An Orbit Tracking Algorithm in Quantum Systems

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***Abstract*—A convergence algorithm for the orbit tracking of the free-evolutionary target system in closed quantum system is studied in this paper. The unitary transformation is used to change the problem of orbit tracking into the one of state transferring. The Lyapunov function with a virtual mechanical quantity *P* is employed to design a Lyapunov–based controller for such a state transferring. The target states are divided into two classes: diagonal and non-diagonal. In the first class, the specific convergent conditions for the target state of diagonal states are studied; in the second class, the target states with non-diagonal superposition states and non-diagonal mixed-states are treated separately by two quite different ways. At last, the system simulation experiments are performed on a two-level quantum system and the tracking process is illustrated on the Bloch sphere.**

Keywords-orbit tracking; state transferring; the Lyapunov stability theorem; convergence

I. INTRODUCTION

In recent years, the closed quantum system control theory has been increasingly developed. Quantum state transferring and quantum system tracking have been fully researched. The effective control algorithm has been designed based on certain control theory, such as optimal control [1-3], Bang-bang control [4, 5], the Lyapunov-based control [6-9] and so on. Among them, optimal control produces iterative control algorithm, while bang-bang control is realized experimentally by pulses. Further demand for accuracy may results in the unacceptable number of pulses. The Lyapunov stability theorem may obtain an analytical algorithm, which can help us analyze the characteristics of the system.

In this paper, the quantum system orbit tracking is investigated. For the orbit tracking, the target system is the free-evolutionary of the quantum system to be controlled. What we expect is to make the controlled system track the target system. The control goal is carried on in two steps: firstly, the system orbit tracking problem is changed into the state transferring one; secondly, a convergent control algorithm is designed to complete the goal.

In fact, the control algorithm designed by the Lyapunov stability theorem is only stable, which cannot guarantee the quantum system converges to desired target state. For this purpose, a convergent rather than just stable control algorithm is needed to manipulate the quantum system. Some papers on this topic have been reported [9, 10]. Among them, complete state transferring with target state of eigenstates and diagonal mixed-states in closed quantum system has been proved to be convergent conditionally [11, 12]. However, there is an open problem on the convergence of the non-diagonal target states including the superposition states and some mixed-states. In [12], a Lyapunov function based on virtual mechanical quantity *P* was proposed to get the convergence conditions for the diagonal mixed-states. However, it did not give specific instructions on how to design *P*. We will discuss the detailed convergence conditions of diagonal target states and non-diagonal target states in this paper.

The rest of the paper is organized as follows. In Section II, the system model is described by the Liouville equation. The Lyapunov stability theorem is used to design the trajectory tracking control algorithm in Section III. Section IV is divided into two parts, the first part is to handle the convergence of initial target state with diagonal mixes-state and the second part is the one for non-diagonal initial target state. In Section V, numerical simulation experiments are performed on a two-level system. Finally, Section VI concludes this paper.

II. DESCRIPTION OF THE CONTROL SYSTEM MODEL

In this paper, we use quantum-Liouville equation to describe the control system model

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t)=\left[H_0+\sum_m f_m(t)H_m,\hat{\rho}(t)\right]\quad \hat{\rho}(0)=\hat{\rho}_0 \tag{1a}$$

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}_f(t)=\left[H_0,\hat{\rho}_f(t)\right]\quad \hat{\rho}_f(0)=\hat{\rho}_{f0} \tag{1b}$$

where H_0 is the free Hamiltonian representing the energy of the system and H_m represent system's control Hamiltonians, all of them will be assumed to be time-independent. $f_m(t)$ are time-dependent external control fields. The Planck constant is chosen as $\hbar=1$ for convenience. The system state is denoted by density matrix $\hat{\rho}(t)$, the initial states of which at $t=0$ is $\hat{\rho}_0$. Similarly, $\hat{\rho}_f(t)$ is the target system state and its initial value is $\hat{\rho}_{f0}$.

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The control objective is to design a convergent control algorithm to make the state $\hat{\rho}(t)$ of (1a) follow target state $\hat{\rho}_f(t)$ of system (1b).

Firstly, to change the orbit tracking problem into a state transferring one, a unitary transformation $U(t)=\exp(-itH_0)$ is performed on system (1): $\rho(t)=U^{\dagger}(t)\hat{\rho}(t)U(t)$, $\rho_f(t)=U^{\dagger}(t)\hat{\rho}_f(t)U(t)$, where "†" denotes conjugate transpose, "^" denotes states before unitary transformation. The system (1) after this transformation is represented by

$$i\frac{\partial}{\partial t}\rho(t)=[\sum_m f_m(t)H_m(t),\rho(t)] \quad \rho(0)=\hat{\rho}_0 \quad (2a)$$

$$i\frac{\partial}{\partial t}\rho_f(t)=0 \qquad \rho_f(0)=\hat{\rho}_{f0} \qquad (2b)$$

where $H_m(t)=U^{\dagger}(t)H_mU(t)$.

Calculate (2b), one gets

$$\rho_f(t)=\hat{\rho}_{f0}. \qquad (3)$$

After the above transformation, we have changed the tracking problem into the state transferring one. At present, the control goal of system (1a) following target system (1b) becomes the one of regulating state of (2a) to target state (3).

III. CONTROL ALGORITHM DERIVATION

Among many control methods, the Lyapunov-based method is simpler and easy to design. The basic idea of the Lyapunov method is to select a Lyapunov function $V(x)$ which satisfies the following three conditions: a) $V(x)$ is continuous and its first-order partial derivatives is also continuous in its definition; b) $V(x)$ is positive semi-definite, i.e., $V(x)\geq 0$; c) The first order time derivative of the Lyapunov function is negative semi-definite, i.e., $\dot{V}(x)\leq 0$.

There are usually three kinds of Lyapunov functions [11]. Here, the Lyapunov function based on virtual physical quantity is chosen

$$V(\rho)=tr(P\hat{\rho}), \qquad (4)$$

where P is virtual physical observable operator.

To obtain a convergent control algorithm, the first-order time derivation of function (4) is obtained as

$$\dot{V}=-\sum_m f_m(t)tr(iH_m(t)[\rho(t),P]). \qquad (5)$$

For the sake of simplicity and availability, we let each item on the right side of (5) of summation sign be non-positive in order to ensure $\dot{V}\leq 0$. The control algorithm can be derived as

$$f_m(t)=-k_m tr(iH_m(t)([\rho(t),P])), \ k_m>0 \qquad (6)$$

where, $k_m>0$ is the control gain to adjust the convergence speed of the system state.

Comparing system (1) with (2a) and (3), one can see that: the system (1) is autonomous and the system (2) is not, however, the time-dependent target state $\rho_f(t)$ in (1b) becomes a stationary state $\hat{\rho}_{f0}$ in (3).

As we know, the control algorithm (6) designed by the Lyapunov stability theorem is usually only a stable one, which can not guarantee that the system converges to target state. For this reason, we need to do further study to get the convergence condition $\dot{V}<0$, which may guide people to design a convergent control algorithm. Next, we study this problem in detail.

IV. CONTROL ALGORITHM CONVERGENCE ANALYSIS

In order to implement effective control on a quantum system, a convergent control algorithm is even more important than the control algorithm itself. In the process of state transferring, the desired target is a time-invariant state. The variety of quantum states such as eigenstates, superposition states and mixed-states makes the different expressions of target state. On the one hand, from the system control perspective, the initial state of target system will be grouped into stationary state and the non-stationary one. For the stationary target state, according to (1b), $i\hbar\frac{\partial}{\partial t}\hat{\rho}_f(t)=0$ holds, and so does $\hat{\rho}_f(t)=\hat{\rho}_f(0)$. In such a case, one can get $[H_0,\hat{\rho}_f(0)]=0$. The system tracking of (1a) is equivalent to the state transferring. For the non-stationary initial target state, the unitary transformation has changed the tracking problem into the state transferring one. Therefore, we can always change the system tracking into the state transferring. On the other hand, from the convergence of view, the initial target states are divided into two kinds. The first one is diagonal initial target state including eigenstates and some mixed-states. It can be represented by a diagonal density matrix. The second kind is non-diagonal initial target state, including superposition state and some mixed-states, whose convergence has not been resolved fully so far.

For autonomous system (1), the LaSalle's invariant principle can be used to analyze the convergence, where two assumptions are needed [13]: Assumption 1: H_0 is strongly regular, i.e., all the transition frequencies (differences of pairs of energy levels) are different, viz. $\Delta_{jk}\neq\Delta_{pq}$, $(j,k)\neq(p,q)$, where $\Delta_{jk}=\lambda_j-\lambda_k$ and λ_j is an

eigenvalues of H_0. Assumption 2: Control Hamiltonian H_m is full-connected, viz. $H_m \in \{\hbar h_{jk} \mid h_{jk} = |j\rangle\langle k| + |k\rangle\langle j|, j > k\}$, where $|j\rangle$ is the eigenstate associated with λ_j. Because of the unitary evolution of closed quantum system, if being reachable then target state must be unitarily equivalent to the initial state, i.e., there exists a unitary transformation U such that $\hat{\rho}_0 = U\hat{\rho}_{f0}U^{\dagger}$. We make it as Assumption 3.

The LaSalle's invariant principle is not able to deal with the non-autonomous system (2), however, the Barbalat lemma can be applied based on above three assumptions, whose content is [14]: If scalar function $V(x,t)$ satisfies: (1) $V(x,t)$ is lower bounded; (2) $\dot{V}(x,t)$ is negative semi-definite; (3) $\dot{V}(x,t)$ is uniformly continuous in time, then $\dot{V}(x,t) \to 0$ as $t \to \infty$. One can see from the Lyapunov function (4) that (4) satisfies all the three conditions of Barbalat lemma: (1) $V = tr(P\rho) \geq 0$ is lower bounded for a positive P; (2) Its first order derivative is negative semi-definite under control algorithm (6); (3) The third condition can be replaced by the existence and continuity of the second derivation of $V(x,t)$: $\ddot{V}(\rho,t) = -\sum_m f_m(t)\{tr(i\dot{H}_m(t)[\rho,P]) + tr(iH_m(t)[\dot{\rho},P])\}$ is bounded for a bounded input.

According to the Barbalat Lemma, the first derivation of the Lyapunov function converges to zero for $t \to \infty$, viz., $\dot{V}(\rho(\infty),\infty) = 0$. So a limitation states set at $t \to \infty$ is deduced by the Barbalat Lemma. We defined it as a stable set $\mathcal{R}$, which is a concept similar to invariant set. According to the formula (6), the states in stable set satisfy $f = 0$. For the non-autonomous system (2), if $\rho \in \mathcal{R}$, then $\dot{\rho}$=0 holds for $f = 0$. It means that, once the system evolves into the stable set, it will stop at this set.

The stable set $\mathcal{R}$ is the set of critical points on any dynamic trajectory, viz.

$$\mathcal{R} \equiv \{\rho_s : tr(iH_m(t)[\rho_s, P]) = 0, \forall m, t\}, \quad (7)$$

where ρ_s denote critical stable points of (2), and $\rho_f \in \mathcal{R}$. The controlled system may converge to any one of states in stable set $\mathcal{R}$. Now P is to be constructed to make the system converge to the target state. According to Assumption 2, (7) is rewritten as $\mathcal{R} \equiv \{\rho_s : [\rho_s, P] = D\}$, where D is a diagonal matrix. Obviously, if P is chosen as a diagonal matrix, D is zero one, otherwise we can always design a set of proper eigenvalues of P to simplify (7) as [15]:

$$\mathcal{R} \equiv \{\rho_s : [\rho_s, P] = 0\}. \quad (8)$$

Equation (8) is the stable set to be discussed in this paper. It is known that (4) is a function of state and the system (2) will converge to stable set (8). Whether the system converges to target state or not depends on the relative position among target state, the controlled initial state, and all other stable states than target state. To make the system converge to the target state, the following condition is needed, viz. the relationship among initial state, target state, and other stable states must satisfy [12]

$$v(\rho_f) < v(\rho_0) < v(\rho_s). \quad (9)$$

Equation (9) is the condition to ensure the convergence of the controlled system. How to realize (9) is another key point which needs to be solved. In the following we'll focus on this topic by designing the suitable *P*. The solutions are given by two cases.

A. The target state is diagonal mixed-state

The convergent conditions for diagonal target states have been investigated adequately [11-13]. However, the concrete construction of P has not been mentioned in previous results. In this section, we'll discuss how to design P in (4) to satisfy (9).

Suppose the target state ρ_f is a diagonal target state and $\{\lambda_i, i = 1, 2 \cdots n\}$ is the eigen-spectrum of ρ_0. The target state ρ_f should be one permutation of $\{\lambda_i, i = 1, 2 \cdots n\}$, viz. $\rho_f = diag(\lambda_1, \lambda_2 \cdots \lambda_n)$. The other states ρ_s in $\mathcal{R}$ are the different permutations of eigen-spectrum. In order to construct a P satisfied (9), three steps are performed:

Firstly, P is constructed to make ρ_f be the point corresponding to the minimum value of (4), which is realized by the following lemma:

Lemma 1: If the diagonal target state is $\rho_f = diag(\lambda_1, \lambda_2 \cdots \lambda_n)$, the matrix P corresponding to ρ_f is $P = diag(p_1, p_2, \cdots p_n)$, then ρ_f is the point for the Lyapunov function (4) to be the minimum if the diagonal element p_i of P meets $(\lambda_i - \lambda_j)(p_i - p_j) < 0, \forall i \neq j$.

The proof of Lemma 1 is in Appendix 1.

Secondly, based on Lemma 1, a further study on P is carried out and (9) is divided into two parts:

Part 1: $v(\rho_f) < v(\rho_0)$

The condition $v(\rho_f) < v(\rho_0)$ indicates that the Lyapunov function value of initial state is larger than that of target state. Otherwise it is inconsistent with the monotonically decreasing of (4) and the target state will be unreachable. It is easy to obtain

$v(\rho_f)-v(\rho_0)=\sum_{i=1}^{n}(P)_{ii}\left(\lambda_i-(\rho_0)_{ii}\right)$. Based on the relationship between eigenvalues and matrix diagonal elements, the expression $\sum_{i=1}^{n}\lambda_i=\sum_{i=1}^{n}\mu_i=\sum_{i=1}^{n}(\rho_0)_{ii}=1$ holds, where $(\rho_0)_{ii}$ is the *i*-th diagonal element of the initial state ρ_0 . So there must be at least one *k* to make $\lambda_k<(\rho_0)_{kk}$ hold. If one wants to make $v(\rho_f)-v(\rho_0)=(P)_{kk}\left(\lambda_k-(\rho_0)_{kk}\right)+\sum_{i=1,i\neq k}^{n}(P)_{ii}\left(\lambda_i-(\rho_0)_{ii}\right)<0$ hold, where $(P)_{kk}$ is the *k*-th diagonal element of *P*, then we choose a certain *k* satisfied $\lambda_k<(\rho_0)_{kk}$, one gets

$$(P)_{kk}>\sum_{i=1,i\neq k}^{n}(P)_{ii}\left(\lambda_i-(\rho_0)_{ii}\right)\Big/\left((\rho_0)_{kk}-\lambda_k\right). \quad (10)$$

Maybe there are more than one *k* to satisfy $\lambda_k<(\rho_0)_{kk}$, we usually choose the one which makes $(P)_{kk}$ correspond to a larger value such that $v(\rho_f)<v(\rho_0)$ holds.

Part 2: $v(\rho_0)<v(\rho_s)$

ρ_s should be one of the permutations of eigen-spectrum. According to Assumption 3, $\rho_0=U\rho_f U^{\dagger}$, so one has

$$tr(P\rho_0)=tr(PU\rho_f U^{\dagger})=\sum_{i=1}^{n}(P)_{ii}\sum_{j=1}^{n}(\rho_f)_{jj}\left(U_{ij}\right)^2$$

$$tr(P\rho_s)=\sum_{i=1}^{n}(P)_{ii}(\rho_s)_{ii}$$

$$tr(P\rho_0)-tr(P\rho_s)=\sum_{i=1}^{n}(P)_{ii}\left(\sum_{j=1}^{n}(\rho_f)_{jj}\left(U_{ij}\right)^2-(\rho_s)_{ii}\right)$$

ρ_s and ρ_f have the same spectrums, there must be $(\rho_f)_{kk}=(\rho_s)_{ii}$. For any unitary matrix *U* , there exist $UU^{\dagger}=U^{\dagger}U=I$ and $\sum_{j=1}^{n}\left(U_{ij}\right)^2=1$. So one can get

$$tr(P\rho_0)-tr(P\rho_s)=\sum_{i=1}^{n}(P)_{ii}\left(\sum_{j\neq k}^{n}(\rho_f)_{jj}\left(U_{ij}\right)^2-(\rho_f)_{kk}\sum_{j\neq k}^{n}\left(U_{ij}\right)^2\right)$$

$$=\sum_{i=1}^{n}(P)_{ii}\sum_{j\neq k}^{n}\left((\rho_f)_{jj}-(\rho_f)_{kk}\right)\left(U_{ij}\right)^2$$

For the above equation, there is at least one *l* to make $(\rho_f)_{ll}-(\rho_f)_{kk}<0$ hold. To make $tr(P\rho_0)-tr(P\rho_s)<0$, the following expression is needed

$$(P)_{ll}>(\sum_{i\neq l}^{n}(P)_{ii}\sum_{j\neq k}^{n}\left((\rho_f)_{jj}-(\rho_f)_{kk}\right)\left(U_{ij}\right)^2+(P)_{ll}\bullet$$
$$\sum_{j\neq k,j\neq l}^{n}\left((\rho_f)_{jj}-(\rho_f)_{kk}\right)\left(U_{lj}\right)^2)/\left((\rho_f)_{kk}-(\rho_f)_{ll}\right) \quad (11)$$

The above process is to construct *P* for diagonal target states. We conclude that: if the target state is of diagonal, a Hermite and positive diagonal matrix *P* is selected. To ensure the convergence, the diagonal elements of *P* must satisfy Lemma 1, (10) and (11) simultaneously.

B. The target state is of non-diagonal density matrix

It is more complicated to analyze the convergence for the non-diagonal target state. The idea is as follows: let the non-diagonal target state be changed into diagonal one and the virtual physical quantity *P* is designed as that in *A*. However, the superposition state is one kind of pure states, which can be represented by wave functions as $\rho_f=|\psi_f\rangle\langle\psi_f|$. In this case, the diagonalization of target state is not necessary. Next, we go onto the analysis of non-diagonal superposition state and mixed-state in detail.

1) In the case of non-diagonal superposition state

Prior to analysis, another lemma is introduced.

Lemma 2 [16]**:** For the *n*-level Hermite matrix *A* and *B*, if they are commutive, viz. [*A, B*] =0, then *A* and *B* own the same eigenstates. We rewrite *P* according to its eigen-decomposition as $P=\sum_k p_k|\psi_k\rangle\langle\psi_k|$, where $|\psi_k\rangle$ is its eigenstate and p_k is eigenvalue. According to $\rho_f\in\mathcal{R}$ and **Lemma 2**: *P* is demonstrated as

$$P=p_1|\psi_f\rangle\langle\psi_f|+\sum_{k=2}^{n}p_k|\psi_k\rangle\langle\psi_k|\text{, where }|\psi_1\rangle=|\psi_f\rangle\text{. And}$$
$$\langle\psi_i|\psi_j\rangle=0,\ \text{for } i\neq j\,. \quad (12)$$

And ρ_s should be

$$\rho_s=\lambda_1|\psi_f\rangle\langle\psi_f|+\sum_{k=2}^{n}\lambda_k|\psi_k\rangle\langle\psi_k| \quad \sum_{k=1}^{n}\lambda_k=1 \quad (13)$$

It is known that the states ρ_0 and ρ_s have the same spectrum under the unitary evolution, and therefore ρ_s has the same eigenvalues with ρ_0, so does the target state ρ_f.

Substituting $\rho_f=|\psi_f\rangle\langle\psi_f|$ into (13), the eigen-spectrum of ρ_f is $\{1,0,\cdots 0\}$. Then for ρ_s, there is only one eigenvalue λ_i to be non-zero, viz. $\rho_s=\lambda_i|\psi_i\rangle\langle\psi_i|$ $(\lambda_i=1)$.

If it denotes $\rho_j=|\psi_j\rangle\langle\psi_j|$ in (12), then

$$\begin{cases} v(\rho_f)=tr(P\rho_f)=p_1 \\ v(\rho_0)=p_1 tr(\rho_f\rho_0)+\sum_{k=2}^{n} p_k tr(\rho_k\rho_0) \\ v(\rho_s)=p_j \quad (j\neq 1) \end{cases} \tag{14}$$

Combined (14) with (9), a suitable P must be constructed to satisfy

$$0<p_1<p_1 tr(\rho_f\rho_0)+\sum_{k=2}^{n} p_k tr(\rho_k\rho_0)<p_j \ (j\neq 1) \tag{15}$$

It can be seen from (15) that the virtual mechanical quantity P may not be a diagonal matrix for non-diagonal superposition target state. The P constructed based on (12) and (15) can guarantee the convergence of non-diagonal superposition target state, where (12) describes how to construct the eigenstates and (15) is to determine the eigenvalues. Moreover, the eigenvalue p_1 of P, whose corresponding eigenstate is the target state, is the smallest one.

2) In the case of non-diagonal mixed-state

For system (1), suppose the initial target state $\hat{\rho}_{f0}$ is a non-diagonal mixed-state. The solution of (1b) is $\hat{\rho}_f(t)=e^{-iH_0t}\hat{\rho}_{f0}e^{iH_0t}$. To deal with this situation, a unitary transformation has transformed the tracking problem of (1) into the state transferring one of (2). We follow the idea of changing the non-diagonal $\hat{\rho}_{f0}$ into a diagonal one by another unitary transformation and then a convergent control algorithm can be designed based on *A*.

In system (2), the target state $\hat{\rho}_{f0}$ is a Hermite matrix, so it exists another unitary transformation U_f to meet $U_f\hat{\rho}_{f0}U_f^{\dagger}=D_f$. It is performed on system (2), viz., $\rho'=U_f\rho U_f^{\dagger}$, $\rho'_f=U_f\rho_f U_f^{\dagger}=D_f$, then the system (2) becomes

$$i\hbar\frac{\partial}{\partial t}\rho'(t)=\left[H_0+\sum_{m=1}^{M} f_m(t)H_{mt},\rho'(t)\right] \quad \rho'(0)=U_f\hat{\rho}_0U_f^{\dagger} \tag{16a}$$

$$i\hbar\frac{\partial}{\partial t}\rho'_f(t)=0 \qquad \rho'_f(0)=D_f \tag{16b}$$

where $H_{mt}=U_fH_mU_f^{\dagger}$.

After unitary transformation U_f, the tracking of target system with non-diagonal initial state $\hat{\rho}_{f0}$ in (2) can be changed into the tracking of a diagonal stationary state in (16). According to (8), the stable set of (16) is still $\mathcal{R}\equiv\{\rho_s:[\rho_s,P]=0\}$. The convergence analysis is the same as that in *A*.

In conclusion, we have acquired the convergence conditions for non-diagonal target states.

V. APPLICATIONS AND EXPERIMENTAL RESULTS ANALYSES

In this part, a two level atom system controlled by a single control field is considered. Take superposition target state for example, the effectiveness of the proposed method will be illustrated.

The free Hamiltonian of the controlled system (1) is $H_0=\omega\sigma_z$ and the control Hamiltonian is $H_1=\sigma_x$, where $\sigma_i(i=x,y,z)$ denotes Pauli matrix and $\sigma_x=[0\ \ 1;1\ \ 0]$, $\sigma_z=[1\ \ 0;0\ \ -1]$. Obviously, this example satisfies the three conditions in Section IV.

The initial state of (1a) is $|\psi_0\rangle=\frac{1}{\sqrt{3}}|0\rangle+\frac{\sqrt{2}}{\sqrt{3}}|1\rangle$ and the initial target state of (1b) is $|\psi_f\rangle=\frac{1}{\sqrt{8}}|0\rangle+\frac{\sqrt{7}}{\sqrt{8}}|1\rangle$. They are both non-diagonal superposition states. The design process of a convergent control algorithm is as follows:

1) Construct P

To construct *P*, a set of linearly independent vector $|\psi_k\rangle(k=1,2)$ is prepared. In this example, we choose $|\psi_1\rangle=|\psi_f\rangle$, $|\psi_2\rangle=e_1$. Then the Schmidt orthogonalization is performed. Suppose the orthogonalized vectors are $|s_1\rangle$ and $|s_2\rangle$, where $|s_1\rangle=|\psi_f\rangle$. According to (12), $P=p_1|s_1\rangle\langle s_1|+p_2|s_2\rangle\langle s_2|$ holds. The state except target state in $\mathcal{R}$ is $\rho_s=|s_2\rangle\langle s_2|$. Here, we choose $p_1=0.2, p_2=2$, then $P=[1.775\ \ -0.595;-0.595\ \ 0.425]$.

2) System Simulation results

The control gain in (6) is selected as $k=0.1$. The simulation results are showed in Fig. 1, where the red circle denotes the controlled initial state and the blue circle is the target state; the red line is the controlled trajectory and the arrow indicates its direction. Fig. 1(a) shows the state transferring process during $t\in[0,50]$. Fig. 1(b) is the control field.

To illustrate better the control strategy, the control field Fig. 1(b) is applied to the origianl system (1). The tracking results are showed in Fig. 2, where the red dashed line is the evolution curve of controlled state in (1a) and the blue solid line is the one of target state in target system (1b); the red circle and the blue circle indicate the initial location at the current period of the controlled state and the target state respectively; the arrow indicates the direction. In Fig. 2(a), the evolution trajectory at $t\in[0,8]$ is showed, from which one can see that the controlled system is asymptotically stable with respect to the target system on the Bloch sphere. Fig. 2(b) is the state trajectory at $t\in[8,30]$ and Fig. 2(c) is the magnified bottom view of Fig. 2 (b). We have specially labeled the different locations with black box. It can be seen

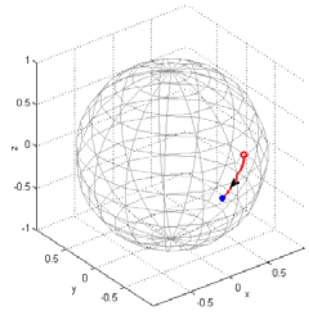


Figure 1(a). State evolution of (2)

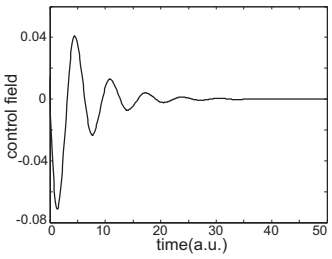
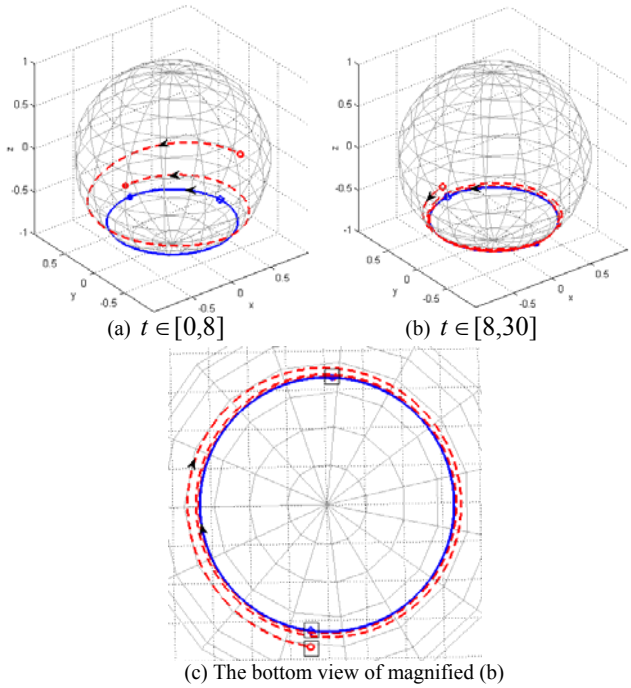


Figure 1(b). Control field



(a) $t \in [0,8]$ (b) $t \in [8,30]$

(c) The bottom view of magnified (b)

Figure 2. The state trajectory tracking process of non-diagonal superposition target state

from Fig. 2(c) that the red circle overlapped the blue one at t=30 (the top box), so the tracking is completed at the moment. Since then, the controlled system would follow the target state in the target orbit. All the three figures demonstrated completely how the system (1a) to track the system (1b). If the index performance $v = \left\| \hat{\rho}(t) - \hat{\rho}_f(t) \right\|^2 = tr\left(\left(\hat{\rho} - \hat{\rho}_f \right)^{\dagger} \left(\hat{\rho} - \hat{\rho}_f \right) \right)$ is used to measure the tracking accuracy, then $v = 9.41 * 10^{-5}$ holds at t=50.

In summary, for the initial target states including diagonal and non-diagonal target states, the controlled system will converge to its target system under the control algorithm (6) with P designed as in Section IV.

VI. CONCLUSION

We have proposed a convergent orbit tracking control algorithm for the free-evolutionary target quantum system in this paper. The unitary transformation was used to change the tracking problem into the regulation one. For the convergence analysis in state regulation, the target states were divided into diagonal and non-diagonal ones. For the former, we continued to perfect the convergence analysis of diagonal mixed-states. The explicit convergence conditions of P have been obtained. For the non-diagonal target state, if the superposition state was considered, a specific non-diagonal P was designed to ensure the convergence. If the target state was non-diagonal mixed-state, there must be a unitary transformation to change the Hermitian non-diagonal matrix into the diagonal one, and the convergence conditions could been obtained as that of diagonal mixed-state.

Appendix 1: The proof of Lemma 1.

Prove: P is a diagonal matrix, one gets $\dot{v}(\rho_f) = 0$ from (10).

$$\ddot{v}(\rho) = -i * \sum_m f_m \left\{ tr\left(\dot{H}_{mt} [\rho, P] \right) + tr\left(H_{mt} [\dot{\rho}, P] \right) \right\}$$

$$\ddot{v}(\rho_f) = -\sum_m f_m^2 tr\left(\left[H_{mt}, \rho_f \right] * [P, H_{mt}] \right)$$

$$= \sum_m f_m^2 tr\left(\left[H_{mt}, \rho_f \right] * [H_{mt}, P] \right)$$

Let $A = \left[H_{mt}, \rho_f \right], B = [H_{mt}, P]$, then $(A)_{ij} = (\lambda_j - \lambda_i)(H_{mt})_{ij}, (B)_{ij} = (p_j - p_i)(H_{mt})_{ij}$, so

$$tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_j - \lambda_i)(p_i - p_j)(H_{mt})_{ij}^2$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_j - \lambda_i)(p_j - p_i)(H_{mt})_{ij}^2 .$$

If ρ_f is a stable state, then $\ddot{v}(\rho_f) > 0$, one gets: $(\lambda_i - \lambda_j)(p_i - p_j) < 0, \forall i \neq j$.

Let $\{\mu_1, \mu_2 \cdots \mu_n\}$ be the spectrum of ρ_f with μ_i arranged in a non-increasing order, viz $\mu_1 < \mu_2 < \cdots < \mu_n$. Then the corresponding P is $P = diag(p_1, p_2, \cdots p_n)$ and $p_1 > p_2 > \cdots > p_n$ is obtained by the above description. Any other states ρ_s in stable set $\mathcal{R}$ can be obtained by m times swapping arbitrary two elements of $\{\mu_1, \mu_2 \cdots \mu_n\}$. We let $bool = tr(P\rho_f) - tr(P\rho_s)$.

Suppose the spectrum from smallest to largest of target state is $\{\mu_1, \mu_2, \cdots, \mu_i, \cdots, \mu_j, \cdots, \mu_k, \cdots \mu_n\}$, then:

$i \leftrightarrow j$:

$$bool = p_i(\mu_i - \mu_j) + p_j(\mu_j - \mu_i) = (p_i - p_j)(\mu_i - \mu_j) < 0$$

$i \leftrightarrow j, j \leftrightarrow k$:

$$bool = p_i(\mu_i - \mu_j) + p_j(\mu_j - \mu_k) + p_k(\mu_k - \mu_i)$$
$$= p_i(\mu_i - \mu_j) + p_j(\mu_j - \mu_i + \mu_i - \mu_k) + p_k(\mu_k - \mu_i)$$
$$= (p_i - p_j)(\mu_i - \mu_j) + (p_j - p_k)(\mu_i - \mu_k) < 0$$

$i \leftrightarrow j, j \leftrightarrow k, k \leftrightarrow l$:

$$bool = p_i(\mu_i - \mu_j) + p_j(\mu_j - \mu_k) + p_k(\mu_k - \mu_l) + p_l(\mu_l - \mu_i)$$
$$= p_i(\mu_i - \mu_j) + p_j(\mu_j - \mu_i + \mu_i - \mu_k) + p_k(\mu_k - \mu_i + \mu_i - \mu_l) + p_l(\mu_l - \mu_i)$$
$$= (p_i - p_j)(\mu_i - \mu_j) + (p_j - p_k)(\mu_i - \mu_k) + (p_k - p_l)(\mu_i - \mu_l) < 0$$

and so on. Finally, we get $v(\rho_f) < v(\rho_s)$. Lemma 1 is proved.

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