.2656	17.9859	9.8821
8		25	25	250	250	2127.5	2719.5	55.6479	3.9328	20.8000	2288.7	62.2031	0.3094	21.7517	7.0396
9	75	25	25	100	100	760.4551	986.0295	31.6953	2.5156	25.9000	807.3817	36.6528	0.2172	22.8722	5.7832
10		25	25	100	250	1497.7	1998.7	54.0413	2.6828	17.4000	1573.3	62.6726	0.2578	25.0623	4.7602
11		25	25	250	100	1388.2	1743.4	60.6976	4.1031	27.3000	1572.8	59.4721	0.2938	20.3690	11.7179
12		25	25	250	250	2131.8	2761.1	60.2416	4.9328	22.5000	2311.9	73.1240	0.3250	22.7843	7.7919
13	100	25	25	100	100	772.6953	1001.2	84.1361	11.4016	37	826.4422	65.5359	0.2500	22.6190	6.3321
14		25	25	100	250	1472.6	1966.2	71.4463	2.7703	16.1000	1535.6	56.7641	0.2641	25.0811	4.0855
15		25	25	250	100	1377.4	1726.4	41.9370	4.9156	26.5000	1552.3	35.1868	0.3234	20.2008	11.2598
16		25	25	250	250	2102.2	2708.9	76.7820	5.5844	22.1000	2283.2	84.8015	0.3625	22.3871	7.8996
17	25	50	50	100	100	604.5257	781.5844	58.1403	6.0828	33.6000	612.4399	55.7202	0.2172	22.6608	1.2270
18		50	50	100	500	2540.1	3455.1	71.2097	6.6953	14.9000	2585.6	65.6119	0.4031	26.4760	1.7418
19		50	50	500	100	2376.6	2822.8	58.5527	14.5063	45.9000	2645.4	62.4349	0.4484	15.7803	10.1482
20		50	50	500	500	4275.7	5460.6	89.9955	26.5719	33.7000	4594.6	122.9921	0.6375	21.6995	6.9387
21	50	50	50	100	100	626.8187	814.5498	67.9897	6.5563	31.9000	638.3389	69.7229	0.2250	23.1182	1.6959
22		50	50	100	500	2583.6	3489.2	73.0974	7.5297	15.6000	2631.8	88.4050	0.4188	25.9651	1.8243
23		50	50	500	100	2272.6	2747.3	68.7120	20.6469	54.5000	2581.8	66.2905	0.5047	17.2523	11.9420
24		50	50	500	500	4229.5	5455.9	68.9315	28.6516	34.5000	4577	80.0359	0.6859	22.4812	7.5911
25	75	50	50	100	100	622.8990	794.1815	78.9943	3.9438	12.6000	637.6912	70.7837	0.2422	21.3467	2.1283
26		50	50	100	500	2573.2	3484.7	90.6039	8.0594	15.8000	2632	121.2170	0.4266	26.1898	2.2405
27		50	50	500	100	2289	2757.2	54.4159	27.1641	57.5000	2588.4	70.9270	0.5641	16.9958	11.5822
28		50	50	500	500	4284.8	5507.7	143.5341	28.1391	31.8000	4641.4	126.0542	0.7375	22.1955	7.6747
29	100	50	50	100	100	663.1588	865.8819	88.8559	8.8281	30.7000	684.2872	77.8045	0.2531	23.3672	2.9595
30		50	50	100	500	2532.7	3456.1	59.4832	9.0188	16.4000	2592.8	71.4109	0.4422	26.7103	2.2995
31		50	50	500	100	2305.5	2821.9	74.0348	25.1313	49.4000	2649.8	66.3914	0.6234	18.2569	12.9559
32		50	50	500	500	4182	5405.5	109.3389	30.8578	32	4537.1	131.9653	0.8078	22.6406	7.8144
33	25	100	100	200	200	1246.5	1583	62.0970	54.3094	50.6000	1252.3	47.2326	0.4969	21.2346	0.4482
34		100	100	200	500	2634.2	3560.8	56.6990	54.5812	35	2688.9	35.4166	0.7688	26.0167	2.0287
35		100	100	500	200	2709	3153.9	130.3294	84.5500	77	2838.2	139.4445	0.8281	14.1019	4.4996
36		100	100	500	500	4074.9	5194.1	140.2969	118.9000	57.2000	4306	124.6958	1.0531	21.5359	5.3538
37	50	100	100	200	200	1260.4	1579.9	77.4642	52.2313	44.4000	1270.7	67.9033	0.5281	20.1980	0.7712
38		100	100	200	500	2697.9	3641.6	125.9619	42.6219	28.4000	2756	153.8828	0.7813	25.9101	2.0966
39		100	100	500	200	2562.8	3044.8	73.0194	148.5656	101.4000	2696	102.6904	0.8500	15.8426	4.9312
40		100	100	500	500	3978.4	5096.3	105.5789	125.0719	62.4000	4209.8	82.0509	1.1375	21.9291	5.4933
41	75	100	100	200	200	1255.6	1612.9	42.5691	85.2406	63.8000	1275.4	67.2191	0.5375	22.1537	1.5483
42		100	100	200	500	2711.9	3683.5	29.0749	51.3500	32.6000	2775.8	99.9843	0.8000	26.3931	2.3101
43		100	100	500	200	2586.1	3039.6	131.0485	108.6281	72.8000	2769.2	95.2660	0.9031	14.8680	6.6020
44		100	100	500	500	3983.2	5108.9	96.7893	113.1469	55.2000	4223.8	111.6474	1.1406	22.0329	5.6904
45	100	100	100	200	200	1318.8	1649.6	70.3429	90.2156	60.4000	1340.7	63.9094	0.5594	20.0475	1.6257
46		100	100	200	500	2681.1	3622.4	76.0711	57.9453	35.2500	2733.7	108.2080	0.8164	26.0016	1.8915
47		100	100	500	200	2549.2	3121.6	128.1110	154.1375	93.2000	2778	119.7769	0.9344	18.2931	8.1927
48		100	100	500	500	3992.3	5062.4	116.0704	122.7125	56.2000	4200.6	149.1135	1.1844	21.1434	4.9305

the size of the instances and that they are very tight when compared to the optimal solution of the problem. On the other hand, the cpu times show that the LP relaxations are faster than IMM algorithm. For the latter, we observe a rapid growth which is directly related to the size of the instances. Concerning the average number of LPs IMM needs to converge, we notice a slightly increasing trend. Then, the growth in cpu time can be explained by the size of the LPs it solves within each iteration. Finally, we can see that the standard deviations show a constant behavior when compared to the average upper bounds in both cases, for the IMM and for the LP relaxation. The numerical results in table II, are a little bit different. Here, we observe that the relative gaps are not as tight as in table I for the LP relaxations, but still better than those obtained with IMM algorithm. Although, they become tighter as the size of the instances increase which is an interesting result. We can also see that the effect of increasing the number of scenarios in the probabilistic constraints does not have a significant impact on the numerical results. It is easy to note that these gaps are tighter when $n_1 < n_2$. Concerning the cpu times, we observe an increasing trend for the Lagrangian approach while for the LP relaxation they almost remain unchanged. The average number of LPs solved by IMM shows a slight increasing trend. Finally, we observe that the standard deviation behaviors are similar.

VII. CONCLUSION AND FUTURE WORK

In this paper, we computed upper bounds and optimal solutions for a deterministic linear bilevel programming problem and a probabilistic constrained linear bilevel counterpart due to [7]. The upper bounds were computed using the iterative minmax algorithm proposed in [10] and also using linear programming relaxations we derived according to the approach proposed in [1].

To this end, we transformed all the linear bilevel models into the so called Global Linear Complementarity problems from which we derived equivalent MIP and LP formulations. Our numerical results showed tight relative gaps for the upper bounds obtained with the LP relaxations. On the opposite, those obtained with IMM algorithm were considerably larger in all the instances we tested. In particular, we obtained better gaps on deterministic instances rather than for the stochastic ones, which means that probabilistic constraints decrease the effectiveness of the LP relaxations.

Finally, we argue that even when the LP relaxations give tighter bounds on these specific problems, IMM algorithm still provides a more general framework as it can be used to handle any type of non-linear constraints. Therefore, future research should also be devoted to strengthen IMM while testing it on different types of problems.

ACKNOWLEDGMENT

The first author is grateful for the financial support given by Conicyt Chilean government through the Insertion project number: 79100020.

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