Optimal Selection of Sampling Rate in Multiple H_2 Control Loops

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***Abstract*—In this paper, the scheduling of the sampling frequencies of a set of independent controllers that share a limited resource is addressed. Bandwidth or CPU time limitations are assumed to be translated to constraints on the sum of the sampling frequencies. For the individual loops, H_2 sampled-data controllers are proposed, whose performance indexes can be calculated for the different sampling frequencies. A weighted sum of the individual loop performance conforms a global cost index. The problem is then posed as an optimization one, and some sensible simplifying alternatives are proposed, based on a grid of frequency points, that allow to solve it with Linear Programming (and hence with a low computing cost).**

***Keywords*-network control; optimal sampling frequency; network resource sharing**

I. Introduction

In digitally controlled systems, limitations on the frequency of the control computations are frequent. They may arise from multiple tasks running on the same processor so that a higher frequency of the control tasks would saturate the CPU load; they may also arise when a communication network between process and controllers is present and it has a limited bandwidth to be shared between multiple controllers, Programmable Logic Controllers and other information-processing elements. Apart from bandwidth limitations, increasing the network or computer load also gives rise to increased delays and sampling jitter, which might as well result in a performance loss in the tasks requiring the limited resource.

In most control literature, criteria for selection of sampling time do not usually consider the underlying resource limitation. Basically, the desired settling time, performance attenuation level, etc. result in a recommended sampling period, in most cases with practical "rules of thumb" (see [1] or [2]); it is left to the underlying real-time scheduler to achieve such a period with a reduced jitter, by dedicating to the task whichever computing/network resources are necessary.

The so-called co-design research line [3] tries to consider the design of both the control system and the communication and multi-tasking structures as a joint problem. Basically, the idea is combining restrictions on the sampling period arising from schedulability issues and bandwidth limitation (computation and transmission cost) and sampling-period dependent controller performance measures in order to solve a joint optimization whose results are the scheduler sampling periods and the controllers to be applied.

For instance Branicky et al. [3] proposed an optimality-based choice of sampling period for a multiple-loop control over a network based on a performance measure for each loop and some schedulability constraints. In [4] the objective was stability robustness, although first-order systems were only considered. Integral of Absolute Error as a function of sampling period was considered in [5]. Interestingly, Cervin et al. [6] discuss a generic approach in which the sampled-data cost of a controller is evaluated.

This paper roots on the last of the above cited works, formalizing the ideas to sampled-data output-feedback $\mathcal{H}_2$ control, and proposing a linear programming approximation on a finite grid of sampling frequency points.

The structure of the paper is as follows. Next section discusses some preliminary ideas and states the problem to be solved. Section III reviews sampled-data $\mathcal{H}_2$ control. Off-line (fixed rate) scheduling is discussed on Section IV. An example section and some conclusions are also provided.

II. Preliminaries and problem statement

Consider a network that is shared by several control loops (controllers, sensors and actuators).

The network resources used by each control loop and the achieved performance depend on the sampling frequency of that loop and the controller designed for it. Hence, each loop will be characterized by:

- Its sampling frequency f_i
- A controller scheduling policy $C(f_i)$
- A theoretical performance measure with that controller $J_i(f_i)$

Therefore, due to the overall bandwidth limitations, if the performance of one loop needs to be improved by increasing its sampling frequency, other loops must reduce their frequency and, hence, their performance. The problem to be discussed in this paper is how to apportion the limited resource between loops while trying to maximize the overall performance by minimizing a global cost index (composed

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of the weighted sum of individual indices), such as:

$$J(f_1,\ldots,f_r)=\sum_{i=1}^{r} h_i J_i(f_i)^p \quad (1)$$

where h_i are weights allowing the designer to emphasize the need of more accurate control of some processes. As commented in the introduction, this idea has been previously explored in literature. The result is an optimal sampling frequency distribution given the network constraints.

This paper has chosen the $\mathcal{H}_2$ performance measure as cost index (minimum-variance controller when subject to white-noise inputs). Indeed, if $\mathcal{H}_2$ sampled-data optimal controllers are used in the loops, the optimal performance of each loop can be calculated as a function of its sampling period via well-known sampled-data $\mathcal{H}_2$ formulae.

III. Sampled-data optimal control

The main issues in $\mathcal{H}_2$ sampled-data optimal control are reviewed next.

Consider a linear time-invariant continuous-time process given by:

$$\dot{x}=A_1^c x+B^c u+G_1^c v \quad (2)$$

$$\dot{\psi}=A_2^c\psi+G_2^c w \quad (3)$$

$$z=Cx+Du \quad (4)$$

$$y=C_2x+C_3\psi+D_2u \quad (5)$$

so that it has a transfer function representation given by:

$$z=G_{11}(s)v+G_{13}(s)u \quad (6)$$

$$y=G_{21}(s)v+G_{22}(s)w+G_{23}(s)u \quad (7)$$

where z denotes the variables to be controlled, u are the manipulated inputs, y are the measurements and v, w are assumed to be white-noise disturbances to be denoted as process noise and measurement noise, respectively, with unit variance (all variance information is included in matrices G_1^c ad G_2^c). The state variables x are denoted as process state, whereas the state variables ψ are states of the measurement noise generator subsystem, assumed to evolve uninfluenced by x.

Given a sampling period T, a sampled-data controller will be designed so that its input will be the sequence of sampled outputs y_k and its output will be a sequence of control actions u_k to be fed to the continuous-time plant via a zero-order hold.

The control objective is to obtain the controller that minimizes the variance of z, $tr(E(zz^T))$ for a given, fixed, sampling period T. Such a problem is denoted in literature as the $\mathcal{H}_2$ sampled-data optimal control problem. It was shown in [7], [8] that such a problem can be cast as a pure discrete-time $\mathcal{H}_2$ optimal control problem for the discretized model given by:

$$x_{k+1}=A_1x_k+Bu_k+G_1v_k \quad (8)$$

$$\psi_{k+1}=A_2\psi_k+G_2w_k \quad (9)$$

$$z_k=C_1x_k+D_1u_k \quad (10)$$

$$y_k=C_2x_k+C_3\psi_k+D_2u_k \quad (11)$$

where the above discrete-time matrices are given by:

$$A_1=e^{A_1^cT}, A_2=e^{A_2^cT} \quad (12)$$

$$B=\int_0^T e^{As}Bds \quad (13)$$

and G_1, G_2, C_1 and D_1 are any matrices satisfying:

$$G_iG_i^T = \int_0^T e^{A_i^cs}G_i^c(G_i^c)^Te^{(A_i^c)^Ts}ds \quad (14)$$

$$(C_1D_1)^T(C_1D_1) = \int_0^T e^{\underline{\mathbf{A}}^Ts}(CD)^T(CD)e^{\underline{\mathbf{A}}s}ds \quad (15)$$

where:

$$\underline{\mathbf{A}}=\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix} \quad (16)$$

and all of the above matrices can be obtained from matrix exponential formulae without the need of actually carrying out any integration [9].

The obtained variance approaches that of the continuous-time $\mathcal{H}_2$ controller when the sampling period tends to zero [10]. Indeed, as a piecewise-constant control is a valid possibility for the optimal control action $u(t)$, the continuous-time solution of the above minimum-variance problem will be equal or better than any sampled-data optimal solution. The sampled-data system will have a closed-loop $\mathcal{H}_2$ norm given by that of the above discrete system plus a factor given by [10]:

$$\frac{1}{T}\int_0^T\int_0^{T-s} trace(C_2e^{A_i^c\tau}G_i^c(G_i^c)^Te^{(A_i^c)^T\tau}C_2d\tau ds \quad (17)$$

It is well known that the optimal controllers have the form of a Kalman filter observer plus a static state feedback, and that optimal control weights in classical linear quadratic regulator setups can be translated to the above $\mathcal{H}_2$ problems by a suitable choice of C and D.

If the measurement-noise dynamics G_{22} is sufficiently fast, the measurement noise will appear as a constant-variance stationary process when sampled at all except very small sampling periods; in that case, the dynamics of the states ψ can be eliminated in practice and $C_3\psi$ replaced by a discrete stochastic process with a constant variance equal to the stationary variance of the continuous one. This yields the classical "measurement noise variance" in discrete stochastic process models.

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IV. Resource scheduling

Basically, the computing and network resources required by a control task will be proportional to the sampling frequency. Hence, the objective is achieving maximum performance (in terms to be later detailed) given a finite "bandwidth" bound, β, set up from computer and network load analysis.

The objective of this paper is proposing an scheduling methodology that allows an efficient use of the assigned control bandwidth by devoting more resources to processes that need a better control and, hence, must be operated at a higher sampling rate. The ideas in the previous section allow to easily obtain the optimal performance as a function of the sampling rate of a particular process. Considering now r independent control loops, which should share a computer or a network, we will denote as $\gamma_i(f)$ the optimal disturbance-rejection $\mathcal{H}_2$ performance obtained for process i by a controller operating at frequency f (i.e., with sampling period $T = 1/f$).

Taking into account all loops simultaneously, an overall performance measure may be defined as:

$$J(f_1, \dots, f_r) = \sum_{i=1}^{r} h_i \gamma_i(f_i)^p \qquad (18)$$

where h_i are weights allowing the designer to emphasize the need of more accurate control of some processes. The selection of these weights should depend on the disturbances acting on each loop, and on the economic cost derived from the resulting loop error. The higher the disturbance and the cost, the higher the weight.

The needed resources as a function of controller frequency may be expressed as:

$$R(f_1, \dots, f_r) = \sum_{i=1}^{r} d_i f_i \qquad (19)$$

for some given constants d_i to be determined based on processor load and number of bits transmitted by each control task (transmission time plus computation time). On the sequel, the vector of frequencies for each control loop will be denoted as F.

The goal of the bandwidth scheduling will be to obtain the sampling frequencies f_i for each of the control loops taking into account the performance and resource measures defined above.

Then, some scheduling problems of interest may be conceived:

- Given a resource constraint
$$R(F) \leq \beta \qquad (20)$$
obtain the lowest $J(F)$.
- Given a performance objective J_0, obtain the lowest level of resources needed to achieve it: minimize $R(F)$ constrained to $J(F) < J_0$.
- variations of the above problems including some performance requirements for individual loops $\gamma_i(f_i) \leq J_{0,i}$ or multi-criteria settings (obtaining, for instance, a Pareto front on performance vs. available bandwidth).

A. Alternatives for the optimization problems

Depending on the shape of γ_i, the allowed values for the decision variables F, the value of the exponent parameter p in (18) and the chosen problem formulation from the options above, the required optimization algorithm will be different.

Several interesting options are discussed below:

1) Discrete optimization over a finite set of alternatives: Set up two or three performance levels for each process, say: high-frequency, normal-frequency, low-frequency sampling. Then, the problem gets transformed to a choice between a finite set of decision variables and it can be explored by brute force if the number of controlled loops is small.

2) Linear (approximate) optimization: Set up a dense enough grid of points f_j^*. Then, for each individual $\gamma_i(\cdot)^p$ function, compute the linear interpolation between the available frequency points, giving rise to a piecewise-linear $\gamma_i^*(\cdot)$ interpolation function. Subsequently, determine an interval of interest $[f_i^-, f_i^+]$ where individual performances $\gamma_i^*(\cdot)$ are convex functions. Doing this for all the controlled loops, the controller cost will then be the sum of univariate convex functions and, hence, a convex piecewise-affine function.

It is well known that piecewise-affine functions can be optimized via Linear Programming. Let us describe the basic idea below:

Denote as f_k^*, $k = 0, \dots, \bar{k}$ the grid points in the above interval $[f_i^- = f_0^*, f_1^*, \dots, f_i^+ = f_{\bar{k}}^*]$.

In that interval, $\gamma_i^*(f)$ may be approximately expressed as the piecewise-linear interpolation between grid points, to be denoted as $\bar{\gamma}_i^*(f) \approx \gamma_i^*(f)$. Conveniently, such linear interpolation can be rewritten as a linear-programming optimization:

$$\bar{\gamma}_i^*(f) = \gamma_i(f_i^-) + \min_{\epsilon_k} \sum_{k=0}^{\bar{k}} \alpha_k \epsilon_k \qquad (21)$$

subject to the linear constraints

$$\epsilon_k \leq f_k^* - f_{k-1}^*, \quad \sum_{k=0}^{\bar{k}} \epsilon_k = f - f_i^- \qquad (22)$$

and where α_k are the piecewise slopes, defined as

$$\alpha_k = \frac{\gamma_i(f_{k+1}^*) - \gamma_i(f_k^*)}{f_{k+1}^* - f_k^*}$$

that fulfils the condition $\alpha_{k+1} \geq \alpha_k$ due to the assumed convexity of $\gamma_i^*(\cdot)$ in the given interval.

Carrying out a similar derivation for each of the loop performances (choosing a gridding with $\bar{k}_i$ intervals for each

i), the overall cost can be expressed as:

$$J(F) = \min_{\epsilon_{i,k}} \sum_{i=1}^{r} h_i \left(\gamma_i(f_i^-) + \sum_{k=0}^{\bar{k}_i} \alpha_{i,k}\epsilon_{i,k} \right) \quad (23)$$

subject to $\epsilon_{i,k} \leq f_{i,k}^* - f_{i,k-1}^*$, $\sum_{k=0}^{\bar{k}_i} \epsilon_{i,k} = f_i - f_i^-$. Hence, the optimization problem to be solved consists of the above problem constrained to the additional condition $\sum d_i f_i \leq \beta$. Such problem is a linear programming one that can be efficiently solved.

Remark: An interesting particular case results when the weights $d_i = 1$, and the grid points f_j^* are uniformly spaced so the distance between the grid points is an exact divisor of the bound β. In that case, as LP optimal solutions always lie at slope changes or constraint bounds, the result of the LP optimization will always lie at one of the grid points, i.e., the optimum frequencies always belong to a predefined set. This would be especially useful for an on-line scheduling, where the switching between a finite set of controllers could be a simple solution. This issue will be studied in future works.

3) Generic nonlinear optimization.: If a non-linear (polynomial, spline, etc.) interpolation were chosen to approximate $\gamma_i(\cdot)$, the scheduling problem would be a nonlinear optimization problem. That would also be the case if the intervals $[f_i^-, f_i^+]$ were too small for the particular application. If the number of simultaneous loops were small, a subdivision of the interpolation table in a finite number piecewise convex (or concave) regions would allow for solving a linear programming problem for each of such regions and computing the global minimum as the minimum of the local optimizers (details omitted for brevity).

V. EXAMPLES

Considerer the simple case of controlling two identical SISO systems whose disturbance inputs might, however, be not identical. Under limited resources, the effort should concentrate on the process subject to larger disturbances, which must be known *a priori* and cast into the optimization index in the off-line scheduling case.

Each of the systems can be represented by the state space model (24) where x are the state variables, y is the output, v are the white noise variables and z represents the weighted controlled variables.

$$\begin{aligned} \dot{x} &= Ax + Bu + Gv \\ y &= Cx \\ z &= C_z x + D_z u \end{aligned} \quad (24)$$

where

$$A = \begin{pmatrix} -20 & -12.5 \\ 8 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 16 \\ 0 \end{pmatrix} \quad (25)$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

$$C = \begin{pmatrix} 0 & 15.625 \end{pmatrix}, \ C_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ D_z = \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix} \quad (27)$$

In order to select the discrete dynamic controllers for both systems, we obtain the sampled models at different frequencies f following the procedure commented in Section II. Once converted we obtain for each frequency the $\mathcal{H}_2$ controller and the minimum $\mathcal{H}_2$-gain bound for disturbance rejection. These values are presented in Figure 1. As it can be seen at low frequencies the closed-loop $\mathcal{H}_2$ norm approaches the open loop value (5.867), as it was expected, and at high frequencies $f \simeq 1KHz$ it approaches the norm of the continuous-time $\mathcal{H}_2$ optimal controller, whose value is 1.576.

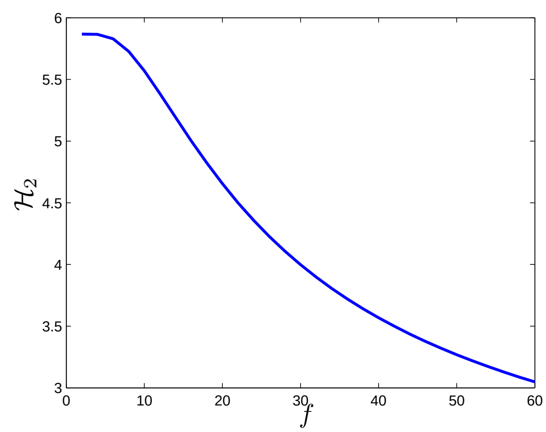

Figure 1. Minimum $\mathcal{H}_2$ rejection at frequency f

The selection of the frequencies for each controller (f_1 and f_2) will be done taking into account the resource constraint (28) and the performance index (29).

$$f_1 + f_2 \leq K \quad (28)$$

$$J = h_1\gamma_1^2(f_1) + h_2\gamma_2^2(f_2) \quad (29)$$

where the chosen values have been $h_1 = 4$, $h_2 = 1$ indicating that the expected value of the disturbance inputs for process 1 is double than that for process 2. The frequency limit has been set to $K = 60$.

As it can be seen in Figure 1, the cost function J will not be convex at low frequency values, but it will be convex for frequencies larger or equal than 10 Hz. However, the part of the plot on the left of the inflection point corresponds to almost open-loop behaviour for very low sampling rates, which are not relevant for the tested cases. They would only be relevant for a setup in which one of the disturbances is extremely larger than the other one, almost requiring controlling one of the processes at the maximum frequency

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and the other one in open-loop. This has not been the case for the simulations in this paper's examples.

In order to solve a convex problem we approximate the cost function J following the methodology in Section IV-A2 at the convex range of J as

$$\gamma_1^{*2} = \gamma_1^2(f_1^-) + \min_{\epsilon_{1,k}} \sum_k \alpha_k \epsilon_{1,k} \tag{30}$$

$$\gamma_2^{*2} = \gamma_1^2(f_2^-) + \min_{\epsilon_{2,k}} \sum_k \beta_k \epsilon_{2,k} \tag{31}$$

Note that, in this case $\alpha_k = \beta_k$ because the systems have the same dynamic model. The candidate sampling frequencies vectors F_1 and F_2 are taken also identical for both systems, uniformly distributed from 12 to $60Hz$, computing approximation points every 2 Hz (i.e., $f_1^- = f_2^- = 12$, $f_1^+ = f_2^+ = 60$, $f_{i,k}^* = 12 + 2k$). Then the optimization problem can be approached by the linear programming problem procedure presented in the referred section.

As a result, we obtain the optimal $\mathcal{H}_2$ norm bound at frequencies $f_1 = 42$ and $f_2 = 18$. The controllers' state space gains (K_i) and Kalman filter gains (L_i) that minimize J^* are presented in (32) and (33) below, respectively.

$$K_1 = (-4.228 \ \ -25.16), \quad L_1 = (-0.014 \ 0.0622)^T \tag{32}$$

with the sampling time $T_1 = 1/42$.

$$K_2 = (-1.839 \ \ -6.4171), \quad L_2 = (-0.0275 \ 0.0548)^T \tag{33}$$

with the sampling time $T_2 = 1/18$.

A contour plot of the cost function $J(f_1, f_2)$ is represented at Figure 2 with its constraints.

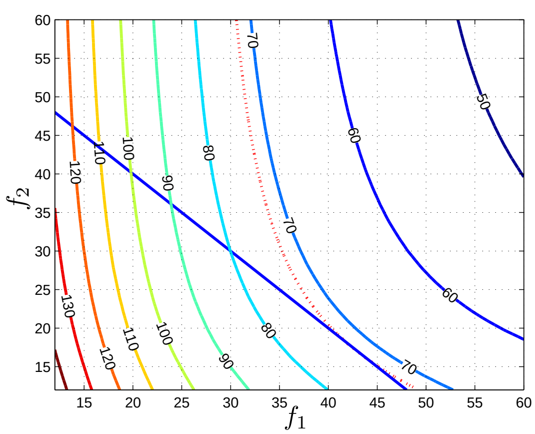

Figure 2. Contour plot of the cost function J and the constraint $f_1 + f_2 \leq 60$.

VI. Conclusion and Future Work

This paper has presented an optimal scheduling of a set of independent $\mathcal{H}_2$ sampled-data controllers operating on a shared resource, which gives rise to sampling frequency constraints. The problem has been posed as an optimization one and some sensible simplifying alternatives have been proposed, based on a grid of frequency points, that allow to solve it with Linear Programming, and hence, with a low computing cost. Some examples have illustrated the approach. As a future work, the online scheduling of the sampling rates will be studied. The idea will be to adapt the scheduling to changes in the available resources or in the process disturbances. The simplified Linear Programming based optimization presented in this paper will be a key point in that work.

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