Chance Constrained Portfolio Optimization using Loan

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***Abstract*—In this paper, portfolio optimization using loan is formulated as a chance constrained problem in which the borrowing money from a loan can be invested in risk assets. Then, the chance constrained problem is transformed into an equivalence problem. Furthermore, the equivalence problem is proven to be a convex optimization problem and solved efficiently by using an interior point method. Experimental results show that the use of the loan depends on acceptable risk and improves the efficient frontier.**

Keywords–Portfolio optimization; risk management; modeling.

I. INTRODUCTION

Portfolio optimization is the process of determining the best proportion of investment in different assets according to some objective. Portfolio optimization requires knowledge about the returns of assets. Thereby, the objective typically maximizes factors such as expected return, and minimizes costs like financial risk. Since portfolio optimization is one of the most challenging problems in the field of finance, a large number of works about portfolio optimization have been reported. In these works, according to some precondition, portfolio optimization is formulated as an optimization problem. Then, an appropriate method is used to solve the optimization problem [1].

Optimization methods of mathematical programming have been used to solve portfolio optimization problems based on Markowitz's model [2]. Evolutionary Algorithms (EAs) have been also used to solve portfolio optimization problems recently. Cardinality constraints limit a portfolio to have a specified number of assets. Genetic Algorithm (GA), Tabu Search (TS), and Simulated Annealing (SA) have been applied, respectively, to a portfolio optimization problem that includes cardinality constraints [3]. An extended GA has been proposed to solve a portfolio optimization problem considering the costs for selling and buying assets to change portfolio structures in multiple periods [4]. An Artificial Bee Colony (ABC) algorithm [5] has been proposed for solving a portfolio optimization problem based on the efficient frontier model [3]. Furthermore, in order to predict the future returns of assets from market data for portfolio optimization, an Artificial Intelligence (AI) method based on deep learning has been reported [6].

In our prior work [7], portfolio optimization using bank deposit and loan is formulated as a chance constrained problem in which a non-risk asset called bank deposit is included in the portfolio and the borrowing money called bank loan can be invested in risk assets. The chance constrained problem is transformed into an equivalence problem. The equivalence problem is proven to be a multimodal optimization problem having multiple optimal solutions. Therefore, an Adaptive Differential Evolution using Directed mutation (ADED) is proposed to solve the equivalence problem effectively.

In this paper, the effect of the loan is studied intensively. Portfolio optimization using loan is formulated as a chance constrained problem in which borrowing money from a loan can be invested in risk assets. Then, the chance constrained problem is transformed into an equivalence problem. The equivalence problem is analyzed mathematically. As a result, it is proven that the equivalence problem is a convex optimization problem. Besides, the condition of borrowing money from a loan up to the limit is revealed. An interior point method [8] is used to solve the portfolio optimization problem using loan because the interior point method is effective for convex optimization problems. Experimental results show that the use of the loan depends on acceptable risk and loan interest. If the loan is used properly, the efficient frontier is improved.

The remainder of this paper is organized as follows. Section II defines the portfolio considered in this paper. Besides, the basic models of portfolio optimization are explained. Section III formulates the portfolio optimization problem using loan. Section IV analyzes the optimization problem mathematically. Section V presents and discusses the results of experiments. Section VI concludes this paper and mentions future work.

II. PORTFOLIO OPTIMIZATION

A. Definition of Portfolio

We invest money in n assets. Let $x_i \in \Re$, $i = 1, \cdots, n$ be the proportion of i-asset normalized by owned capital. A portfolio is defined as $\boldsymbol{x} = (x_1, \cdots, x_n)$. Since $\boldsymbol{x} \in \Re^n$ is a long-only portfolio of single-period, it is constrained as

$$x_1 + x_2 + \cdots + x_n = 1 \tag{1}$$

where $0 \leq x_i$, $i = 1, \cdots, n$.

The unit investment in the i-asset provides return $\xi_i \in \Re$ over a single period operation [3]. Each $\xi_i \in \Re$ is modeled by a random variable following a normal distribution as

$$\xi_i \sim \mathrm{Normal}(\mu_i, \sigma_i^2). \tag{2}$$

Let ρ_{ij} be the correlation coefficient between ξ_i and ξ_j, $i \neq j$. The mean μ_i and the standard deviation σ_i in (2), ρ_{ij} are estimated statistically from historical data.

From (2), the vector of random returns $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)$ obeys a multivariable normal distribution as

$$\boldsymbol{\xi} \sim \mathrm{Normal}(\boldsymbol{\mu}, \boldsymbol{C}) \tag{3}$$

where the mean is given as $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n) \in \Re^n$.

 ISBN: 978-1-61208-765-8

The covariance matrix $\boldsymbol{C}$ in (3) is derived as follows. First of all, the matrix $\boldsymbol{D}$ is defined by using σ_i in (2) as

$$\boldsymbol{D} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}. \quad (4)$$

From the correlation coefficient ρ_{ij} between asset returns ξ_i and ξ_j, the coefficient matrix $\boldsymbol{R}$ is also defined as

$$\boldsymbol{R} = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{pmatrix}. \quad (5)$$

From $\boldsymbol{D}$ in (4) and $\boldsymbol{R}$ in (5), $\boldsymbol{C}$ in (3) is obtained as

$$\boldsymbol{C} = \boldsymbol{D}\,\boldsymbol{R}\,\boldsymbol{D}. \quad (6)$$

The return of a portfolio $\boldsymbol{x} \in \Re^n$ is defined as

$$r(\boldsymbol{x},\,\boldsymbol{\xi}) = \sum_{i=1}^{n} \xi_i\, x_i = \boldsymbol{\xi}\,\boldsymbol{x}^T. \quad (7)$$

From the reproductive property of normal distribution [9], the return in (7) also obeys a normal distribution as

$$r(\boldsymbol{x},\,\boldsymbol{\xi}) \sim \text{Normal}(\mu_r(\boldsymbol{x}),\,\sigma^2(\boldsymbol{x})) \quad (8)$$

where the mean and the variance are given as

$$\mu_r(\boldsymbol{x}) = \sum_{i=1}^{n} \mu_i\, x_i = \boldsymbol{\mu}\,\boldsymbol{x}^T \quad (9)$$

$$\sigma^2(\boldsymbol{x}) = \boldsymbol{x}\,\boldsymbol{C}\,\boldsymbol{x}^T. \quad (10)$$

B. Models for Portfolio Optimization

By using the portfolio $\boldsymbol{x} \in \Re^n$ stated above, we explain basic models used to formulate portfolio optimization.

1) Markowitz's Model [10]: The risk of the portfolio is evaluated by the variance $\sigma^2(\boldsymbol{x})$ in (10). The risk is minimized keeping an expected return $\mu_r(\boldsymbol{x})$ larger than $\gamma \in \Re$ as

$$\left[\begin{array}{ll} \text{min} & \sigma^2(\boldsymbol{x}) = \boldsymbol{x}\,\boldsymbol{C}\,\boldsymbol{x}^T \\ \text{sub. to} & \mu_r(\boldsymbol{x}) = \boldsymbol{\mu}\,\boldsymbol{x}^T \geq \gamma, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \quad (11)$$

2) Efficient frontier Model [3]: By using the risk aversion indicator $\lambda \in [0,\,1]$, Markowitz's model is modified as

$$\left[\begin{array}{ll} \text{min} & \lambda\,\sigma^2(\boldsymbol{x}) - (1-\lambda)\,\mu_r(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \quad (12)$$

By changing the value of $0 \leq \lambda \leq 1$, we can obtain the efficient frontier, which is a continuous curve illustrating the tradeoff between expected return and risk (variance).

3) Roy's Model [11]: The risk of portfolio is evaluated by the probability that the return $r(\boldsymbol{x},\,\boldsymbol{\xi})$ in (7) falls below a desired value $\gamma \in \Re$. For minimizing the risk α, portfolio optimization is formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \text{min} & \alpha \\ \text{sub. to} & \Pr(r(\boldsymbol{x},\,\boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \quad (13)$$

4) Kataoka's Model [12]: Contrary to Roy's model in (13), the desired value $\gamma \in \Re$ of the return $r(\boldsymbol{x},\,\boldsymbol{\xi})$ is maximized for a given risk $\alpha \in (0,\,0.5)$. Thereby, portfolio optimization is also formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \text{max} & \gamma \\ \text{sub. to} & \Pr(r(\boldsymbol{x},\,\boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \quad (14)$$

III. Problem Formulation

The Portfolio Optimization Problem using Loan (POPL) is an extended version of Kataoka's Model shown in (14).

A. Portfolio including Loan

The money borrowed from a loan is invested in risk assets. Let $x_0 \in \Re$ be the proportion of the loan used for a portfolio $\boldsymbol{x} \in \Re^n$. Let $m \in \Re$, $m > 0$ be the upper limit of the loan, which is specified by a multiple of owned capital. If the loan is not used, the proportion of the loan is $x_0 = 0$. On the other hand, if the loan is used up to the limit, the proportion of the loan is $x_0 = -m$. Therefore, the constraints of POPL are

$$\left(\begin{array}{l} x_0 + x_1 + x_2 + \cdots + x_n = 1, \\ -m \leq x_0 \leq 0,\ 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \quad (15)$$

From the first constraint in (15), the proportion of the loan $x_0 \in \Re$ used for a portfolio $\boldsymbol{x} \in \Re^n$ can be evaluated as

$$x_0 = 1 - \mathbb{1}\,\boldsymbol{x}^T \quad (16)$$

where $\mathbb{1} \in \Re^n$ is a vector defined as $\mathbb{1} = (1, \cdots, 1)$.

Let $L \in \Re$ be the interest rate of the loan. The interest rate $L \in \Re$, $L \geq 0$ is a constant value. Considering the loan, the return $r(\boldsymbol{x},\,\boldsymbol{\xi})$ of a portfolio $\boldsymbol{x} \in \Re^n$ in (7) is revised as

$$\begin{array}{rcl} g(\boldsymbol{x},\,\boldsymbol{\xi}) & = & r(\boldsymbol{x},\,\boldsymbol{\xi}) + L\,x_0 \\ & = & \boldsymbol{\xi}\,\boldsymbol{x}^T + L\,(1 - \mathbb{1}\,\boldsymbol{x}^T) \\ & = & (\boldsymbol{\xi} - L\,\mathbb{1})\,\boldsymbol{x}^T + L. \end{array} \quad (17)$$

From the reproductive property of normal distribution [9], the return in (17) also obeys a normal distribution as

$$g(\boldsymbol{x},\,\boldsymbol{\xi}) \sim \text{Normal}(\mu_g(\boldsymbol{x}),\,\sigma^2(\boldsymbol{x})) \quad (18)$$

where the mean is given as

$$\mu_g(\boldsymbol{x}) = (\boldsymbol{\mu} - L\,\mathbb{1})\,\boldsymbol{x}^T + L. \quad (19)$$

The variance $\sigma^2(\boldsymbol{x})$ in (18) is also given by (10).

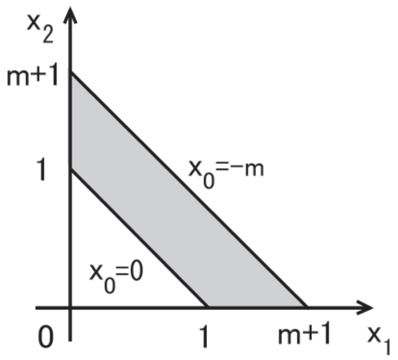

Figure 1. Feasible region of POPL in (23).

B. Portfolio Optimization using Loan

As stated above, POPL is formulated as an extended version of Kataoka's Model in (14). A risk $\alpha \in (0,\, 0.5)$ is given in advance. From (15) and (17), POPL is also formulated as a chance constrained problem:

$$\left[\begin{array}{ll} \max & \gamma \\ \text{sub. to} & \Pr(g(\boldsymbol{x},\, \boldsymbol{\xi}) \leq \gamma) \leq \alpha, \\ & x_1 + x_2 + \cdots + x_n = \mathbb{1}\, \boldsymbol{x}^T \leq m+1, \\ & x_1 + x_2 + \cdots + x_n = \mathbb{1}\, \boldsymbol{x}^T \geq 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n \end{array}\right. \tag{20}$$

where the proportion of the loan $x_0 \in [-m,\, 0]$ is eliminated from the constraints in (15) by using the equation in (16).

C. Equivalence Problem

It is hard to solve the chance constrained problem that contains probabilistic constraints [13]. Therefore, we transform the above POPL in (20) into an equivalence problem.

Since the return $g(\boldsymbol{x},\, \boldsymbol{\xi})$ follows the normal distribution in (18), we can standardize the chance constraint in (20) as

$$\Pr\left(\frac{g(\boldsymbol{x},\, \boldsymbol{\xi}) - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})} \leq \frac{\gamma - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})}\right) \leq \alpha. \tag{21}$$

The probability in (21) can be written as

$$\Phi\left(\frac{\gamma - \mu_g(\boldsymbol{x})}{\sigma(\boldsymbol{x})}\right) \leq \alpha \tag{22}$$

where $\Phi : \Re \to [0,\, 1]$ denotes the Cumulative Distribution Function (CDF) of the standard normal distribution.

From (22), we can derive the equivalence problem of the chance constrained problem in (20) as

$$\left[\begin{array}{ll} \max & \gamma(\boldsymbol{x}) = \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\, \sigma(\boldsymbol{x}) \\ \text{sub. to} & x_1 + x_2 + \cdots + x_n = \mathbb{1}\, \boldsymbol{x}^T \leq m+1, \\ & x_1 + x_2 + \cdots + x_n = \mathbb{1}\, \boldsymbol{x}^T \geq 1, \\ & 0 \leq x_i,\ i = 1, \cdots, n. \end{array}\right. \tag{23}$$

Since the equivalence problem in (23) is a deterministic one, we do not need to evaluate the chance constraint in (20). The optimization problem in (23) is also called POPL.

Figure 1 illustrates the feasible region of POPL for the case of $n = 2$. The feasible region is shown by the gray area between two hyper-planes. If a portfolio $\boldsymbol{x} \in \Re^n$ does not use the loan as $x_0 = 0$, it exists on the lower plane: $\mathbb{1}\, \boldsymbol{x}^T = 1$. On the other hand, if a portfolio $\boldsymbol{x} \in \Re^n$ uses the loan up to the limit as $x_0 = -m$, it exists on the upper plane: $\mathbb{1}\, \boldsymbol{x}^T = m+1$.

IV. Problem Analysis

We analyze POPL in (23) mathematically.

Lemma 1: The standard deviation $\sigma(\boldsymbol{x})$ in (10) is convex.

Proof: Since the covariance matrix $\boldsymbol{C}$ in (6) is positive semi-definite, it is decomposed with $\boldsymbol{y} = \boldsymbol{x}\, \boldsymbol{A} \in \Re^n$ as

$$\sigma(\boldsymbol{x}) = \sqrt{\boldsymbol{x}\, \boldsymbol{C}\, \boldsymbol{x}^T} = \sqrt{\boldsymbol{x}\, \boldsymbol{A}\, \boldsymbol{A}^T\, \boldsymbol{x}^T} = \sqrt{\boldsymbol{y}\, \boldsymbol{y}^T}. \tag{24}$$

From (24), $\sigma(\boldsymbol{x})$ is a norm. Therefore, for any $\theta \in [0,\, 1]$ and $\hat{\boldsymbol{x}} \in \Re^n$, $\hat{\boldsymbol{x}} \neq \boldsymbol{x}$, the triangle inequality holds as

$$\sigma(\theta\, \boldsymbol{x} + (1-\theta)\, \hat{\boldsymbol{x}}) \leq \sigma(\theta\, \boldsymbol{x}) + \sigma((1-\theta)\, \hat{\boldsymbol{x}}). \tag{25}$$

The right side of (25) can be transformed as

$$\sigma(\theta\, \boldsymbol{x}) = \sqrt{\theta\, \boldsymbol{y}\, (\theta\, \boldsymbol{y})^T} = \theta\, \sqrt{\boldsymbol{y}\, \boldsymbol{y}^T} = \theta\, \sigma(\boldsymbol{x}). \tag{26}$$

From (25) and (26), we have

$$\sigma(\theta\, \boldsymbol{x} + (1-\theta)\, \hat{\boldsymbol{x}}) \leq \theta\, \sigma(\boldsymbol{x}) + (1-\theta)\, \sigma(\hat{\boldsymbol{x}}). \tag{27}$$

From (27), $\sigma(\boldsymbol{x})$ in (10) is a convex function. ■

Theorem 1: The objective function $\gamma(\boldsymbol{x})$ of POPL in (23) is concave. In other words, $-\gamma(\boldsymbol{x})$ is convex.

Proof: From (19) and $\gamma(\boldsymbol{x})$ in (23), we have

$$\begin{aligned} & \theta\, \gamma(\boldsymbol{x}) + (1-\theta)\, \gamma(\hat{\boldsymbol{x}}) - \gamma(\theta\, \boldsymbol{x} + (1-\theta)\, \hat{\boldsymbol{x}}) \\ & = \Phi^{-1}(\alpha) \times \\ & (\theta\, \sigma(\boldsymbol{x}) + (1-\theta)\, \sigma(\hat{\boldsymbol{x}}) - \sigma(\theta\, \boldsymbol{x} + (1-\theta)\, \hat{\boldsymbol{x}})). \end{aligned} \tag{28}$$

From Lemma 1 and $\Phi^{-1}(\alpha) < 0$ for $\alpha \in (0,\, 0.5)$, the right side of (28) is negative. Therefore, we have

$$\gamma(\theta\, \boldsymbol{x} + (1-\theta)\, \hat{\boldsymbol{x}}) \geq \theta\, \gamma(\boldsymbol{x}) + (1-\theta)\, \gamma(\hat{\boldsymbol{x}}). \tag{29}$$

From (29), $\gamma(\boldsymbol{x})$ in (23) is a concave function. ■

Since all constraints of POPL in (23) are linear, the feasible region of POPL is convex. Therefore, from Theorem 1, POPL in (23) is a convex optimization problem [14].

Theorem 2: Let $\boldsymbol{x}^\star \in \Re^n$ be a local optimum solution of a convex optimization problem. Then, the solution $\boldsymbol{x}^\star \in \Re^n$ is a global optimum solution for the optimization problem.

Proof: See arguments in [14]. ■

The gradient of $\gamma(\boldsymbol{x})$ in (23) can be derived as

$$\nabla\gamma(\boldsymbol{x}) = (\boldsymbol{\mu} - L\, \mathbb{1}) + \Phi^{-1}(\alpha) \frac{\boldsymbol{x}\, \boldsymbol{C}}{\sqrt{\boldsymbol{x}\, \boldsymbol{C}\, \boldsymbol{x}^T}}. \tag{30}$$

According to Karush-Kuhn-Tucker (KKT) conditions [14], the optimum solution (portfolio) $\boldsymbol{x}^\star \in \Re^n$ of POPL in (23) satisfies either of the following two conditions.

- $\nabla\gamma(\boldsymbol{x}^\star) = \boldsymbol{0}$ holds.
- Some constraints in (23) are active with $\boldsymbol{x}^\star \in \Re^n$.

Theorem 3: A portfolio $\boldsymbol{x} \in \Re^n$ of POPL borrows money from the loan up to the limit such as $x_0 = -m$ if

$$\gamma(\boldsymbol{x}) > L. \tag{31}$$

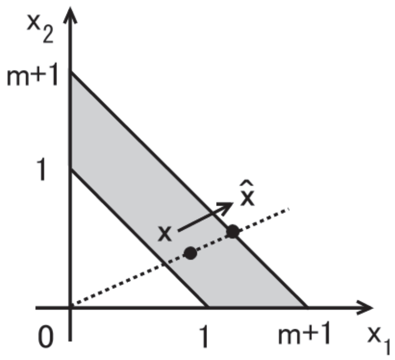

Figure 2. Illustration of Theorem 3.

Proof: Let us consider a new portfolio $\hat{\boldsymbol{x}} = \kappa\,\boldsymbol{x}$, $\kappa > 1$. The portfolio $\hat{\boldsymbol{x}} \in \Re^n$ borrows much money than $\boldsymbol{x}$ as

$$\begin{aligned} \hat{x}_0 &= 1 - \mathbb{1}\,\hat{\boldsymbol{x}}^T = 1 - \kappa\,\mathbb{1}\,\boldsymbol{x}^T \\ &< 1 - \mathbb{1}\,\boldsymbol{x}^T = x_0 \le 0 \end{aligned} \tag{32}$$

where $\hat{x}_0 \in \Re$ is the proportion of the loan for the new portfolio $\hat{\boldsymbol{x}} \in \Re^n$, while $x_0 \in \Re$ is the proportion of the loan for the current portfolio $\boldsymbol{x} \in \Re^n$.

The objective function value of $\boldsymbol{x} \in \Re^n$ is

$$\begin{aligned} \gamma(\boldsymbol{x}) &= \mu_g(\boldsymbol{x}) + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}) \\ &= (\boldsymbol{\mu} - L\,\mathbb{1})\,\boldsymbol{x}^T + L + \Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}). \end{aligned} \tag{33}$$

The objective function value of $\hat{\boldsymbol{x}} \in \Re^n$ is

$$\begin{aligned} \gamma(\hat{\boldsymbol{x}}) &= \mu_g(\hat{\boldsymbol{x}}) + \Phi^{-1}(\alpha)\,\sigma(\hat{\boldsymbol{x}}) \\ &= \kappa\,(\boldsymbol{\mu} - L\,\mathbb{1})\,\boldsymbol{x}^T + L + \kappa\,\Phi^{-1}(\alpha)\,\sigma(\boldsymbol{x}). \end{aligned} \tag{34}$$

From (33) and (34), the gap between them is

$$\varepsilon = \gamma(\hat{\boldsymbol{x}}) - \gamma(\boldsymbol{x}) = (\kappa - 1)\,(\gamma(\boldsymbol{x}) - L). \tag{35}$$

From (35) and $\kappa > 1$, if the condition in (31) is satisfied, the new portfolio $\hat{\boldsymbol{x}}$ is better than the current one $\boldsymbol{x}$ as

$$\varepsilon = \gamma(\hat{\boldsymbol{x}}) - \gamma(\boldsymbol{x}) > 0. \tag{36}$$

Since the value of ε increases in proportion to the value of κ, the portfolio $\boldsymbol{x} \in \Re^n$ that satisfies the condition in (31) is improved by borrowing money as much as possible. ∎

Figure 2 illustrates the two portfolios in Theorem 3 for the case of $n = 2$. The portfolio $\hat{\boldsymbol{x}} \in \Re^2$ on the upper hyper-plane is better than any portfolios $\boldsymbol{x} \in \Re^2$ within the feasible region of POPL (gray area) if the condition in (31) is satisfied.

V. Numerical Experiment

In order to solve POPL in (23), the interior point method provided by MATLAB [15] is employed. The information of the gradient $\nabla\gamma(\boldsymbol{x}) \in \Re^n$ in (30) is used effectively by the interior point method for improving its performance. As stated above, the optimality of the obtained solution is verified.

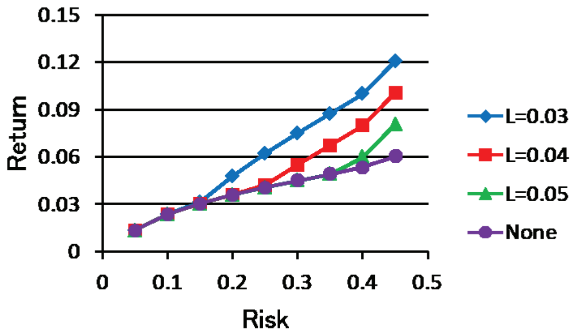

Figure 3. Efficient frontier (port0, $m = 2$).

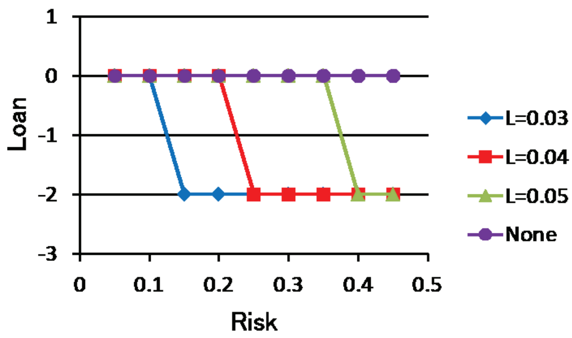

Figure 4. Proportion of the loan x_0 (port0, $m = 2$).

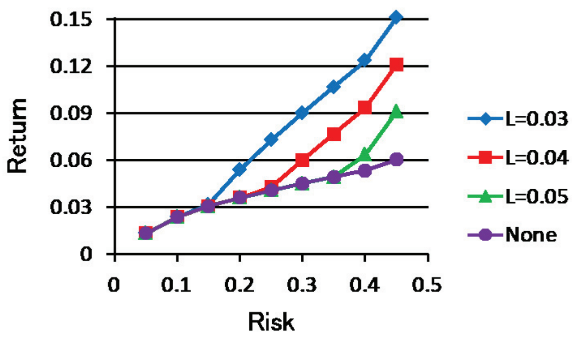

Figure 5. Efficient frontier (port0, $m = 3$).

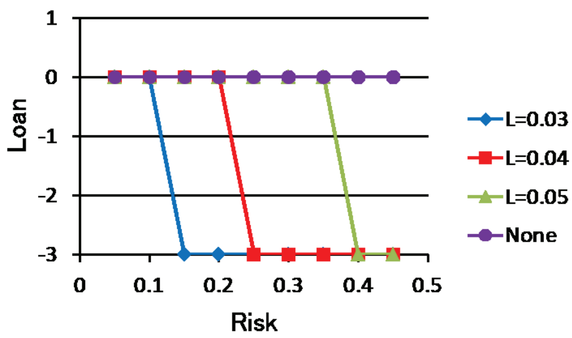

Figure 6. Proportion of the loan x_0 (port0, $m = 3$).

TABLE I. MEAN AND VARIANCE OF ASSET RETURN (PORT0).

ξ_i	ξ_1	ξ_2	ξ_3	ξ_4
μ_i	0.05	0.06	0.07	0.08
σ_i^2	0.10^2	0.20^2	0.15^2	0.25^2

TABLE II. CORRELATION BETWEEN ASSET RETURNS (PORT0).

ρ_{ij}	ξ_1	ξ_2	ξ_3	ξ_4
ξ_1	1.0	−0.7	0.1	−0.4
ξ_2	−0.7	1.0	−0.5	0.2
ξ_3	0.1	−0.5	0.1	−0.3
ξ_4	−0.4	0.2	−0.3	1.0

TABLE III. INDEX OF DATA SET AND NUMBER OF ASSETS.

Data set	Index	n
port1	Hang Seng	31
port2	DAX	85
port3	FTSE	89
port4	S&P	98

A. *Case Study 1*

An instance of POPL called port0 is given by Table I and Table II. The port0 consists of $n = 4$ assets. Table I shows the mean and variance of asset returns $\xi_i \in \Re$, $n = 1, \cdots, n$. Table II shows the correlation coefficient between them.

We evaluate the effect of the interest rate of the loan L on the return $\gamma(\boldsymbol{x})$ of POPL. Figure 3 shows the efficient frontier, or the trade-off between the return $\gamma(\boldsymbol{x}) \in \Re$ and the risk $\alpha \in (0, 0.5)$, when the upper limit of the loan is given as $m = 2$. Three different interest rates of the loan, $L = 0.03$, 0.04, and 0.05, are compared in Figure 3. "None" denotes the efficient frontier when the loan is not used. Figure 4 shows the proportion of the loan $x_0 \in [-m, 0]$ for each portfolio shown in Figure 3. In the case of "None", $x_0 = 0$ always holds.

From Figure 3, the efficient frontier is improved by using the loan. Specifically, the smaller the interest rate of the loan is, the higher the return. Besides, the portfolio $\boldsymbol{x}$ taking a high risk uses the loan even if the interest rate of the loan is high.

From Figure 4, as we have proven in Theorem 3, the loan is always used up to the limit ($x_0 = -m$) when it is used.

Figure 5 shows the efficient frontier when the upper limit of the loan is given as $m = 3$. Figure 6 shows the proportion of the loan $x_0 \in [-m, 0]$ for each portfolio in Figure 5.

From Figure 3 and Figure 5, we can confirm that the efficient frontier is improved by borrowing more money from the loan. From Figure 4 and Figure 6, the loan is always used up to the limit ($x_0 = -m$) regardless of the value of m.

B. *Case Study 2*

Instances of POPL are defined by using the data sets called port1 to port4, which are available from OR-Library [16]. Each of the data sets contains the means, variances, and a coefficient matrix of n asset returns. They are evaluated for real-life data sets, or historical data sets about asset returns. Table III shows the capital market indices and their numbers of assets. By using port1 to port4, we construct the models of asset returns as shown in (3) for the instances of POPL.

For all the instances of POPL, the upper limit of the loan is given as $m = 3$. Thereby, three different interest rates of

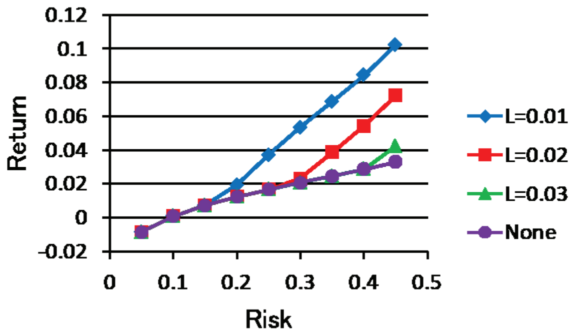

Figure 7. Efficient frontier (port1, $m = 3$).

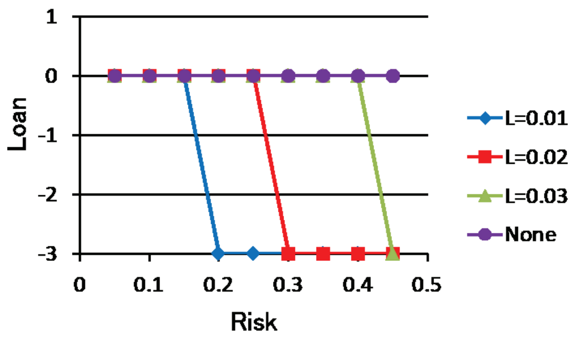

Figure 8. Proportion of the loan x_0 (port1, $m = 3$).

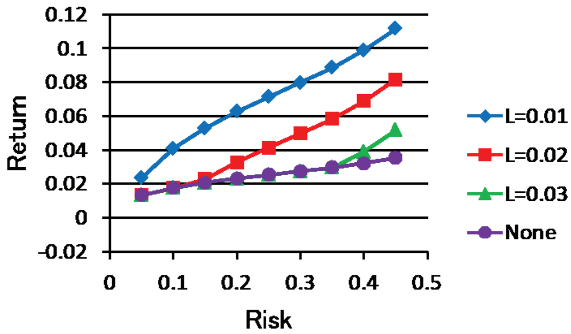

Figure 9. Efficient frontier (port2, $m = 3$).

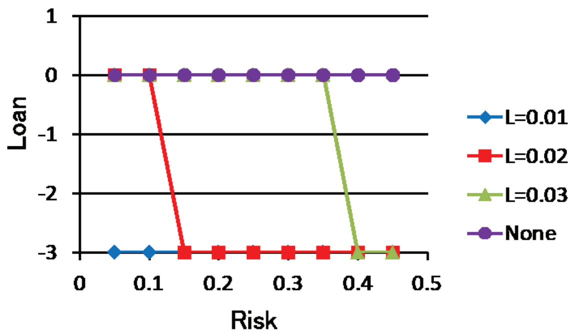

Figure 10. Proportion of the loan x_0 (port2, $m = 3$).

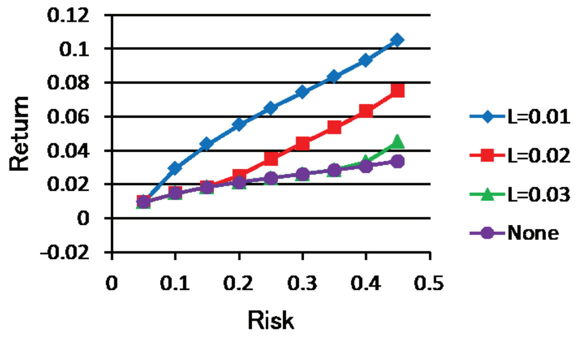

Figure 11. Efficient frontier (port3, $m = 3$).

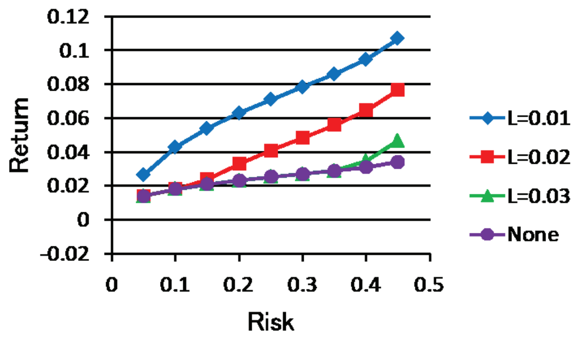

Figure 12. Efficient frontier (port4, $m = 3$).

the loan, namely $L = 0.01$, $L = 0.02$, and $L = 0.03$, are examined in each of the instances of POPL.

Figure 7 and Figure 8 show the results of the experiment conducted on port1. Figure 7 shows the efficient frontier, or the trade-off between the return and the risk, while Figure 8 shows the proportion of the loan for each portfolio shown in Figure 7. As stated above, “None” denotes the result when the loan is not used. Similarly, Figure 9 and Figure 10 show the results of the experiment conducted on port2. Figure 9 shows the efficient frontier, while Figure 10 shows the proportion of the loan for each portfolio shown in Figure 9.

Figure 11 shows the efficient frontier obtained for port3. Figure 12 also shows the efficient frontier obtained for port4. The proportions of the loan for the portfolios shown in Figure 11 and Figure 12 are omitted for want of space. However, the results of the experiments conducted on port1 to port4 are almost the same as the result of the experiment on port0.

From Figure 7 to Figure 12, we can confirm that the use of the loan works well for improving the efficient frontier. Especially, the lower interest rate of the loan provides higher return and benefits borrowers. On the other hand, the higher interest rate of the loan does not benefit lenders because such a loan is not used. Of course, the higher interest rate of the loan does not benefit borrowers, either. Furthermore, as we have proven by Theorem 3, we can see that the loan is always used up to the limit when it is used. From the results of the experiments, the loan seems to be used up to the limit immediately when the condition in (31) is satisfied.

VI. Conclusion

Portfolio optimization using loan is formulated as POPL. The emphasis of our work is on the theoretical characterization of POPL. From the analysis of POPL in (23), it is proven that POPL is a convex optimization problem. Furthermore, the condition of borrowing money from the loan up to the limit is revealed. Through numerical experiments, it can also be confirmed that the risk $\alpha \in (0, 0.5)$ and the interest rate of loan L play an important role in the decision whether the loan is used or not. If the money borrowed from the loan is invested in assets, the efficient frontier is improved by higher return.

For future work, we will think about a suitable interest rate of the loan that benefits both borrowers and lenders. Besides, we would like to include cardinality constraints in POPL.

Acknowledgment

This work was supported by JSPS KAKENHI (17K06508).

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