BER Performance of Binary Transmitted Signal for Power Line Communication under Nakagami-like Background Noise

Youngsun Kim, Yong-Hwa Kim, Hui-Myoung Oh, and Sungsoo Choi
Power Telecommunication Research Center
Korea Electrotechnology Research Institute (KERI)
Ansan-city, Gyeonggi-do, Republic of Korea
Email: yskim, yongkim, hmoh, sschoi@keri.re.kr,

***Abstract*—Power line communication is an emerging communication technology for the home area network of Smart Grid. To evaluate the performance of power line communication, in our previous work, we derived the closed-form probability density function for the real part of amplitude distribution of Nakagami-like background noise. With this result, in this paper, we investigated the bit error rate performance of binary modulated signal with a single channel transmission. We derived an expression of bit error rate performance and verified its validity through simulations.**

***Keywords*-Power Line Communication; Background Noise; Nakagami Distribution; Bit Error Rate Performance.**

I. Introduction

Smart Grid becomes the focus of public attention as a next-generation energy efficiency optimization. In Smart Grid, various communication technologies enable a two-way exchange of energy consumption and control data via wired/wireless medium. Power line communication (PLC) has been adopted as a communication technology for the automatic meter reading (AMR) [1] and, recently, massively deployed in Korea. Now, PLC is a candidate technology for Smart Grid as a home area network (HAN) communication infrastructure.

To adopt the PLC technology effectively, there is a need for the channel modeling of background noise, impulsive noise, etc. [2], [3]. The channel modeling for the power-line channel has been extensively analyzed and simulated by many researchers. They tried to find the exact (or approximated) channel parameters such as noise, impedance, and attenuation. Among the various parameters for channel modeling, noise is important to evaluate the performance of PLC system. However, due to the nature of power-line channel such as various topologies, connected electrical appliances, types of electrical loads, etc. , the noise modeling could not be easily described as a mathematical expression.

Recently, it was proposed that the amplitude of background noise in power-line channel follows the Nakagami-m distribution [4] [5]. The bit error rate (BER) performance of the PLC system was also evaluated for both single and multi-channel transmission. In [5], however, the BER performances are not expressed as a closed-form mathematical expression, there is thus work to be done yet. We briefly review the previous work in Section II and derive the BER performance in Section III. The mathematical expression gives the system designer a better understanding of the system design and a performance prediction for his system.

II. PDF of Background Noise for PLC Channel

We derived the closed-form probability density function (PDF) of the real part noise, y, for power-line channel [6],

$$\begin{aligned} f(y) &= \frac{1}{\sqrt{\pi}\Gamma(m)}\sqrt{\frac{m}{\Omega}}e^{-\frac{my^2}{\Omega}}\left\{\frac{\Gamma(\frac{1}{2}-m)}{\Gamma(1-m)}\left(\frac{my^2}{\Omega}\right)^{m-\frac{1}{2}}\right. \\ &\quad \times {}_1F_1\left(\frac{1}{2},\frac{1}{2}+m,\frac{my^2}{\Omega}\right)+\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}} \\ &\quad \left.\times {}_1F_1\left(1-m,\frac{3}{2}-m,\frac{my^2}{\Omega}\right)\right\}, \end{aligned} \tag{1}$$

for $0 < m < 1$ and $m \neq \frac{1}{2}$, where $\Gamma(\cdot)$ is the Gamma function, m and Ω are parameters defined as $\Omega = E[\alpha^2] = \overline{\alpha^2}$, $E[x] = \overline{x}$ denoting the expected value of x and $m = \frac{\left(\overline{\alpha^2}\right)^2}{(\alpha^2-\overline{\alpha^2})^2} > 0$, which comes from the amplitude PDF of a power-line background noise characterized by Nakagami-m distribution:

$$f(\alpha) = \frac{2m^m\alpha^{2m-1}}{\Gamma(m)\Omega^m}e^{-(m/\Omega)\alpha^2}, \qquad \alpha \geq 0. \tag{2}$$

Ω denotes the power of the amplitude, α. Originally, the Nakagami-m distribution represents the fading amplitude of wireless communication channel. It spans the one-sided Gaussian, Rayleigh, Hoyt, Rice and non-fading channel as m varies [7]. Furthermore, the confluent hypergeometric function of the first kind, ${}_1F_1$, as [8, Chap. 9.2] :

$${}_1F_1(a,b,z) = 1+\frac{a}{b}\frac{z}{1!}+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!}\cdots \tag{3}$$

is used for (1). For special case of $m = \frac{1}{2}$, we get

$$f(y) = \frac{1}{\pi}\sqrt{\frac{1}{2\Omega\pi}}e^{-\frac{y^2}{4\Omega}}K_0\left(\frac{y^2}{4\Omega}\right), \tag{4}$$

where $K_0(z)$ is the modified Bessel function of the second kind of order 0. The accuracy of the analyzed closed-form expression in (1) has been verified in [6]. Eqn. (1) shows

 ISBN: 978-1-61208-136-6

well evaluation of the noise modeling for Nakagmi-like background noise.

III. Derivation of the BER performance

In this section, we derive the BER performance of binary data transmission with the noise as described in section II. Fig. 1 demonstrates the system model for binary transmission. A transmitter sends binary data, A or $-A$, then, the decision metric at the receiver, r, is defined as

$$r = \pm A + y, \tag{5}$$

which is added by Nakagami-like background power-line noise, y. We assumed the perfect time and carrier synchronization in demodulation for the ease of analysis.

Then, the average bit error rate (BER), P_e, can be expressed as

$$P_e = p(A)P(E|s=A) + p(-A)P(E|s=-A) \tag{6}$$

where $p(A)$ is the probability that the transmitter sends data A, and $P(E|s=A)$ is the probability of error decision at the receiver when data A is transmitted. With equal probability of transmitting A and $-A$, Eqn. (6) is represented as

$$P_e = \frac{1}{2}P(E|s=A) + \frac{1}{2}P(E|s=-A). \tag{7}$$

Since the noise PDF, $p(y)$, is symmetric about $y=0$, we get

$$P_e = P(E|s=A) = P(E|s=-A) = \int_A^\infty f(y)dy. \tag{8}$$

By substituting Eqn.(1) into Eqn. (8), we obtain

$$\begin{aligned} P_e &= \int_A^\infty f(y)dy \\ &= \int_A^\infty \frac{1}{\sqrt{\pi}\Gamma(m)}\sqrt{\frac{m}{\Omega}}e^{-\frac{my^2}{\Omega}}\left\{\frac{\Gamma(\frac{1}{2}-m)}{\Gamma(1-m)}\left(\frac{my^2}{\Omega}\right)^{m-\frac{1}{2}}\right. \\ &\quad \times {}_1F_1\left(\tfrac{1}{2},\tfrac{1}{2}+m,\tfrac{my^2}{\Omega}\right) + \frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}} \\ &\quad \left.\times {}_1F_1\left(1-m,\tfrac{3}{2}-m,\tfrac{my^2}{\Omega}\right)\right\}dy \\ &= \frac{1}{\sqrt{\pi}\Gamma(m)}\sqrt{\frac{m}{\Omega}}\frac{\Gamma(\frac{1}{2}-m)}{\Gamma(1-m)}\int_A^\infty e^{-\frac{my^2}{\Omega}}\left(\frac{my^2}{\Omega}\right)^{m-\frac{1}{2}} \\ &\quad \times {}_1F_1\left(\tfrac{1}{2},\tfrac{1}{2}+m,\tfrac{my^2}{\Omega}\right)dy + \frac{1}{\sqrt{\pi}\Gamma(m)}\sqrt{\frac{m}{\Omega}}\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}} \\ &\quad \times \int_A^\infty e^{-\frac{my^2}{\Omega}}{}_1F_1\left(1-m,\tfrac{3}{2}-m,\tfrac{my^2}{\Omega}\right)dy. \end{aligned} \tag{9}$$

Letting $\sqrt{\frac{m}{\Omega}}y = x$ gives $\sqrt{\frac{m}{\Omega}}dy = dx$, then, Eqn. (27) is

$$\begin{aligned} P_e &= \frac{1}{\sqrt{\pi}\Gamma(m)}\frac{\Gamma(\frac{1}{2}-m)}{\Gamma(1-m)}\int_{\sqrt{\frac{m}{\Omega}}A}^\infty e^{-x^2}x^{2m-1} \\ &\quad \times {}_1F_1\left(\tfrac{1}{2},\tfrac{1}{2}+m,x^2\right)dx + \frac{1}{\sqrt{\pi}\Gamma(m)}\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}} \\ &\quad \times \int_{\sqrt{\frac{m}{\Omega}}A}^\infty e^{-x^2}{}_1F_1\left(1-m,\tfrac{3}{2}-m,x^2\right)dx. \end{aligned} \tag{10}$$

Applying Kummer Transform [8, (13.1.27)],

$$ {}_1F_1(a,b,z) = e^z{}_1F_1(b-a,b,-z), \tag{11}$$

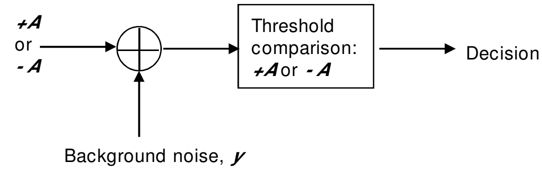

Figure 1. System model for binary transmission

to Eqn. (28) gives

$$\begin{aligned} P_e &= \frac{\Gamma(\frac{1}{2}-m)}{\sqrt{\pi}\Gamma(m)\Gamma(1-m)}\int_{\sqrt{\frac{m}{\Omega}}A}^{\infty} x^{2m-1}{}_1F_1\left(m,\frac{1}{2}+m,-x^2\right)dx \\ &+\frac{\Gamma(m-\frac{1}{2})}{\pi\Gamma(m)}\int_{\sqrt{\frac{m}{\Omega}}A}^{\infty} {}_1F_1\left(\frac{1}{2},\frac{3}{2}-m,-x^2\right)dx. \end{aligned} \tag{12}$$

In Eqn.(31), the first term in the integration is

$$\int_{\sqrt{\frac{m}{\Omega}}A}^\infty x^{2m-1}{}_1F_1\left(m,\frac{1}{2}+m,-x^2\right)dx. \tag{13}$$

By substituting Eqn.(3), Eqn.(13) is represented as

$$\int_{\sqrt{\frac{m}{\Omega}}A}^\infty x^{2m-1}\sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{(-x^2)^n}{n!}dx, \tag{14}$$

where $(m)_n$ is the Pochhammer symbol [9, (6.1.22)], which is defined as

$$\begin{aligned} (m)_n &= \frac{\Gamma(m+n)}{\Gamma(m)} \\ &= m(m+1)\cdots(m+n-1). \end{aligned} \tag{15}$$

Then, Eqn.(14) is represented as

$$\begin{aligned} &\sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{1}{n!}\int_{\sqrt{\frac{m}{\Omega}}A}^\infty x^{2m-1}(-x^2)^n dx \\ &= \sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{(-1)^n}{n!}\int_{\sqrt{\frac{m}{\Omega}}A}^\infty x^{2n+2m-1}dx \\ &= \sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{(-1)^n}{n!}\left[\frac{1}{2(n+m)}x^{2(n+m)}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^\infty \\ &= \left[\sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{(-1)^n}{n!}\frac{1}{2(n+m)}x^{2(n+m)}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^\infty \\ &= \left[\frac{x^{2m}}{2}\sum_{n=0}^\infty \frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{1}{n+m}\frac{(-x^2)^n}{n!}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^\infty. \end{aligned} \tag{16}$$

Since we can replace $\frac{1}{n+m}$ as

$$\begin{aligned} \frac{1}{n+m} &= \frac{m(m+1)\cdots(m+n-1)}{m(m+1)\cdots(m+n-1)(m+n)} \\ &= \frac{1}{m}\frac{m_n}{(m+1)_n}, \end{aligned} \tag{17}$$

$$\begin{aligned} P_e = \; & \frac{\Gamma(\frac{1}{2}-m)}{2m\sqrt{\pi}\Gamma(m)\Gamma(1-m)}\left[x^{2m}\,{}_2F_2\left(m,m;\frac{1}{2}+m,m+1;-x^2\right)\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & +\frac{\Gamma(m-\frac{1}{2})}{\sqrt{\pi}\Gamma(m)}\left[x\,{}_2F_2\left(\frac{1}{2},\frac{1}{2};\frac{3}{2}-m,\frac{3}{2};-x^2\right)\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty}. \end{aligned} \tag{26}$$

the Eqn. (16) can be written as

$$\begin{aligned} & \left[\frac{x^{2m}}{2}\sum_{n=0}^{\infty}\frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{1}{m}\frac{m_n}{(m+1)_n}\frac{(-x^2)^n}{n!}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & =\left[\frac{x^{2m}}{2m}\sum_{n=0}^{\infty}\frac{(m)_n}{(\frac{1}{2}+m)_n}\frac{m_n}{(m+1)_n}\frac{(-x^2)^n}{n!}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & =\left[\frac{x^{2m}}{2m}\,{}_2F_2\left(m,m;\frac{1}{2}+m,m+1;-x^2\right)\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \end{aligned} \tag{18}$$

where

$$\begin{aligned} & {}_2F_2(a_1,a_2;b_1,b_2;z)= \\ & \quad 1+\frac{a_1}{b_1}\frac{a_2}{b_2}\frac{z}{1!}+\frac{a_1(a_1+1)}{b_1(b_1+1)}\frac{a_2(a_2+1)}{b_2(b_2+1)}\frac{z^2}{2!}+\cdots. \end{aligned} \tag{19}$$

Now, we treat the integration of the second term in Eqn.(31),

$$\int_{\sqrt{\frac{m}{\Omega}}A}^{\infty}{}_1F_1\left(\frac{1}{2},\frac{3}{2}-m,-x^2\right)dx. \tag{20}$$

By substituting Eqn.(3) into Eqn.(20), we obtain

$$\int_{\sqrt{\frac{m}{\Omega}}A}^{\infty}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{(-x^2)^n}{n!}dx. \tag{21}$$

Eqn. (21) is represented as

$$\begin{aligned} & \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{(-1)^n}{n!}\int_{\sqrt{\frac{m}{\Omega}}A}^{\infty}x^{2n}dx \\ & =\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{(-1)^n}{n!}\left[\frac{1}{2n+1}x^{2n+1}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & =\left[x\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{(-1)^n}{n!}\frac{x^{2n}}{2n+1}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & =\left[x\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{1}{2n+1}\frac{(-x^2)^n}{n!}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty}. \end{aligned} \tag{22}$$

Since we can replace $\frac{1}{2n+1}$ as

$$\begin{aligned} \frac{1}{2n+1} & =\frac{1}{2}\frac{1}{\frac{1}{2}+n} \\ & =\frac{\frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)}{\frac{3}{2}(\frac{3}{2}+1)\cdots(\frac{3}{2}+n-2)(\frac{3}{2}+n-1)} \\ & =\frac{(\frac{1}{2})_n}{(\frac{3}{2})_n}, \end{aligned} \tag{23}$$

the Eqn. (22) can be written as

$$\begin{aligned} & \left[x\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2}-m)_n}\frac{(\frac{1}{2})_n}{(\frac{3}{2})_n}\frac{(-x^2)^n}{n!}\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty} \\ & =\left[x\,{}_2F_2\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2}-m,\tfrac{3}{2};-x^2\right)\right]_{x=\sqrt{\frac{m}{\Omega}}A}^{\infty}. \end{aligned} \tag{24}$$

Using the relation,

$$\int z^{\alpha-1}{}_1F_1\left(a,b,cz\right)dz=\frac{z^{\alpha}{}_2F_2\left(a,\alpha;b,\alpha+1;cz\right)}{a}, \tag{25}$$

and substituting Eqns.(16) and (24) into Eqn. (31) gives Eqn. (26). Eqn. (26) represents the BER performance of binary transmitted signal without integration for the Nakagami-like power-line background noise.

Now, we derive the BER performance for $m=\frac{1}{2}$ as a special case. By substituting Eqn.(4) into Eqn. (8), we obtain

$$\begin{aligned} P_e & =\int_A^{\infty}f(y)dy \\ & =\int_A^{\infty}\frac{1}{\pi}\sqrt{\frac{1}{2\Omega\pi}}e^{-\frac{y^2}{4\Omega}}K_0\left(\frac{y^2}{4\Omega}\right)dy \\ & =\frac{1}{\pi}\sqrt{\frac{1}{2\Omega\pi}}\int_A^{\infty}e^{-\frac{y^2}{4\Omega}}K_0\left(\frac{y^2}{4\Omega}\right)dy. \end{aligned} \tag{27}$$

Letting $\frac{y^2}{4\Omega}=x$ gives $\frac{y}{2\Omega}dy=dx$, then, Eqn. (27) is

$$\begin{aligned} Pe & =\frac{1}{\pi}\sqrt{\frac{1}{2\Omega\pi}}\int_{\frac{A^2}{4\Omega}}^{\infty}2\Omega y^{-1}e^{-x}K_0\left(x\right)dx \\ & =\frac{1}{\pi}\sqrt{\frac{1}{2\Omega\pi}}\int_{\frac{A^2}{4\Omega}}^{\infty}\sqrt{\Omega}x^{-1}e^{-x}K_0\left(x\right)dx \\ & =\frac{1}{\pi}\sqrt{\frac{1}{2\pi}}\int_{\frac{A^2}{4\Omega}}^{\infty}x^{-1}e^{-x}K_0\left(x\right)dx. \end{aligned} \tag{28}$$

Eqn. (28) can be divided into two terms;

$$\begin{aligned} P_e & =\frac{1}{\pi}\sqrt{\frac{1}{2\pi}}\int_0^{\infty}x^{-1}e^{-x}K_0\left(x\right)dx \\ & \quad -\frac{1}{\pi}\sqrt{\frac{1}{2\pi}}\int_0^{\frac{A^2}{4\Omega}}x^{-1}e^{-x}K_0\left(x\right)dx. \end{aligned} \tag{29}$$

Then, applying [9, (6.621.4)],

$$\begin{aligned} & \int_0^{\infty}x^{\mu-1}e^{-\alpha x}K_{\nu}(\beta x)dx \\ & \quad =\frac{\sqrt{\pi}(2\beta)^{\nu}}{(\alpha+\beta)^{\mu+\nu}}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2})} \\ & \qquad \times F\left(\mu+\nu,\nu+\tfrac{1}{2};\mu+\tfrac{1}{2};\tfrac{\alpha-\beta}{\alpha+\beta}\right), \end{aligned} \tag{30}$$

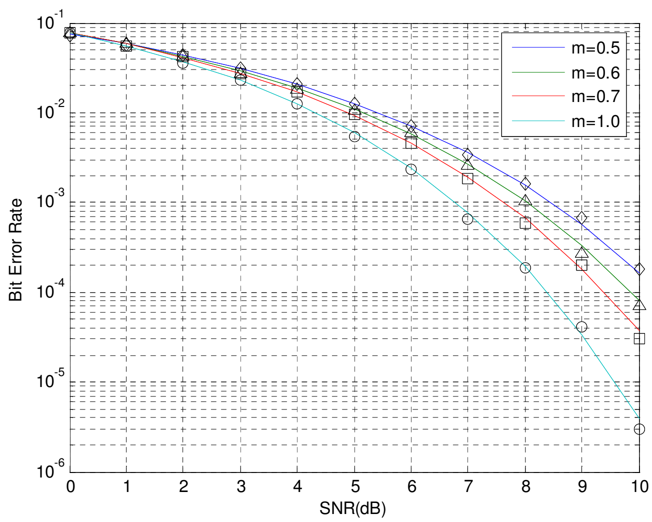

Figure 2. Simulated and analyzed BER performance under Nakagami-like background noise with $m = 0.5, 0.6, 0.7, 1.0$

where Re $\mu >|$ Re $\nu|$, Re $(\alpha+\beta) > 0$ for $\mu = \frac{1}{2}$, $\nu = 0$, $\alpha = 1$, $\beta = 1$, to Eqn. (29) gives

$$\begin{aligned} P_e &= \frac{1}{\pi}\sqrt{\frac{1}{2\pi}}\left(\pi\sqrt{\frac{\pi}{2}} - \int_0^{\frac{A^2}{4\Omega}} x^{-1}e^{-x}K_0\left(x\right)dx\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\sqrt{\frac{1}{2\pi}}\int_0^{\frac{A^2}{4\Omega}} x^{-1}e^{-x}K_0\left(x\right)dx. \end{aligned} \tag{31}$$

Eqn. (31) represents the BER performance of binary transmitted signal with Nakagami-like power-line background noise for $m = \frac{1}{2}$.

IV. NUMERICAL RESULTS

Numerical examples for the BER performance of binary modulated receiver under Nakagami-like background noise are presented through both analysis and simulation. The analyzed BER performance is obtained using Eqn.(26). For the ease of analysis, we assumed the transmitted data $A = 1$. The simulated performances are obtained by using the Monte Carlo method with 10^7 binary transmitted data. Fig. 2 compares the simulated and analyzed BER performance with $m = 0.5, 0.6, 0.7$ and 1. As m increases, the BER performance improves since the Nakagami-like background noise, y, has the shape of PDF close to Gaussian PDF, with a result similar to [5]. Thus, the signal-to-noise ratio increases as m increases. Simulation results with various m values show good agreement between simulations and analysis up to the SNR of 10 dB.

V. CONCLUSION

In this paper, the BER performance of binary transmitted signal under Nakagami-like background noise was presented. We derived the BER performance with conventional functions such as the hypergeometric function and the Gamma function. Simulated results show the validity of the derived BER performances. With these results, a PLC system designer can easily predict and analyze the performance of HAN system. The BER performance with impulsive noise and the transmission of multi-channel modulated signal would be the focus of the future work.

REFERENCES

[1] Y. Zhang and S. Cheng, "Power Line Communications," IEEE Potentials, vol. 23, no. 4, pp. 4-8, Oct.-Nov. 2004.

[2] M. Zimmermann and K. Dostert, "A multipath model for the powerline channel," IEEE Trans. Commun., vol. 50, no. 4, pp. 553-559, April 2002.

[3] M. Zimmermann and K. Dostert, "Analysis and modeling of impulsive noise in broadband powerline communications," IEEE Trans. Electromagn. Compat., vol. 44, no. 1, pp. 249-258, Feb. 2002.

[4] M. Nakagami, "The m-distribution - A general formula of intensity distribution of rapid fading," in Statistical Methods in Radio Wave Propagation, pp. 3-36, Pergamon Press, Oxford, U.K, 1960.

[5] H. Meng, Y. L. Guan and S. Chen, "Modeling and analysis for noise effects on broadband power-line communications," IEEE Trans. Power Del., vol. 20, no. 2, pp. 630-637, April 2005.

[6] Y. Kim, H-M. Oh and S. Choi, "Closed-form expression of Nakagami-like background noise in power-line channel," IEEE Trans. Power Del., vol. 23, no. 3, pp. 1410-1412, Oct. 2008.

[7] M. K. Simon and M.-S. Alouini, Digital Communications Over Generalized Fading Channels : A Unified Approach to Performance Analysis, New York: Wiley, 2000.

[8] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 6th ed., San Diego, CA: Academic, 2000.

[9] M. Abromowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York, 1970.

ISBN: 978-1-61208-136-6