Compositions of Fuzzy Weights for Double Inner Dependence AHP

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Abstract - **Analytic Hierarchy Process (AHP) is major method for decision making now and decision maker's answers are strongly related to his or her cognitive abilities. Inner dependence AHP is used for cases in which criteria or alternatives are not independent enough. In the process, calculations and compositions of weights are very important steps. Using the original AHP or inner dependence AHP may cause results losing reliability because the comparison matrix is not necessarily sufficiently consistent. In such cases, fuzzy representation for weighting criteria or alternatives using results from sensitivity analysis is useful. In the previous papers, we defined local weights of criteria and alternatives for inner dependence AHP via fuzzy sets. In this paper, we deal with overall weights of alternatives for double inner dependence structure AHP (among criteria and among alternatives respectively). Results show the fuzziness of double inner dependence structure AHP in different way of composition of weights.**

Keywords - decision making; AHP; fuzzy sets; sensitivity analysis.

I. INTRODUCTION

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1977 [1][2] is widely used in decision making, because it reflects humans feelings "naturally". A normal AHP assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. Inner dependence method AHP [3] is used to solve this problem even for criteria or alternatives having dependence.

A comparison matrix may not, however, have enough consistency when AHP or inner dependence is used because, for instance, a problem may contain too many criteria or alternatives for decision making, meaning that answers from decision-makers, i.e., comparison matrix components, do not have enough reliability, they are too ambiguous or too fuzzy [4]. To avoid this problem, we usually have to revise again or abandon the data, but it takes a lot of time and costs[2][3].

Then, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights using fuzzy sets.

Our research first applied sensitivity analysis [5] to inner dependence AHP to analyze how much the components of a pairwise comparison matrix influence the weights and consistency of a matrix [6]. This may enable us to show the magnitude of fuzziness in weights. We previously proposed new representation for criteria and alternatives weights in AHP, also representation for criteria weights for inner dependence, as L-R fuzzy numbers [7]. In the next step, we stated deal with double inner dependence structure [8]. In this paper, we consider composition of weights to obtain over all alternative weights for double inner dependence structure AHP, using results from sensitivity analysis and fuzzy operations. We then consider fuzziness as a result of double inner dependence AHP when a comparison matrix among alternatives does not have enough consistency.

In Section II, we introduce AHP and its inner dependence method. The sensitivity analyses for AHP are described in Section III. Then the fuzzy weight representation is defined in Section IV, and Section V is a example and conclusions.

II. INNER DEPENDENCE AHP AND CONSISTENCY

In this section, we introduce process of AHP, consistency of data (comparison matrix) and inner dependence method.

A. Process of Normal AHP

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. At the middle levels, there are multiple criteria. Alternative elements are put at the lowest level of the hierarchy.

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level, the upper triangular components of the matrix a_{ij} ($i < j = 1,…,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from element i to j. The lower triangular components a_{ji} are described with reciprocal numbers, for diagonal elements, let $a_{ii} = 1$.

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grade of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in this paper.

(Process 4) Priority of an alternative by a composition of weights. With repetition of composition of weights, the overall weights of the alternative, which are the priorities of

the alternatives with respect to the overall objective, are finally found.

B. Consistency

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n - 1},\tag{1}$$

where n is the order of comparison matrix A, and λ_A is its maximum eigenvalue (Frobenius root).

If the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. The omparison matrix is consistent if the following inequality holds.

$$\text{C.I.} \le 0.1\tag{2}$$

C. Inner Dependence Structure

The normal AHP ordinarily assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. Inner dependence AHP [3] is used to solve this type of problem even for criteria or alternatives having dependence.

In the method, using a dependency matrix $F = \{ f_{ij} \}$, we can calculate modified weights $w^{(n)}$ as follows,

$$w^{(n)} = Fw\tag{3}$$

where w is weights from independent criteria or alternatives, i.e., normal weights of normal AHP and dependency matrix F is consist of eigenvectors of influence matrices showing dependency among criteria or alternatives.

If there is dependence both lower levels, i.e., not only among criteria but also among alternatives, we call such kind of structure "double inner dependence". In the double inner dependence structure, we have to calculate modified weights of criteria and alternatives, $w^{(n)}$ and $u_i^{(n)}$. Then we composite these 2 modified weights to obtain overall weights of alternative k, $v_k^{(n)}$ as follow:

$$v_k^{(n)} = \sum_i^m w_i^{(n)} u_{ik}^{(n)}\tag{4}$$

where m is number of criteria.

III. SENSITIVITY ANALYSES

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence on its consistency or on the weights. To analyse how results are influenced when a certain variable has changed, we can use sensitivity analysis.

In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change a structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds. From Perron-Frobenius theorem, following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let $A = (a_{ij})$, $(i, j = 1, ..., n)$ denote a comparison matrix and let $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij} d_{ij})$ denote a matrix that has been perturbed. Let λ_A be the Frobenius root of A, w be the eigenvector corresponding to λ_A, and v be the eigenvector corresponding to the Frobenius root of A'. Then, a Frobenius root $\lambda(\varepsilon)$ of $A(\varepsilon)$ and a corresponding eigenvector $w(\varepsilon)$ can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon),\tag{5}$$

$$w(\varepsilon) = w + \varepsilon w^{(1)} + o(\varepsilon),\tag{6}$$

where

$$\lambda^{(1)} = \frac{v' D_A w}{v' w},\tag{7}$$

$w^{(1)}$ *is an n-dimension vector that satisfies*

$$(A - \lambda_A I) w^{(1)} = -(D_A - \lambda^{(1)} I) w,\tag{8}$$

where $o(\varepsilon)$ denotes an n-dimension vector in which all components are $o(\varepsilon)$.

A. Analysis for Consistency of Pairwise Comparison

About a fluctuation of the consistency index, following corollary can be obtained from Theorem 1.

Corollary 1 *Using appropriate g_{ij}, we can represent the consistency index C.I.(ε) of the perturbed comparison matrix $A(\varepsilon)$ as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon).\tag{9}$$

To see g_{ij} in the equation (9) in Corollary 1, how the components of a comparison matrix impart influence on its consistency can be found.

B. Analysis for Weights of AHP

About the fluctuation of the weights, following corollary also can be obtained from Theorem 1.

Corollary 2 *Using appropriate $h_{ij}^{(k)}$, we can represent the fluctuation $w^{(1)}=(w_k^{(1)})$ of the weight (i.e., the eigenvector corresponding to the Frobenius root) as follows*

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \qquad (10)$$

From the equation (6) in Theorem 1, the component that has a great influence on weight $w(\varepsilon)$ is the component which has the greatest influence on $w^{(1)}$. Accordingly, from Corollary 2, how components of a comparison matrix impart influence on the weights, can be found, to see $h_{ij}^{(k)}$ in the equation (10).

Calculations or proofs of these theorem and corollaries are shown in [7].

IV. FUZZY WEIGHTS REPRESENTATIONS

A comparison matrix often has poor consistency (i.e., 0.1<C.I.<0.2) because it encompasses several criteria or elements. In these cases, comparison matrix components are considered to be fuzzy because they are results from human fuzzy judgment. Weights should therefore be treated as fuzzy numbers.

A. L-R Fuzzy Number

L-R fuzzy number

$$M = (m, \alpha, \beta)_{LR} \qquad (11)$$

is defined as fuzzy sets whose membership function is as follows.

$$\mu_M(x) = \begin{cases} R\left(\dfrac{x-m}{\beta}\right) & (x > m), \\ L\left(\dfrac{m-x}{\alpha}\right) & (x \le m). \end{cases}$$

where $L(x)$ and $R(x)$ are shape function .

B. Fuzzy Weights of Criteria or Alternatives of Normal AHP

From the fluctuation of the consistency index, the multiple coefficient $g_{ij}h_{ij}^{(k)}$ in Corollary 1 and 2 is considered as the influence on a_{ij}.

Since g_{ij} is always positive, if the coefficient $h_{ij}^{(k)}$ is positive, the real weight of criterion or alternative k is considered to be larger than w_k. Conversely, if $h_{ij}^{(k)}$ is negative, the real weight of criterion or alternative k is considered to be smaller. Therefore, the sign of $h_{ij}^{(k)}$

represents the direction of the fuzzy number spread. The absolute value $g_{ij} \mid h_{ij}^{(k)}\mid$ represents the size of the influence.

On the other hand, if C.I. becomes bigger, then the judgment becomes fuzzier.

Consequently, multiple C.I. $g_{ij}\mid h_{ij}^{(k)}\mid$ can be regarded as a spread of a fuzzy weight concerned with a_{ij}.

Definition 1 (fuzzy weight) Let $w_k^{(n)}$ be a crisp weight of criterion or alternative k of inner dependence model, and $g_{ij}\mid h_{ij}^{(k)}\mid$ denote the coefficients found in Corollary 1 and 2. If $0.1<C.I.<0.2$, then a fuzzy weight $\tilde{w}_k$ is defined by

$$\tilde{w}_k = (w_k, \alpha_k, \beta_k)_{LR} \qquad (12)$$

where

$$\alpha_k = C.I. \sum_i^n \sum_j^n s(-, h_{kij}) g_{ij} \mid h_{kij} \mid, \qquad (13)$$

$$\beta_k = C.I. \sum_i^n \sum_j^n s(+, h_{kij}) g_{ij} \mid h_{kij} \mid, \qquad (14)$$

$$s(+,h) = \begin{cases} 1, (h \ge 0) \\ 0. (h < 0) \end{cases}, \quad s(-,h) = \begin{cases} 1, (h < 0) \\ 0. (h \ge 0) \end{cases}$$

C. Fuzzy Weights for double inner dependence AHP

For double inner dependence structure, we can define and calculate modified fuzzy local weights of a criteria $\tilde{w}^{(n)} = (\tilde{w}_i^{(n)})$, $i = 1, ..., n$ and also weights of alternatives $\tilde{u}_i^{(n)} = (\tilde{u}_{ik}^{(n)})$, $k = 1, ..., m$ with only respect to criterion i using an dependence matrix F_C, F_A, as follows

$$\tilde{w}_i^{(n)} = (w_i^{(n)}, \alpha_i^{(n)}, \beta_i^{(n)})_{LR} \qquad (15)$$

$$\tilde{u}_{ik}^{(n)} = (u_{ik}^{(n)}, \alpha_{ik}^{(n)}, \beta_{ik}^{(n)})_{LR} \qquad (16)$$

where

$$w^{(n)} = (w_i^{(n)}) = F_C w \qquad (17)$$

$$u_i^{(n)} = (u_{ik}^{(n)}) = F_A u_i \qquad (18)$$

w is crisp weights of criteria and u_i is crisp local alternative weights with only respect to criterion i. $\alpha_i, \beta_i, \alpha_{ik}, \beta_{ik}$ are calculated by fuzzy multiple operations, equation(3) and definition 1.

Fuzzy overall weights of alternative k in double inner dependence AHP can be also calculated as follows using fuzzy multiple $\otimes$ and fuzzy summation operations:

$$\tilde{v}_k^{(n)} = \sum_i^m \tilde{w}_i^{(n)} \otimes \tilde{u}_{ik}^{(n)} \qquad (19)$$

Fuzzy weights $\tilde{w}_i^{(n)}$ becomes crisp weights $w_i^{(n)}$ if there is good consistency among criteria. Therefore

$$\tilde{v}_k^{(n)} = \sum_i^m w_i^{(n)} \otimes \tilde{u}_{ik}^{(n)} \qquad (20)$$

In any cases we can evaluate fuzzy overall weights of alternatives with their centers and spreads.

V. EXAMPLE AND CONCLUSIONS

In this section, we introduce an example of the leasure. Criteria are {congestion, good for rain (rain), trouble, cost} and alternatives are {Theme park (park), Indoor theme park (indoor), Movie, Zoo}.

Table I shows a comparison matrix of criteria and weights, where its consistency is not so good (C.I. >0.1). Then using results of sensitivity analyses of consistency and weights, we can calculate fuzzy weights. Next using a dependency matrix, modified fuzzy weights are obtained as shown in Table II. There is bad dependency and dependency between alternatives with only respect to criterion "congestion", we also calculate fuzzy modified weights of alternatives shown in Table III. Finally, we evaluate overall fuzzy weights of alternatives as in Table IV.

On the other hand, TableV shows a comparison matrix and weights of criteria with enough consistency.

TABLE I.　COMPARISON OF CRITERIA

	congestion	rain	trouble	cost	Weights
congestion	1	1/3	5	1	0.206
rain		1	2	1/2	0.165
trouble			1	1/3	0.499
cost				1	0.129

C.I.=0.169

TABLE II.　FUZZY MODIFIED WEIGHTS OF CRITERIA

	Center	Left	Right
congestion	0.105	0.0170	0.0051
rain	0.202	0.0080	0.0098
trouble	0.523	0.0080	0.0056
cost	0.162	0.0140	0.0126

TABLE III.　FUZZY MODIFIED WEIGHTS OF ALTERNATIVES WITH ONLY RESPECT TO "CONGESTIONS"

	Center	Left	Right
Park	0.257	0.0079	0.0094
Indoor	0.087	0.0029	0.0031
Movie	0.304	0.0113	0.0110
Zoo	0.351	0.1210	0.0097

TABLE IV.　FUZZY OVERALL MODIFIED WEIGHTS OF ALTERNATIVES

	Center	Left	Right
Park	0.182	0.0091	0.0091
Indoor	0.277	0.0089	0.0089
Movie	0.405	0.0094	0.0094
Zoo	0.129	0.0050	0.0031

TABLE V.　COMPARISON OF CRITERIA

	congestion	rain	trouble	cost	Weights
congestion	1	2	5	3	0.491
rain		1	2	3	0.266
trouble			1	2	0.140
cost				1	0.103

C.I.=0.044

TABLE VI.　FUZZY OVERALL MODIFIED WEIGHTS OF ALTERNATIVES WI

	Center	Left	Right
Park	0.177	0.0012	0.0014
Indoor	0.296	0.0004	0.0005
Movie	0.378	0.0017	0.0017
Zoo	0.150	0.0018	0.0015

In this case, overall fuzzy weights are shown in Table VI. (lower weights are same as fuzzy criteria weight case).

There are a lot of cases that data of AHP do not have enough consistent or reliable. We propose these 2 compositions of weights, and our approach can show how to represent weights and will be efficient to investigate how the result of AHP has fuzziness when data is not sufficiently consistent or reliable.

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