Weighted Branching Preorders and Distances: Logical Characterization and Complexity

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***Abstract*—We investigate branching bisimulation for weighted transition systems. It is known that branching bisimulation is characterized by computational tree logic without the next operator in the non-deterministic case. We demonstrate that the weighted version of this logic characterizes a weighted version of branching bisimulation, for which the decidability is NP-complete. This leads us to investigating two fragments of this logic allowing only upper bounds and either existential or universal quantification. The resulting existential and universal simulation relations are decidable in polynomial time. We consider distance-based analogues of weighted branching bisimulation and existential simulation and characterize these using fragments of the aforementioned logic.**

Keywords–Weighted transition systems; Weighted computational tree logic; Characterization; Weighted branching bisimulation.

I. Introduction

Classical process algebras, such as CCS, CSP and ACP [1]–[3] provide formalisms for describing the behavior of concurrent and interacting systems essential in terms of labeled transition systems. To capture the semantic equality of processes several behavioral preorders and equivalences have been considered, including the by now classical notion of bisimulation equivalence introduced by Milner [4] and Park [5]. Alongside the development of behavioral equivalences, an overall quest has been identification of corresponding temporal or modal logics, in the sense that the behavioral equivalence between two processes is in complete agreement with equality between the sets of logical properties they satisfy [6], [7].

Another important issue has been identification of behavioral preorders and equivalences that permit internal activities of processes to be abstracted away. The original notion of observational equivalence by Milner [1] serve this precise purpose, as does the later notion of branching bisimulation introduced by Weijland and Van Glabbeck [8]. Branching bisimulation equivalence has the remarkable additional property of being completely characterized by several different and natural modal logics, one of which is computational tree logic (CTL) without the next operator [9].

Whereas labeled transition systems suffice for describing the reactive and functional behavior of processes, they lack information about quantitative and non-functional aspects such as time or resource consumption. This has motivated the introduction and study of weighted transition systems, where transitions are labeled with quantities [10], [11], e.g., real, rational or integer values, allowing for the modeling of consumption or production of resources.

In this paper, we revisit weighted transition systems to identify useful behavioral relationships that are sensitive to quantities while permitting abstraction from internal activities. As a motivational example consider the following processes s and t both ending in the inactive process 0:

$$s \rightarrow_5 0 \text{ and } t \rightarrow_3 t' \rightarrow_2 0$$

Assuming that the states s, t, t' have the same atomic propositions, the intermediate state t' may be considered unobservable, and consequently s and t may be considered behaviorally equivalent as they end up in 0 with the same overall weight. To capture this situation in more generality, we extend in various ways the idea of branching bisimulation and simulation with weights. Aiming at extending [9], we consider a weighted version of CTL [12] without the next operator with the purpose of identifying interesting fragments and the various weighted versions of branching bisimulation and simulation they characterize. We allow for the systems to have reals as their weights, but the logic can only have rationals in the parameters, it is notable that we are, however, still able to capture the entire behavior of the systems. The study of those fragments are of importance because their weighted simulations are decidable in polynomial time (in contract to the NP-complete decidability of the full logic) while the logics can still specify interesting properties. This is essential for developing efficient tools.

Finally, we consider weighted branching bisimulation distances. Consider that the process s has a slightly perturbed weighted transition, e.g., $s \rightarrow_{5+\epsilon} 0$. Then, s and t are expected no longer to be weighted branching bisimilar. However, following the recent trends in replacing equivalences with metrics, and boolean answers with quantities [13]–[16], we shall introduce and logically characterize notions of weighted branching distance such that the distance between s and t will decrease for decreasing values of ϵ.

The structure of this paper will be as follows; in Section II, we introduce the preliminaries, Sections III-V each introduce a weighted branching (bi)simulation variant, identify a characterizing fragment of WCTL (without next) and determine its complexity. Section VI introduces distances and finally Section VII concludes the paper and describes future work.

II. Preliminaries

A weighted Kripke structure (WKS) is a straightforward extension of Kripke structures, where weights, in the form of non-negative reals, are added to each transition.

Definition 1 (Weighted Kripke Structure). A *weighted Kripke structure* is a tuple $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, where S is a set of states, $\mathcal{AP}$ is a set of atomic propositions, $\mathcal{V} : S \rightarrow \mathcal{P}(\mathcal{AP})$ is a mapping from states to sets of atomic propositions and $\rightarrow \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is a labeled transition relation. ⋆

For simplicity, transitions are denoted by $s \rightarrow_w s'$ instead of $(s, w, s') \in \rightarrow$.

We say that a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$ is *finite*, if S and $\to$ are finite; *non-blocking*, if for all states $s \in S$ there is at least one transition $(s, w, s') \in\to$ that starts in s and *rational* if all weights on transitions belong to $\mathbb{Q}_{\geq 0}$.

Example 1. In Figure 1(a), we show the WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$, where $S = \{s^1, s^1_1, s^1_2\}$, $\mathcal{AP} = \{p, q\}$, $\to$ is defined as $s^1 \to_3 s^1, s^1 \to_5 s^1_1, s^1 \to_2 s^1_2$ and $\mathcal{V}(s^1) = \{p\}, \mathcal{V}(s^1_1) = \{q\}$ and $\mathcal{V}(s^1_2) = \{q\}$.
Note that $\mathcal{K}$ is finite and rational, since it has a finite number of states, a finite number of transitions and all weights are rational. It is, however, not non-blocking, since there are no transitions from either of the states s^1_1 and s^1_2. ⋄

To specify properties of WKSs, we introduce a weighted extension of CTL (WCTL), where intervals are introduced on the next and the until operators.

Definition 2 (Syntax of WCTL). Let $\mathcal{AP}$ be a set of atomic propositions. The syntax of WCTL is given by

$$\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid EX_I\phi \mid AX_I\phi \mid E(\phi_1 U_I \phi_2) \mid A(\phi_1 U_I \phi_2),$$

where $p \in \mathcal{AP}$ and $I = [l, u]$, where $l, u \in \mathbb{Q}_{\geq 0}$, $l \leq u$. ⋆

Note that I can be any type of interval and that we allow that $l = u$, such that the interval can be a single point.
The semantics of WCTL are given by the satisfiability relation, defined inductively for an arbitrary non-blocking WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$ and an arbitrary state $s \in S$, as follows

- $\mathcal{K}, s \models p$ iff $p \in \mathcal{V}(s)$,
- $\mathcal{K}, s \models \neg\phi$ iff $s \not\models \phi$
- $\mathcal{K}, s \models \phi_1 \wedge \phi_2$ iff $s \models \phi_1$ and $s \models \phi_2$,
- $\mathcal{K}, s \models EX_I\phi$ iff there exists $s \to_w s'$, such that $w \in I$ and $s' \models \phi$,
- $\mathcal{K}, s \models AX_I\phi$ iff for all $s \to_w s'$ such that $w \in I$, $s' \models \phi$,
- $\mathcal{K}, s \models E(\phi_1 U_I \phi_2)$ iff there exists a trace $s \to_{w_1} s_1 \to_{w_2} \cdots \to_{w_k} s_k \to_{w_{k+1}} \cdots$, such that there exists a state s_k such that $s_k \models \phi_2$, for all $i < k, s_i \models \phi_1$ and $\sum_{i=1}^k w_i \in I$,
- $\mathcal{K}, s \models A(\phi_1 U_I \phi_2)$ iff for all traces $s \to_{w_1} s_1 \to_{w_2} \cdots \to_{w_k} s_k \to_{w_{k+1}} \cdots$, there exists a state s_k such that $s_k \models \phi_2$, for all $i < k, s_i \models \phi_1$ and $\sum_{i=1}^k w_i \in I$.

We use the other Boolean operators with their usual semantics. Consider the following example.

Example 2. Return to the WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$ in Figure 1(a). The state s^1 satisfies, among others, the following formulae: $AX_{[1,3)}p$ and $E(pU_{[4,9]}q)$, but not for instance $A(pU_{[0,7]}q)$, because of the self-loop. ⋄

A well-known way of comparing WKSs is with *weighted bisimulation* [4], [5], which is defined in a way that ensures complete matching of behavior where transitions are matched one-to-one. Weighted bisimulation is completely characterized by WCTL as proven by [17].
Note that in the models, we allow weights to be reals, while in the logic, we only allow rationals. We describe the models as general as possible, but as the logic has to be countable, we will have to restrict ourselves to rationals. The logic is however still able to encompass the behavior of the model, since we can approximate the reals by rationals.

III. WCTL WITHOUT NEXT

Let us now look at a fragment of WCTL for which we have removed the next operator. As discussed in the introduction we do not wish to reason about single transition steps in our models, which explains the need to remove the next operator. With the remaining operators we are able to reason about a bound of the cost of arriving at some behavior, while preserving behavior along the way.

Definition 3 (Syntax of WCTL$_{-X}$). Let $\mathcal{AP}$ be a set of propositions. The syntax of WCTL$_{-X}$ is given by

$$\phi ::= p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid A(\phi_1 U_I \phi_2) \mid E(\phi_1 U_I \phi_2),$$

where $p \in \mathcal{AP}$ and $I = [l, u]$, where $l, u \in \mathbb{Q}_{\geq 0}$. ⋆

The semantics of WCTL$_{-X}$ is given by the same satisfiability relation as for WCTL.
We will now introduce a notion of weighted branching bisimulation and observe that the bisimulation and the logic induces the same relation on finite WKSs. We allow for a weighted transition to be matched by a sequence of transitions with identical accumulated weight, behavior is preserved in each intermediate state and the end behavior is the same.

Definition 4. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$ a *weighted branching bisimulation* (WBB) is a relation $R \subseteq S \times S$, such that whenever $(s, t) \in R$

- $\mathcal{V}(s) = \mathcal{V}(t)$
- for all $s \to_w s'$ there exists $t \to_{v_1} t_1 \to_{v_2} \cdots \to_{v_k} t_k$, such that $\sum_{i=1}^k v_i = w$, $(s', t_k) \in R$ and for all $i < k$, $(s, t_i) \in R$
- for all $t \to_v t'$ there exists $s \to_{w_1} s_1 \to_{w_2} \cdots \to_{w_k} s_k$, such that $\sum_{i=1}^k w_i = v$, $(t', s_k) \in R$ and for all $i < k$, $(t, s_i) \in R$

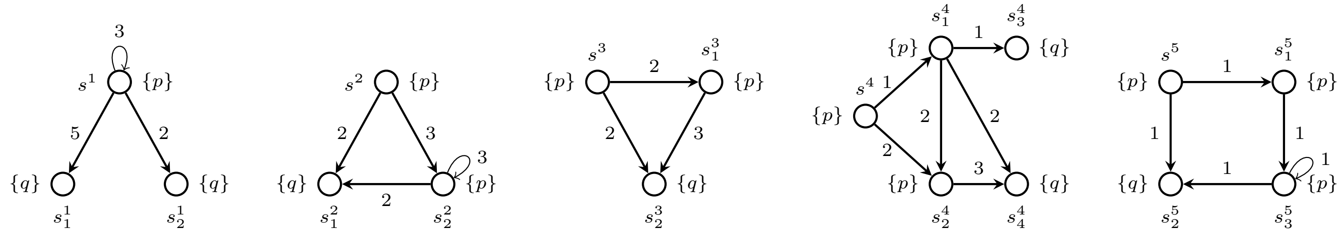
If there exists a weighted branching bisimulation relating s and t, we say that s and t are weighted branching bisimilar and denote it by $s \approx_{\mathbf{I}} t$. The relation $\approx_{\mathbf{I}}$ will henceforth be referred to as weighted branching bisimilarity (WBB). ⋆

The following theorem shows that WCTL$_{-X}$ characterizes WKS up to WBB.

Theorem 1. Let $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \to)$ be a finite WKS. Then for all $s, t \in S$

$$s \approx_{\mathbf{I}} t \text{ iff } [\forall \phi \in \text{WCTL}_{-X}\ s \models \phi \Leftrightarrow t \models \phi].$$

Proof. ($\Rightarrow$) Suppose $s \approx_{\mathbf{I}} t$ and $s \models \phi$. Induction on the structure of ϕ.
The case $\phi = E(\phi_1 U_I \phi_2)$**:** By definition $s \models \phi$ iff there exists $s \to_{w_1} s_1 \to_{w_2} \cdots \to_{w_k} s_k \to \cdots$ s.t. $s_k \models \phi_2$, $\forall i < k, s_i \models \phi_1$ and $\sum_{i=1}^k w_i \in I$. As $s \approx_{\mathbf{I}} t$, we have that for every step $s_i \to_{w_{i+1}} s_{i+1}$ there exists $t^i \to_{v_1^{i+1}} t_1^i \to_{v_2^{i+1}} \cdots \to_{v^{i+1}_{h^{i+1}}} t^{i+1}$ such that $t^{i+1} \approx_{\mathbf{I}} s_{i+1}$, $\forall j < h^{i+1}, t^i_j \approx_{\mathbf{I}} s_i$ and $\sum_{j=1}^{h^{i+1}} v_j^{i+1} = w_{i+1}$. By induction, each state $t^i_j \models \phi_1$ for $j < k$, the final state $t^k \models \phi_2$ and $\sum_{i=1}^k \sum_{j=1}^{h^i} v^i_j = \sum_{i=1}^k w_i$ so by definition, $t \models \phi$.
The case $\phi = A(\phi_1 U_I \phi_2)$**:** If $t \not\to$ trivially $t \models \phi$. Otherwise choose a trace $\pi_1 = t \to_{v_1} t_1 \to_{v_2} \cdots$. For



(a) (b) (c) (d) (e)

Figure 1. Five WKSs. The following relations are true $s^1 \approx_{\mathbf{I}} s^2$, $s^3 \not\approx_{\mathbf{I}} s^4$, $s^3 \leq_{\mathbf{E}} s^5$, $s^4 \not\leq_{\mathbf{E}} s^3$, $s^3 \leq_{\mathbf{A}} s^5$, $s^4 \approx^0_I s^3$ and $s^3 \approx^1_I s^4$.

each step $t_i \rightarrow_{v_{i+1}} t_{i+1}$ by relation, we have that there exist $s^i \rightarrow_{w^i_1} s^i_1 \rightarrow_{w^i_2} \cdots \rightarrow_{w^i_{k^i}} s^{i+1}$, such that $t_{i+1} \approx_{\mathbf{I}} s^{i+1}$, for all $j < k^i$ $t_i \approx_{\mathbf{I}} s^i_j$ and $\sum_{j=1}^{k^i} w^i_j = v_{i+1}$. We therefore know that there exists a trace $\pi_2 = s \rightarrow_{w^0_1} s^0_1 \rightarrow_{w^0_2} \cdots \rightarrow_{w^0_{k^0}} s^1 \rightarrow_{w^1_1} \cdots \rightarrow_{w^{h-1}_{k^{h-1}}} s^h \rightarrow_{w^h_1} \cdots$ such that for all i, we have $t_i \approx_{\mathbf{I}} s^i$, for all $j < k^i$ $t_i \approx_{\mathbf{I}} s^i_j$ and $\sum_{j=1}^{k^i} w^i_j = v_{i+1}$. Since $s \models A(\phi_1 U_I \phi_2)$, we know that for π_2 there must exist some s^h such that $s^h \models \phi_2$, for all $i < h$ and all j, we have $s^i_j \models \phi_1$ and $\sum_{i=0}^{h-1} \sum_{j=1}^{k^i} w^i_j \in I$. By construction of π_2 this means that for π_1 there must exist a t_h such that $t_h \models \phi_2$, for all $i < h$ $t_i \models \phi_1$ and $\sum_{i=1}^{h} v_i = \sum_{i=0}^{h-1} \sum_{j=1}^{k^i} w^i_j \in I$. By definition $t \models A(\phi_1 U_I \phi_2)$.

($\Leftarrow$) Define $(s,t) \in R$ iff $[\forall \phi \in \text{WCTL}_{-X}$ $s \models \phi \Leftrightarrow t \models \phi]$. We show that R is a WBB.
Suppose $s \rightarrow_w s'$ and let $\pi_i = t \rightarrow_{v^i_1} t^i_1 \rightarrow_{v^i_2} \cdots \rightarrow_{v^i_{k^i}} t^i_{k^i}$ such that $\sum_{j=1}^{k^i} v^i_j = w$ be traces out of t of weight equal to w. Without loss of generality we can skip all traces with zero-cycles, which means that since the Kripke structure is finite there is only a finite number of traces π_i of weight w, $i = 1, \ldots, n$. Assume that none of these traces match $s \rightarrow_w s'$, which means that for each trace π_i either $(s', t^i_{k^i}) \notin R$ or there exist a $j < k^i$ such that $(s, t^i_j) \notin R$. For each trace π_i such that $(s', t^i_{k^i}) \notin R$, $i = 1, \ldots, k$, $k \leq n$, there exists a formula ψ_i such that $s' \models \psi_i$ and $t^i_{k^i} \not\models \psi_i$ and for each trace π_i such that $(s, t^i_j) \notin R$, $i = k+1, \ldots, n$, there exists a formula ϕ_i such that $s \models \phi_i$ and $t^i_j \not\models \phi_i$.
For a decreasing series of rationals u^j such that $\lim_{j\to\infty} u^j = w$ and an increasing series of rationals y^j such that $\lim_{j\to\infty} y^j = w$, we can create a series of formulae $\phi^j = E\left(\bigwedge_{i\in[1,k]} \phi_i U_{[y^j,u^j]} \bigwedge_{i\in[k,n]} \psi_i\right)$ for which $s \models \bigwedge_j \phi^j$, but $t \not\models \bigwedge_j \phi^j$, contradicting $(s,t) \in R$. ■

Consider a couple of examples of the use of this theorem.

Example 3. Consider Figure 1(a) and 1(b). Let us verify that s^1 and s^2 are WBB. Transition $s^1 \rightarrow_3 s^1$ and $s^2 \rightarrow_3 s^2_2$ have to match each other, which means, we have to prove that also $s^1 \approx_{\mathbf{I}} s^2_2$. This is trivially proved, since s^1 and s^2_2 are the same except for the transition $s^1 \rightarrow_5 s^1_1$, which is matched by $s^2_2 \rightarrow_3 s^2_2 \rightarrow_2 s^2_1$, since $s^1_1 \approx_{\mathbf{I}} s^1_2 \approx_{\mathbf{I}} s^2_1$ trivially. Transitions $s^1 \rightarrow_2 s^1_2$ and $s^2 \rightarrow_2 s^2_1$ match each other and transition $s^1 \rightarrow_5 s^1_1$ is matched by $s_2 \rightarrow_3 s^2_2 \rightarrow_2 s^2_1$, as we know $s^1 \approx_{\mathbf{I}} s^2_2$.
We therefore know that any formula that is satisfied by either s^1 or s^2 is also satisfied by the other. ◇

Example 4. Consider Figure 1(c) and 1(d). We see that s^3 and s^4 are not weighted branching bisimilar, since the transition $s^4 \rightarrow_1 s^4_1$ cannot be matched from s^3. This means, we can find a distinguishing formula between these two states. We see that for $\phi = A(pU_{[2,5]}q)$, $s^3 \models \phi$, but $s^4 \not\models \phi$, since there is a trace $s^4 \rightarrow_1 s^4_1 \rightarrow_2 s^4_2 \rightarrow_3 s^4_4$ for which the accumulated weight is 6 before it reaches a state that satisfies q. ◇

We can prove that WBB is a maximal fixed point of a suitable function $\mathcal{F}_{\mathbf{I}}$ defined as follows.

Definition 5. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, define a function $\mathcal{F}_{\mathbf{I}} : 2^{S\times S} \rightarrow 2^{S\times S}$ such that given a relation $R \subseteq S \times S$ then $(s,t) \in \mathcal{F}_{\mathbf{I}}(R)$ if and only if

- $\mathcal{V}(s) = \mathcal{V}(t)$
- for all $s \rightarrow_w s'$ there exists $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_k} t_k$, such that $\sum_{i=1}^{k} v_i = w$, $(s', t_k) \in R$ and for all $i < k, (s, t_i) \in R$
- for all $t \rightarrow_v t'$ there exists $s \rightarrow_{w_1} s_1 \rightarrow_{w_2} \cdots \rightarrow_{w_k} s_k$, such that $\sum_{i=1}^{k} w_i = v$, $(t', s_k) \in R$ and for all $i < k, (t, s_i) \in R$ ⋆

The function $\mathcal{F}_{\mathbf{I}}$ takes in a relation R that identifies pairs of states believed to be bisimilar and removes all pairs that fail to be bisimilar in one step, when assuming that the states of R are bisimilar. This means that the resulting relation is not necessarily a bisimulation relation, since the assumption can be wrong. When starting from the assumption that all states are bisimilar, however, and applying $\mathcal{F}_{\mathbf{I}}$ until stability, i.e. finding a maximal fixed point, we are sure to have WBB.

We now determine the complexity of deciding WBB.

Theorem 2. Deciding WBB on finite rational WKSs is NP-hard.

Proof. We use the integer knapsack problem, which is well-known to be NP-complete [18], and show that it is polynomial time reducible to the problem of deciding WBB.
In the integer knapsack problem, we are given a finite set E of elements which each have a value $v_i \in \mathbb{Z}_{\geq 0}$ and a weight $w_i \in \mathbb{Z}_{\geq 0}$. We are further given positive integers p and c. Is there an assignment of positive integers a_i such that $\sum_i a_i v_i \geq p$ and $\sum_i a_i w_i \leq c$? The integer knapsack problem is still NP-complete if for all i $v_i = w_i$, so we want to assign positive integers a_i such that $\sum_i a_i w_i = c$. Let us assume that we have a set $E = \{e_1, \ldots, e_n\}$ of elements with weights (and values) $\{w_1, \ldots, w_n\}$, and our capacity is c.
The reduction generates two WKSs, as seen in Figure 2, such that $\mathcal{K}_1 = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, where $S = \{s\}$, $\mathcal{AP} = \emptyset$, $\mathcal{V}(s) = \emptyset$ and $\rightarrow = \{(s, w_1, s), \ldots, (s, w_n, s)\}$ and $\mathcal{K}_2 = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, where $S = \{t, t'\}$, $\mathcal{AP} = \emptyset$, $\mathcal{V}(t) = \mathcal{V}(t') = \emptyset$ and $\rightarrow=$

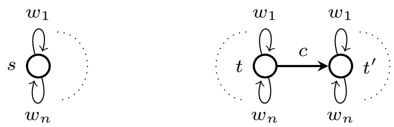


Figure 2. Two WKSs $\mathcal{K}_1$ and $\mathcal{K}_2$ that are generated by reduction.

$\{(t,w_1,t),\dots,(t,w_n,t),(t',w_1,t'),\dots,(t',w_n,t'),(t,c,t')\}$. We demonstrate that there is a solution to the integer knapsack problem if and only if the two states s and t are WBB.

($\Rightarrow$) Suppose there exists a set of integers $\{k_1,\dots,k_n\}$ such that $\sum k_i w_i = c$. If we ignore the transition $t \rightarrow_c t'$ the three states s, t and t' are WBB. Hence the only reason why s and t should not be WBB is if s cannot match the transition $t \rightarrow_c t'$. But since, there exists a set $\{k_1,\dots,k_n\}$ such that $\sum k_i w_i = c$, s can do a series of transitions accordingly and match transition $t \rightarrow_c t'$.

($\Leftarrow$) Suppose the transition $t \rightarrow_c t'$ can be matched by $s \rightarrow_{w_1} s \rightarrow_{w_2} \cdots \rightarrow_{w_k} s$ such that $\sum w_i = c$, note that each $\rightarrow_{w_i}$ can be repeated as many times as necessary. This is equivalent to the existence of a set $\{k_1,\dots,k_n\}$ such that $\sum k_i w_i = c$ providing a solution to the integer knapsack problem. ■

We have demonstrated that deciding WBB is at least NP-hard, now let us prove that the problem is contained in NP. We will need the following theorem, which is a reformulated instance of a theorem proved by Nykänen and Ukkonen [19].

Theorem 3. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$, $s,t \in S$ and a target cost k. The problem of deciding whether there exists a trace in $\mathcal{K}$ from s to t of cost exactly k is NP-complete.

Theorem 4. Deciding WBB on a finite rational WKSs is contained in NP.

Proof. WBB is the maximal fixed point of $\mathcal{F}_\mathbf{I}$, so we apply $\mathcal{F}_\mathbf{I}$ repeatedly on $S \times S$ until the set of bisimilar states stabilizes. Each time $\mathcal{F}_\mathbf{I}^k(S\times S) \neq \mathcal{F}_\mathbf{I}^{k-1}(S \times S)$, meaning at least one pair of states has been removed, we will apply $\mathcal{F}_\mathbf{I}$ once more. This can be done a maximum of $n = |S|$ times, before the resulting set is empty. In each iteration, we have to check for both state in each pair of bisimilar states if every transition is matched. This means that for each state s we will for each other state t (if $(s,t) \in \mathcal{F}_\mathbf{I}^i$) check each transition $s \rightarrow_w s'$ and see if it can be matched by a trace from t as required. This means that we will try to find a path of cost w from t to each of the states t', where $(s',t') \in \mathcal{F}_\mathbf{I}^i$. This means applying Theorem 3 at most n times for each transition $s \rightarrow_w s'$ for each t. Since the problem can be solved by applying an NP-complete problem a polynomial number of times the problem is in NP. ■

By Theorem 2 and Theorem 4, we have the following result.

Theorem 5. Deciding WBB on a finite rational WKSs is NP-complete.

IV. WCTL WITHOUT NEXT, UNIVERSALITY AND LOWER BOUNDS

Using the entire WCTL without the next operator leads to a definition of a bisimulation relation that is NP-complete. We therefore limit ourselves further, to see if we can find a relation which is decidable in polynomial time, while still preserving interesting properties in the logic. We limit ourselves to a fragment of WCTL called $\text{EWCTL}^{\leq}_{-X}$ in which we only concentrate on upper bounds and the existential operator, while still leaving out the next operator. This will enable us to reason about a maximal bound on the cost of arriving to some behavior, while preserving original behavior on the way.

Definition 6 (Syntax of $\text{EWCTL}^{\leq}_{-X}$). Let $\mathcal{AP}$ be a set of atomic propositions. The syntax of $\text{EWCTL}^{\leq}_{-X}$ is given by

$$\phi ::= p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid E(\phi_1 U_{\leq u} \phi_2),$$

where $p \in \mathcal{AP}$ and $u \in \mathbb{Q}_{\geq 0}$. ⋆

The semantics of $\text{EWCTL}^{\leq}_{-X}$ is given by the same satisfiability relation as the semantics for WCTL, since we can substitute any upper bound $\leq u$ with an interval $[0,u]$.
Let us now develop a notion of branching simulation, such that the simulation and the logic induces the same relation on finite WKSs. We define the simulation from the same idea as before, but now demand the cost of the matching trace to be lower than the cost of the transition.

Definition 7. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ an *existential bounded simulation* (EBS) is a relation $R \subseteq S \times S$, such that whenever $(s,t) \in R$

- $\mathcal{V}(s) = \mathcal{V}(t)$,
- for all $s \rightarrow_w s'$ there exists $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_k} t_k$ such that $\sum_{i=1}^k v_i \leq w$, $(s',t_k) \in R$ and $\forall i < k$, $(s,t_i) \in R$.

If there exists an existential bounded simulation relating s and t, we say that s and t are existential bounded similar and denote it by $s \leq_\mathbf{E} t$. The relation $\leq_\mathbf{E}$ will henceforth be referred to as existential bounded similarity (EBS). ⋆

The following theorem shows that indeed $\text{EWCTL}^{\leq}_{-X}$ and EBS induce the same relation on finite WKSs. The proof of the following theorem is as the proof of Theorem 1.

Theorem 6. Let $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ be a finite WKS. Then for all $s,t \in S$

$$s \leq_\mathbf{E} t \text{ iff } [\forall \phi \in \text{EWCTL}^{\leq}_{-X} s \models \phi \Rightarrow t \models \phi].$$

Consider two examples of the use of this theorem.

Example 5. Consider Figure 1(c) and 1(e). We can verify that s^3 and s^5 are EBS. Transition $s^3 \rightarrow_2 s^3_2$ can be matched by $s^5 \rightarrow_1 s^5_2$, since clearly $s^3_2 \leq_\mathbf{E} s^5_2$. Transition $s^3 \rightarrow_2 s^3_1$ can be matched by $s^5 \rightarrow_1 s^5_1$ if $s^3_1 \leq_\mathbf{E} s^5_1$. We only have one transition $s^3_1 \rightarrow_3 s^3_2$ which can be matched by $s^5_1 \rightarrow_1 s^5_3 \rightarrow_1 s^5_2$ if $s^3_1 \leq_\mathbf{E} s^5_3$. There is the same transition $s^3_1 \rightarrow_3 s^3_2$ which can be matched by $s^5_3 \rightarrow_1 s^5_2$. ◇

Example 6. Consider Figure 1(c) and 1(d), s^4 and s^3 are not EBS, since the transition $s^4 \rightarrow_1 s^4_1$ can only be matched from s^3 by doing nothing. Then, we need $s^4_1 \leq_\mathbf{E} s^3$, but the transition $s^4_1 \rightarrow_1 s^4_3$ cannot be matched from s^3. This means, we can find a distinguishing formula, such that $s^4 \models \phi$, but $s^3 \not\models \phi$. We choose $\phi = E(pU_{\leq 1}E(pU_{\leq 1}q))$ to exemplify this. ◇

To compute EBS, we need to compute the maximal fixed point over a suitable function, defined in the canonical way. We can now find all simulating states as $\leq_\mathbf{E} = \mathcal{F}_\mathbf{E}^M(S \times S)$ for some

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natural number M. This gives us a way of computing $\leq_{\mathbf{E}}$ by simply computing the non-increasing sequence $\mathcal{F}^0_{\mathbf{E}}(S \times S) \supseteq \mathcal{F}^1_{\mathbf{E}}(S \times S) \supseteq \mathcal{F}^2_{\mathbf{E}}(S \times S) \supseteq \cdots$ until it stabilizes.

Let us sketch an algorithm based upon this principle.

Algorithm 1. In a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ let n be the number of states and m the number of transitions. We compute $\leq_{\mathbf{E}}$ iteratively as described. The algorithm will stop after at most n^2 iterations, since we will remove at least one pair for each iteration, starting from $\mathcal{F}^0_{\mathbf{E}}(S \times S) = S \times S$.
Let $\mathcal{F}^n_{\mathbf{E}}(S \times S)$ be given. For each state s let $\mathcal{K}^n_s$ be the projection of $\mathcal{K}$ to the set $\{t \mid (s,t) \in \mathcal{F}^n_{\mathbf{E}}(S \times S)\}$. In $\mathcal{K}^n_s$, we may by classic algorithm in $O(n^3)$ compute the shortest path between all pairs of states. We write $t \rightarrow^{n,s}_{w} t'$ if w is the length of the shortest such path between t and t'.
We can now calculate our next iteration; for each $(s,t) \in \mathcal{F}^n_{\mathbf{E}}(S \times S)$, for each $s \rightarrow_w s'$, we check whether there exists a t' such that $t \rightarrow^{s,n}_{v} t'$ and $t' \rightarrow_{v'} t''$ where $v + v' \leq w$ and $(s', t'') \in \mathcal{F}^n_{\mathbf{E}}(S \times S)$. If there exist such t' then $(s,t) \in \mathcal{F}^{n+1}_{\mathbf{E}}(S \times S)$ otherwise $(s,t) \notin \mathcal{F}^{n+1}_{\mathbf{E}}(S \times S)$. The check can be performed in $O(n^2)$. Thus, overall, $\leq_{\mathbf{E}}$ can be computed in $O(n^2 \cdot (n \cdot n^3 + m \cdot n \cdot n^2)) \leq O(n^7)$. We can assume $m \leq n^2$, since we are only interested in the cheapest transition between two states, so if two or more exist we will disregard everyone but the cheapest.

V. WCTL WITHOUT NEXT, EXISTENTIALITY AND LOWER BOUNDS

We have now established a relation for a fragment of WCTL with upper bounds and only the existential quantifier over paths, but we would like to be able to reason about the universal quantifier, which will allow us to model safety properties, e.g., cost-bounded liveness properties. Let us look at a fragment with this quantifier instead.

Definition 8 (Syntax of AWCTL$^{\leq}_{-X}$). Let $\mathcal{AP}$ be a set of atomic propositions. The syntax of AWCTL$^{\leq}_{-X}$ is given by

$$\phi ::= p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid A(\phi_1 U_{\leq u} \phi_2),$$

where $p \in \mathcal{AP}$ and $u \in \mathbb{Q}_{\geq 0}$. ⋆

The semantics of AWCTL$^{\leq}_{-X}$ is given by the same satisfiability relation as the semantics for WCTL, since we can substitute any upper bound $\leq u$ with an interval $[0, u]$.
Let us now develop a notion of weighted branching simulation, such that whatever relation is induced by the simulation is also induced by the logic. The idea of the simulation is basically the same as for EBS except the simulating state has to have a trace that is more expensive than the transition it matches.

Definition 9. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ a *universal bounded simulation* (UBS) is a relation $R \subseteq S \times S$, such that whenever $(s,t) \in R$

- $\mathcal{V}(s) = \mathcal{V}(t)$,
- for all $s \rightarrow_w s'$ there exists $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_k} t_k$ such that $\sum_{i=1}^{k} v_i \geq w$, $(s', t_k) \in R$ and $\forall i < k$, $(s, t_i) \in R$.

If there exists a universal bounded simulation relating s and t, we say that s and t are universal bounded similar and denote it by $s \leq_{\mathbf{A}} t$. The relation $\leq_{\mathbf{A}}$ will henceforth be referred to as universal bounded similarity (UBS). ⋆

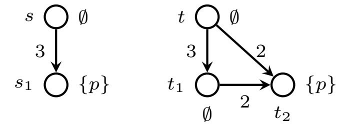

Figure 3. A WKS $\mathcal{K}$. The states s and t are not universally bounded similar, but every formula that t satisfies is also satisfied by s.

The following theorem shows that we can go from UBS to a relation induced by AWCTL$^{\leq}_{-X}$ on finite WKSs. The structure of the proof is as the proof for Theorem 1.

Theorem 7. Let $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ be a finite WKS. Then for all $s, t \in S$

$$\text{if } s \leq_{\mathbf{A}} t \text{ then } [\forall \phi \in \text{AWCTL}^{\leq}_{-X} t \models \phi \Rightarrow s \models \phi].$$

Note that we cannot go from a logic induced relation to our simulation. This is clear from the following counterexample.

Example 7. Consider the WKS $\mathcal{K}$ in Figure 3. We will prove that $[\forall \phi \in \text{AWCTL}^{\leq}_{-X}$ if $t \models \phi$ then $s \models \phi$, but $s \not\leq_{\mathbf{A}} t]$. Look at all possible traces from s and t;

$$\pi^s = s \rightarrow_3 s_1$$
$$\pi^t_1 = t \rightarrow_2 t_2$$
$$\pi^t_2 = t \rightarrow_3 t_1 \rightarrow_2 t_2.$$

We prove that $s \not\leq_{\mathbf{A}} t$. If $s \leq_{\mathbf{A}} t$ the transition $s \rightarrow_3 s_1$ has to be matched from t, this can only be done by π^t_2, since π^t_1 has too low a cost. Then either $s_1 \leq_{\mathbf{A}} t_1$ or both $s \leq_{\mathbf{A}} t_1$ and $s_1 \leq_{\mathbf{A}} t_2$. The first case is impossible since the atomic propositions are different and the second is impossible, since t_1 does not have a transition of enough cost to match $s \rightarrow_3 s_1$.
Now, we prove that $[\forall \phi \in \text{AWCTL}^{\leq}_{-X}$ if $t \models \phi$ then $s \models \phi]$. Induction on the structure of ϕ.
The case $t \models A(\phi_1 U_{\leq u} \phi_2)$**:** Split in cases $u < 3$ and $u \geq 3$. If $u < 3$ then $t \models A(\phi_1 U_{\leq u} \phi_2)$ only if $t \models \phi_2$. By induction $s \models \phi_2$ and therefore $s \models A(\phi_1 U_{\leq u} \phi_2)$.
If $u \geq 3$ then $t \models A(\phi_1 U_{\leq u} \phi_2)$ if either $t \models \phi_2$ or ($t \models \phi_1$, $t_1 \models \phi_1 \wedge \phi_2$ and $t_2 \models \phi_2$). In the first case by induction $s \models \phi_2$, hence $s \models A(\phi_1 U_{\leq u} \phi_2)$. In the second case by induction $s \models \phi_1$ and since $s_1 \leq_{\mathbf{A}} t_2$ by Theorem 7 $s_1 \models \phi_2$, therefore $s \models A(\phi_1 U_{\leq u} \phi_2)$. ◇

Consider an example of how we can use Theorem 7.

Example 8. Consider Figure 1(c) and 1(e). We show that s^3 and s^5 are UBS. Transition $s^3 \rightarrow_2 s^3_2$ can be matched by $s^5 \rightarrow_1 s^5_1 \rightarrow_1 s^5_3 \rightarrow_1 s^5_2$, if $s^3 \leq_{\mathbf{E}} s^5_1$ and $s^3 \leq_{\mathbf{E}} s^5_3$. Let us prove that $s^3 \leq_{\mathbf{E}} s^5_3$. Transition $s^3 \rightarrow_2 s^3_2$ can be matched by $s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_2$. Transition $s^3 \rightarrow_2 s^3_1$ can be matched by $s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3$. Let us prove that $s^3 \leq_{\mathbf{E}} s^5_1$. Transition $s^3 \rightarrow_2 s^3_2$ can be matched by $s^5_1 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_2$, since $s^3 \leq_{\mathbf{E}} s^5_3$. Transition $s^3 \rightarrow_2 s^3_1$ can be matched by $s^5_1 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3$, since $s^3 \leq_{\mathbf{E}} s^5_3$. Let us return to the main problem $s^3 \leq_{\mathbf{E}} s^5$. We only need to prove that transition $s^3 \rightarrow_2 s^3_1$ can be matched. We can match with $s^5 \rightarrow_1 s^5_1 \rightarrow_1 s^5_3 \rightarrow_1 s^5_3$. Therefore s^3 and s^5 satisfy the same formulae. ◇

As before, we can define a suitable function $\mathcal{F}_{\mathbf{A}}$ in the canonical way such that the UBS is a maximal fixed point over this function. Again, we can find all simulating states as $\leq_{\mathbf{A}} = \mathcal{F}^M_{\mathbf{A}}(S \times S)$ for some natural number M. This again gives us a way of computing $\leq_{\mathbf{A}}$ by simply computing the non-increasing sequence $\mathcal{F}^0_{\mathbf{A}}(S \times S) \supseteq \mathcal{F}^1_{\mathbf{A}}(S \times S) \supseteq \mathcal{F}^2_{\mathbf{A}}(S \times S) \supseteq$

$\cdots$ until it stabilizes. To find $\leq_{\mathbf{A}}$, we can use Algorithm 1, only we have to find the longest path between states instead of the shortest.

VI. Distances

Since the notion of simulation and bisimulation is rather restrictive, we turn our attention to distances between systems. When working with weighted systems, it can often be beneficial to describe how well one system approximates another instead of only reasoning about systems that are behaviorally equivalent, since miniscule differences in weight will render two systems not equivalent. The distances are based upon the idea of two systems being the same to a certain degree, but deviating by a percentage ε. To compare behavior for a system at distance ε from another system, we introduce an ε-expansion of formulae, which is defined in the following recursive way.

Definition 10 (ε-expansion). The recursive ε-expansion of formulae is given for an arbitrary $\varepsilon \in \mathbb{Q}_{\geq 0}$ by the following

- If $\phi = x$ then $\phi^\varepsilon = x$, where x is a literal
- If $\phi = \phi_1 \wedge \phi_2$ then $\phi^\varepsilon = \phi_1^\varepsilon \wedge \phi_2^\varepsilon$
- If $\phi = E(\phi_1 U_I \phi_2)$ then $\phi^\varepsilon = E(\phi_1^\varepsilon U_{I^\varepsilon} \phi_2^\varepsilon)$
- If $\phi = A(\phi_1 U_I \phi_2)$ then $\phi^\varepsilon = A(\phi_1^\varepsilon U_{I^\varepsilon} \phi_2^\varepsilon)$,

where I^ε is defined as $I^\varepsilon = [l,u]^\varepsilon = [l(1-\varepsilon), u(1+\varepsilon)]$. ⋆

Consider an ε-relation that takes WBB as a starting point.

Definition 11. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ and an $\varepsilon \in \mathbb{R}_{\geq 0}$ define a relation $R^\varepsilon \subseteq S \times S$ such that whenever $(s,t) \in R^\varepsilon$ then

- $\mathcal{V}(s) = \mathcal{V}(t)$
- for all $s \rightarrow_w s'$ there exists $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_k} t_k$, such that $\sum_{i=1}^k v_i \in [w(1-\varepsilon), w(1+\varepsilon)]$, $(s', t_k) \in R^\varepsilon$ and for all $i < k, (s, t_i) \in R^\varepsilon$

If s and t are in this relation, we denote it by $s \approx_{\mathbf{I}}^\varepsilon t$. ⋆

We can now define a weighted branching distance.

Definition 12. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ and a relation R^ε as described in Definition 11, the *weighted branching distance* (WBD) between two states $s,t \in S$ is given by

$$d^I(s,t) = inf_\varepsilon\{(s,t) \in R^\varepsilon\}$$ ⋆

Let us restrict WCTL$_{-X}$ to only encompass the existential quantifier, which also leads to the removal of negation on formulae, and thereby get the following logic.

Definition 13 (Syntax of EWCTL$_{-X}$). Let $\mathcal{AP}$ be a set of atomic propositions. The syntax of EWCTL$_{-X}$ is given by

$$\phi ::= p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid E(\phi_1 U_I \phi_2),$$

where $p \in \mathcal{AP}$ and $I = [l,u]$, where $l, u \in \mathbb{Q}_{\geq 0}$. ⋆

The semantics of EWCTL$_{-X}$ is given by the same satisfiability relation as the semantics for WCTL$_{-X}$.
We prove that with the ε-expansion of formulae in EWCTL$_{-X}$, we can characterize the properties of WKSs up to WBD.

Theorem 8. Let $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ be a finite WKS. Then for all $s,t \in S$

$$s \approx_{\mathbf{I}}^\varepsilon t \text{ iff } \forall \varepsilon' \in \mathbb{Q}_{\geq 0}, \varepsilon \leq \varepsilon' \; [\forall \phi \in \text{EWCTL}_{-X}\, s \models \phi \Rightarrow t \models \phi^{\varepsilon'}].$$

Proof. ($\Rightarrow$) Suppose $s \approx_{\mathbf{I}}^\varepsilon t$. Induction on the structure of ϕ.
The case $\phi = E(\phi_1 U_I \phi_2)$: Suppose $s \models \phi$. By definition $s \models \phi$ iff there exists $s \rightarrow_{w_1} s_1 \rightarrow_{w_2} \cdots \rightarrow_{w_k} s_k \rightarrow \cdots$, such that $s_k \models \phi_2$, $\forall i < k, s_i \models \phi_1$ and $\sum_{i=1}^k w_i \in I$. As $s \approx_{\mathbf{I}}^\varepsilon t$, we have that for every step $s_i \rightarrow_{w_{i+1}} s_{i+1}$ there exists $t^i \rightarrow_{v_1^i} t_1^i \rightarrow_{v_2^i} \cdots \rightarrow_{v_{h_i}^i} t^{i+1}$ such that $t^{i+1} \approx_{\mathbf{I}}^\varepsilon s_{i+1}$, $\forall j < h^{i+1}, t_j^i \approx_{\mathbf{I}}^\varepsilon s_i$ and $\sum_{j=1}^{h^{i+1}} v_j^{i+1} \in [w_{i+1}(1-\varepsilon), w_{i+1}(1+\varepsilon)]$. By induction, each state $t_j^i \models \phi_1^{\varepsilon'}$, the state $t^k \models \phi_2^{\varepsilon'}$ and $\sum_{i=1}^k \sum_{j=1}^{h^i} v_j^i \in [\sum_{i=1}^k w_i(1-\varepsilon), \sum_{i=1}^k w_i(1+\varepsilon)] \subseteq [\sum_{i=1}^k w_i(1-\varepsilon'), \sum_{i=1}^k w_i(1+\varepsilon')]$ so by definition $t \models \phi^{\varepsilon'}$.
The case $\phi = A(\phi_1 U_I \phi_2)$: Suppose $s \models \phi$. If $t \not\rightarrow$ then trivially $t \models \phi^{\varepsilon'}$. Otherwise suppose $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots$. As $s \approx_{wbb} t$, we have that for every step $t_i \rightarrow_{v_{i+1}} t_{i+1}$ there exists $s^i \rightarrow_{w_1^{i+1}} s_1^i \rightarrow_{w_2^{i+1}} \cdots \rightarrow_{w_{k^i}^{i+1}} s^{i+1}$ such that $s^{i+1} \approx_{wbb} t_{i+1}$, $\forall j < k^i, s_j^i \approx_{wbb} t_i$ and $v_{i+1} \in [\sum_{j=1}^{k^i} w_j^i(1-\varepsilon), \sum_{j=1}^{k^i} w_j^i(1+\varepsilon)] \subseteq [\sum_{j=1}^{k^i} w_j^i(1-\varepsilon'), \sum_{j=1}^{k^i} w_j^i(1+\varepsilon')]$. Since $s \models A(\phi_1 U_I \phi_2)$, by definition for all $s \rightarrow_{w_1} s_1 \rightarrow_{w_2} \cdots \rightarrow_{w_k} s_k \rightarrow \cdots$ exists $s_k \models \phi_2$, such that $\forall j < k, s_j \models \phi_1$ and $\sum_{j=1}^k w_j \in I$. Therefore there exists an h such that $s_k \approx_{\mathbf{I}} t_h$ and $\forall i < h, \exists j < k, t_i \approx_{\mathbf{I}} s_j$ and $\sum_{i=1}^h v_i \in [\sum_{i=1}^k \sum_{j=1}^{k^i} w_j^i(1-\varepsilon), \sum_{i=0}^{k-1} \sum_{j=1}^{k_i} w_j^i(1+\varepsilon)]$. By induction, for all $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_h} t_h \rightarrow \cdots$, there exists t_h, such that $t_h \models \phi_2^{\varepsilon'}$, $\forall i < h, t_i \models \phi_1^{\varepsilon'}$ and $\sum_{i=1}^h v_i \in I^{\varepsilon'}$ so by definition $t \models \phi^{\varepsilon'}$.

($\Leftarrow$) Define $(s,t) \in R$ iff $[\forall \phi \in \text{EWCTL}_{-X}$ if $s \models \phi$ then $t \models \phi^\varepsilon]$. We show that R is a WBD.
Suppose $s \rightarrow_w s'$ and let $\pi_i = t \rightarrow_{v_1^i} t_1^i \rightarrow_{v_2^i} \cdots \rightarrow_{v_{k_i}^i} t_{k_i}^i$ such that $\sum_{j=1}^{k^i} v_j^i \in [w(1-\varepsilon), w(1+\varepsilon)]$ be traces from t of weight within $[w(1-\varepsilon), w(1+\varepsilon)]$. Without loss of generality, we can skip traces with zero-cycles, which means that since the WKS is finite there is a finite number of traces π_i of weight within $[w(1-\varepsilon), w(1+\varepsilon)]$, $i = 1, \ldots, n$. Assume none of these traces match $s \rightarrow_w s'$, which means that for each π_i either $(s', t_{k^i}^i) \notin R$ or there exists a $j < k_i$ such that $(s, t_j^i) \notin R$. For each π_i such that $(s', t_{k^i}^i) \notin R$, $i = 1, \ldots, k$, $k \leq n$, there exists a formula ψ_i such that $s' \models \psi_i$ and $t_{k^i}^i \not\models \psi_i^\varepsilon$ and for each π_i such that $(s, t_j^i) \notin R$, $i = k+1, \ldots, n$, there exists a formula ϕ_i such that $s \models \phi_i$ and $t_j^i \not\models \phi_i^\varepsilon$.
This means that for a decreasing series of rationals u^j such that $\lim_{j \to \infty} u^j = w$ and an increasing series of rationals y^j such that $\lim_{j \to \infty} y^j = w$, we can create a series of formulae $\phi^j = E\left(\bigwedge_{i \in [1,k]} \phi_i U_{[y^j, u^j]} \bigwedge_{i \in [k,n]} \psi_i\right)$ for which $s \models \bigwedge_j \phi^j$, but $t \not\models \bigwedge_j (\phi^j)^\varepsilon$ contradicting $(s,t) \in R$. ■

Consider the following example, illustrating the use of Theorem 8, and the rational of letting the distance be asymmetric.

Example 9. Consider Figure 1(c) and 1(d). In Example 4 we showed that s^3 and s^4 was not WBB. When looking at the distance between them, we see that $d(s^3, s^4) = 0$, as s^4 can exactly match s^3. Notice, however, that $d(s^4, s^3) = 1$. We trivially match $s^4 \rightarrow_2 s_2^4 \rightarrow_3 s_4^4$ with $s^3 \rightarrow_2 s_1^3 \rightarrow_3 s_2^3$. Transition $s^4 \rightarrow_1 s_1^4$ has to be matched with a transition of weight within $[1(1-1), 1(1+1)] = [0,2]$, which is done by matching with no transition from s^3 (cost 0). We have to show that $s_1^4 \approx_I^1 s^3$. The only transition that we cannot trivially match is $s_1^4 \rightarrow_1 s_3^4$, which has to be matched by a transition of weight within $[0, 2]$, it can be matched by $s^3 \rightarrow_2 s^3_2$. This means that whichever formulae in the logic that s^3 satisfies is also satisfied by s^4, but the formulae satisfied by s^4 are only guaranteed to be satisfied by s^3 in their ε-extensions. For instance s^4 satisfies $E(pU_{[0,1]}E(pU_{[0,1]}q))$, but s^3 does not. It does however satisfy the ε-extension of the formula $E(pU_{[0,2]}E(pU_{[0,2]}q))$. $\diamond$

Consider EBS as our starting point to define an ε-relation.

Definition 14. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ and an $\varepsilon \in \mathbb{R}_{\geq 0}$ define a relation $R^\varepsilon \subseteq S \times S$ such that whenever $(s, t) \in R^\varepsilon$ then

- $\mathcal{V}(s) = \mathcal{V}(t)$
- for all $s \rightarrow_w s'$ there exists $t \rightarrow_{v_1} t_1 \rightarrow_{v_2} \cdots \rightarrow_{v_k} t_k$, such that $\sum_{i=1}^{k} v_i \leq w(1 + \varepsilon)$, $(s', t_k) \in R^\varepsilon$ and for all $i < k$, $(s, t_i) \in R^\varepsilon$

If s and t are in this relation, we denote it by $s \leq^\varepsilon_{\mathbf{E}} t$. $\star$

We can now define an existential bounded distance.

Definition 15. Given a WKS $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ and a relation R^ε as described in Definition 14, the *existential bounded distance* (EBD) between states $s, t \in S$ is given by

$$d^{\leq}(s, t) = inf_\varepsilon\{(s, t) \in R^\varepsilon\} \qquad \star$$

Let us return to the logic $\text{EWCTL}^{\leq}_{-X}$ and prove that with ε-expansions the logic characterizes EBD. The proof of the theorem has the same structure as the proof of Theorem 8.

Theorem 9. Let $\mathcal{K} = (S, \mathcal{AP}, \mathcal{V}, \rightarrow)$ be a finite WKS. Then for all $s, t \in S$

$$s \leq^\varepsilon_{\mathbf{E}} t \text{ iff } \forall \varepsilon' \in \mathbb{Q}_{\geq 0}, \varepsilon \leq \varepsilon' \quad [\forall \phi \in \text{EWCTL}^{\leq}_{-X} s \models \phi \Rightarrow t \models \phi^{\varepsilon'}].$$

It should be noted that neither WBD nor EBD are (hemi)metrics as is usually the case, which stems from the relativism in the definition of the distances. Future work should include a classification of these.

VII. Conclusion and Future Work

We have extended the idea of branching bisimulation with weights in three distinct ways, relating to different fragments of WCTL. We initially removed the next operator from the logic, to allow for systems to be related even though they performed specific behavior with different number of transitions. The weighted branching bisimulation relation that was characterized by this logic, turned out to be NP-complete, which prompted us to look into other fragments of the logic. We proved that for fragments allowing only upper bounds and either the existential or the universal quantifier we could decide the resulting simulation relations in polynomial time.
We furthermore expanded these concepts into distance-like relations. The distances build upon the ideas of the different relations and were also characterized by fragments of WCTL, when we introduced a relative expansion on formulae. Even if the distances are not (hemi)metrics, they can however be meaningfully interpreted as relative distances.

Notably, this work demonstrates that the real-valued weights in the models can be described by only involving rational parameters in the logics. The approximation of reals by rationals are enough to describe this more general behavior.

This research opens a few promising future work directions. On one hand, designing a simulation relation which characterizes the same relations as the logic with the universal quantifier is problematic and whether such a relation can be defined at all is an open problem. On the other hand, the distance-like relations inspired by our semantics fail to satisfy the triangle inequality. The characterization of such relations is a promising research direction. Furthermore, computability and complexity results related to these distances are open.

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