

# Simulation of the Deflection of Thin Plates under the Action of Random Loads

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**Abstract**—In this work, new results are obtained using constructed probabilistic representation of the first boundary value problem for polyharmonic equation. It is shown that corresponding solution is presented by the parametric derivative of a solution to the specially constructed Dirichlet problem for Helmholtz equation. On this base, new algorithms of 'random walk by spheres' for solving biharmonic equations are derived.

**Index Terms**—plate bending; Monte Carlo methods; biharmonic equations; 'random walks by spheres'

## I. INTRODUCTION

The deflection of thin plates under the action of loads is satisfied an biharmonic equation with Dirichlet, Neumann, or mixed boundary conditions [1]. Despite the slow rate of convergence of statistical methods for low-dimensional spaces, in comparison with classical numerical methods, their use is advantageous in finding a solution to a small area or for calculating the statistical characteristics of the solutions with random right-hand sides. We can distinguish several approaches of Monte Carlo methods for solving above mentioned problems:

- Approaches based on probabilistic representation of the solution [3], [7], [5]
- Random walk by subdomains methods [8], [9], [10], [13] ("by spheres" is most known)
- Random walk on boundary methods [13]
- SVD-based approaches [14]

Let us consider the pros and cons of each approach. The methods based on probabilistic representation of the solution are often used to find the asymptotic properties of solutions. These methods are difficult to construct numerical algorithms directly and estimates of the simplified approach are used. This methods are more time-consuming in comparison with others. More economical methods are walk by subdomains and walk on boundary based on the reduction of the original equation to a special integral equation with generalized kernel. Walk on boundary methods are derived for a more restricted range of problems, but can solve problems with complex geometry boundaries. New SVD approach allows to construct the most efficient statistical methods for finding solutions of linear equations approximated the original problem.

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In this paper, we consider the biharmonic equation with random inputs functional parameters. The walk by spheres vector estimates of covariance for the solutions were constructed in [6], the corresponding walk on boundary vector estimates were suggested in [13], SVD approach was presented in [14].

Here, a new scalar walk by spheres estimates of covariance for the solution were constructed by an parametric differentiation of well known estimates of solution to special constructed problems. First, this approach was proposed by G. Mikhailov [12]. Besides, Mikhailov and Tolstolytkin [6] proposed scalar estimates had been fully investigated: the finiteness of variances has been proved, absolute errors have been evaluated, laboriousness have been estimated, the problem of optimal choice of method parameters to achieve a given error level have been solved. Compared to [13][14] the offered method can solve a problems with a random spectral parameter. It seems that the approach by Sabelfeld and Mozartova [14] is less time consuming, but a special comparison of methods was not carried out.

We obtain estimates for the Dirichlet BVP and some special Neumann BVP. Further investigation is aimed at building a cost-effective methods for mixed and Neumann BVP.

This work is mostly theoretical. However, the proposed estimates is easy to extend to the real problems with the own geometry of the boundary.

A brief structure of the paper is presented below. Precise mathematical formulation of problems for theory of fluctuations of elastic systems and some auxiliary equations are presented in Section II. General theoretical results for polyharmonic equation, obtained with Mikhailov [4][5], are presented in Section III. New results and model calculations are considered in Section IV.

## II. BOUNDARY VALUE PROBLEMS

### A. Helmholtz equation

Let us consider the Dirichlet problem for the Helmholtz equation in a domain  $D \subset R^3$  with boundary  $\Gamma$ :

$$(\Delta + c)u = -g, \quad u|_{\Gamma} = \varphi. \quad (1)$$

Let us assume that the following conditions hold. The function  $g$  satisfied Holder condition [2] in  $\bar{D}$ ,  $D$  is a bounded open set in  $R^3$  with a regular boundary  $\Gamma$ , the function  $\varphi$  is continuous on  $\Gamma$ ,  $c < c^*$ , where  $-c^*$  is the first eigen value of Laplace

operator defined on the domain  $D$ . These conditions provide the existence and uniqueness of the solution to problem (1), the existence its probabilistic and integral representations in terms of the spherical Green's function [2]. We suppose that above conditions are fulfilled and after change of all parametric functions by their modules.

### B. Elastic BVP

In a domain  $D \in \mathbb{R}^2$  with boundary  $\Gamma$  bending of thin elastic plate satisfies the biharmonic equation [1]

$$\Delta^2 u = f(x, y)/K. \quad (2)$$

In case a plate is lying on an elastic foundation, we have the following equation [1]

$$\Delta^2 u + cu = f(x, y)/K. \quad (3)$$

Here,  $u(x, y)$  is normal flexure of a plate at the point  $(x, y)$ ;  $f(x, y)$  is a strength of normal charge;  $K = Eh^3/12(1 - \sigma^2)$ , where  $E$  is the elastic modulus;  $\sigma$  is the Poisson constant for the stuff of the plate;  $2h$  is plate thickness. Let us consider the following frequently occurring boundary conditions

- the edge of the plate is simply supported:  $u|_{\Gamma} = 0$ ,  $\Delta u - \frac{1-\sigma}{\rho} \frac{\partial u}{\partial n}|_{\Gamma} = 0$ ;
- the edge of the plate is rigid:  $u|_{\Gamma} = 0$ ,  $\frac{\partial u}{\partial n}|_{\Gamma} = 0$ ;
- the edge of the plate is elastically supported:  $u|_{\Gamma} = 0$ ,  $\Delta u + \left(\frac{1-\sigma}{\rho} + k_0\right) \frac{\partial u}{\partial n}|_{\Gamma} = 0$ .

Here,  $n$  is the external normal to the boundary  $\Gamma$  of the plate;  $\rho$  is curvature radius of  $\Gamma$ ;  $k_0$  is a value related to a rigidity of the edge fixity.

### C. Metaharmonic BVP

Let us consider the general problem:

$$\begin{cases} (\Delta + c)^{p+1} u = -g, \\ (\Delta + c)^k u|_{\Gamma} = \varphi_k, \quad k = 0, \dots, p. \end{cases} \quad (4)$$

In this work, the following results will be used [5].

**Theorem 1.** *Let conditions of the part A are satisfied then the  $p$ -th parametric derivative of solution  $u$  to the problem (1) with a functional parameters*

$$\varphi = \sum_{k=0}^p \frac{(-1)^k \lambda^{p-k}}{p!} \varphi_k, \quad g_1 = \frac{(-1)^p}{p!} g \quad (5)$$

is the solution to the problem (4).

## III. ALGORITHMS OF 'RANDOM WALKS BY SPHERES'

### A. General algorithm

Further considered estimators of the Monte Carlo method are associated with a 'random walks by spheres' in the domain  $D$  [8]. For simplicity, we designate:  $\bar{D}$  is a closure of domain  $D$ ;  $d(P)$  is a distance from the point  $P$  to the boundary  $\Gamma$ ;  $\Gamma_{\varepsilon} = \{P \in \bar{D} : d(P) < \varepsilon\}$  is the  $\varepsilon$ -neighborhood of the boundary;  $S(P) = \{Q \in \bar{D} : |Q - P| = d(P)\}$  is a sphere of radius  $d(P)$  with its center at the point  $P$  lying in  $\bar{D}$ . In the 'random walk by spheres' we chose the successive  $P_{k+1}$

uniformly on the sphere  $S(P_k)$ ; the walk is terminated if the point  $P_{k+1}$  occurs in  $\Gamma_{\varepsilon}$ . Let  $N = \min\{m : r_m \in \Gamma_{\varepsilon}\}$ .

It is well known [12] that solution to the problem (1) satisfies the following equation  $u(r_0) = E \eta_{\varepsilon}$  in  $\mathbb{R}^n$ , where

$$\eta_{\varepsilon} = \sum_{i=0}^N \left[ \prod_{j=0}^{i-1} s(c, d_j) \right] \int_{D(r_i)} G(\rho; c, d_i) g(\rho) d\rho + \left[ \prod_{j=0}^{N-1} s(c, d_j) \right] u(r_N). \quad (6)$$

Here,  $d_j = d(r_j)$ ,  $D(r_i)$  is a ball of radius  $d_i$  with its center at the point  $r_i$ ,

$$s(c, d) = \frac{(d\sqrt{c}/2)^{(n-2)/2}}{\Gamma(n/2) J_{(n-2)/2}(d\sqrt{c})}, \quad (7)$$

$G(\rho; c, d)$  is a spherical Green's function,  $J(\cdot)$  is a Bessel function,  $\Gamma(\cdot)$  is a Gamma function.

Therefore, using Theorem 1, we have following assertion (all derivatives are considered at the point  $c = c_0$ ) [4].

**Theorem 2.** *Under the conditions of Theorem 1 the following representation holds true for the solution to the problem (3)  $u = E\left(\frac{\partial \eta_{1,\varepsilon}}{\partial c^p}\right) = E(\eta_{1,\varepsilon}^{(p)}) \quad \forall p \geq 0$ , where  $\eta_{1,\varepsilon}$  is derived from  $\eta_{\varepsilon}$  by the substitute*

$$g \rightarrow g_1 = \frac{(-1)^p}{p!} g, \\ u(r_N) \rightarrow \varphi(r_N, c) = \sum_{k=0}^p \frac{(-1)^k (c - c_0)^{p-k}}{p!} u_k(r_N).$$

Here,  $u_k$  is a solution to the problem (4) with  $\varphi \equiv \varphi_k$ ,  $k = 0, \dots, p$ , function  $g$  is equal to zero for  $k = 0, \dots, p-1$ .

**Theorem 3.** *If  $c < c^*/2$  then  $D(\eta_{1,\varepsilon}^{(p)}) < C_p < +\infty \quad \forall p \geq 0$*

### B. Practical estimators

Let us consider now practically realizing estimate  $\tilde{\eta}_{1,\varepsilon}^{(p)}$  which is obtained after substitute of variables  $u_k(r_N)$  to  $\varphi_k(P)$ , where  $P$  is a nearest to  $r_N$  point of boundary. From (6), we obtain that  $\tilde{\eta}_{1,\varepsilon}^{(p)} = \sum_{i=1}^N [Q_i(c) \int_{D(r_i)} G(\rho; c, d_i) g_1(\rho) d\rho]^{(p)} + [Q_N(c) \varphi(P, c)]^{(p)}$ . The following theorems hold true [4].

**Theorem 4.** *If  $c < c^*$  and first spatial derivatives of the function  $\{u_k^{(i)}\}$ ,  $i = 1, \dots, p+1$ , are uniformly bounded in  $\bar{D}$  then  $|u(r) - E\tilde{\eta}_{1,\varepsilon}^{(p)}| \leq C_p \varepsilon$ ,  $r \in D, \varepsilon > 0$ .*

**Theorem 5.** *Under conditions of Theorem 4, for  $c < c^*/2$  it holds, that*

$$D\tilde{\eta}_{1,\varepsilon}^{(p)} < C_p < +\infty, \quad \forall \varepsilon > 0.$$

In the expression of  $\tilde{\eta}_{1,\varepsilon}^{(p)}$ , we may to estimate the integrals by one random point as follows

$$\begin{aligned} & \int_{D(r)} G(\rho; c, d) g(\rho) d\rho = \\ & = \frac{d^2}{2n} \int_{D(r)} p_0(r, \rho) [G(\rho; c, d) / G(\rho; 0, d)] g(\rho) d\rho = \\ & = \frac{d^2}{2n} E \left\{ \frac{G(\rho; c, d)}{G(\rho; 0, d)} g(\rho) \right\}, \end{aligned} \quad (8)$$

where  $\rho$  is a random point in  $D(r)$ , its distribution function is equal to (under  $n > 2$ )

$$p_0(r, \rho) = 2nd^{-2}G(\rho; 0, d) = \frac{2n}{(n-2)d^2\omega_n} \left( \frac{1}{(\rho-r)^{n-2}} - \frac{1}{d^{n-2}} \right), \quad |\rho - r| \leq d \quad (9)$$

It is clearly that  $\int p_0(r, \rho)d\rho = 1$ . Under  $n=2$ , we obtain

$$p_0(r, \rho) = 4d^{-2}G(\rho; 0, d) = \frac{2}{\pi d^2} \ln \frac{d}{|\rho - r|}, \quad |\rho - r| \leq d$$

Since under various values  $c$  the Green's functions have at the point  $\rho = 0$  a same order of poles the ratio of the function  $G$  is bounded in (8) Then

$$\tilde{\eta}_{1,\varepsilon}^{(p)} = \sum_{i=0}^N \left\{ \left[ \prod_{j=0}^{i-1} s(c, d_j) \right] g_1(\rho_i) \frac{d_i^2 G(\rho; c, d_i)}{2nG(\rho; 0, d_i)} \right\}^{(p)} + \left\{ \left[ \prod_{j=0}^{N-1} s(c, d_j) \right] \varphi(r_N, c) \right\}^{(p)}, \quad (10)$$

besides  $E\tilde{\eta}_{1,\varepsilon}^{(p)} = E\tilde{\eta}_{1,\varepsilon}^{(p)}$ . It is clear that proof of the Theorem 6 still is valid after substitution of  $\tilde{\eta}$  by  $\tilde{\eta}$ , i.e.  $D\tilde{\eta}_{1,\varepsilon}^{(p)} < C_p < +\infty, \quad \forall \varepsilon > 0$ .

#### IV. CALCULATIONS

##### A. Biharmonic equation solving

Consider the first boundary value problem for the inhomogeneous biharmonic equation

$$\Delta^2 u = -g, \quad u|_{\Gamma} = \varphi_0, \quad \Delta u|_{\Gamma} = \varphi_1. \quad (11)$$

in a domain  $D \subset R^n$ .

Under  $n = 3$ , the corresponding estimators  $\tilde{\eta}_{1,\varepsilon}^{(1)}$  has the following form

$$\tilde{\eta}_{1,\varepsilon}^{(1)} = \frac{1}{36} \sum_{i=0}^N \left[ - \sum_{j=0}^i d_j^2 + (d_i - \nu_i)^2 \right] d_i^2 g(\rho_i) - \frac{1}{6} \left( \sum_{j=0}^{N-1} d_j^2 \right) \varphi_1(r_N) + \varphi_0(r_N). \quad (12)$$

The random variable  $\nu_i$  distributed in interval  $(0, d_i)$  with probability density  $6x(1-x/d_i)d_i^{-2}$  and isotropic unit vector  $\omega_i$  are simulated by well known formulas [11].

Under  $n = 2$ , we obtain that the estimator to the solution to problem(14) has form

$$\tilde{\eta}_{1,\varepsilon}^{(1)} = \frac{1}{16} \sum_{i=0}^N \left[ - \sum_{j=0}^{i-1} d_j^2 - \frac{d_i^2 - \nu_i^2 - \nu_i^2 \ln(d/\nu_i)}{\ln(d/\nu_i)} \right] d_i^2 \times g(\nu_i, \omega_i) - \frac{1}{4} \left( \sum_{j=0}^{N-1} d_j^2 \right) \varphi_1(r_N) + \varphi_0(r_N) = \sum_{i=0}^N Q_i g(\rho_i) + \hat{Q}_N \varphi_1(r_N) + \varphi_0(r_N), \quad (13)$$

where  $\omega_i$  is a isotropic unit vector,  $\nu_i/d_i$  is a random variable is distributed in interval  $(0, 1)$  with a density  $-4x \ln x$ .

In case  $g \equiv 0$ , the representation (12) may to get from known estimate [13].

1) *Numerical results:* Let us consider the first boundary value problem for the inhomogeneous biharmonic equation

$$\Delta^2 u = 9 \exp(x) \exp(y) \exp(z),$$

$$u|_{\Gamma} = \exp(x) \exp(y) \exp(z), \quad \Delta u|_{\Gamma} = 3 \exp(x) \exp(y) \exp(z).$$

in the unit cube  $D = [0, 1] \times [0, 1] \times [0, 1] \subset R^3$ . The solution to problem is  $u = \exp(x) \exp(y) \exp(z)$ . The solution to problem is calculated numerically by formula (12). The numerical results are given in the table 1.

TABLE I  
CALCULATIONS FOR THREE-DIMENSIONAL BIHARMONIC EQUATION

$\varepsilon$	$S \cdot 10^{-4}$	$u(r)$	$\tilde{u}(r)$	$ u(r) - \tilde{u}(r)  \pm \sqrt{\frac{\sigma^2}{N}}$
$10^{-2}$	1	4.48169	4.5120	$0.030 \pm 0.025$
$10^{-2}$	4	4.48169	4.4875	$0.006 \pm 0.012$
$10^{-2}$	16	4.48169	4.4882	$0.0066 \pm 0.0063$
$10^{-3}$	16	4.48169	4.4837	$0.002 \pm 0.063$
$10^{-3}$	256	4.48169	4.4815	$0.0001 \pm 0.0016$

In the Table 1, we assume:  $r = (0.5; 0.5; 0.5)$  are the coordinates of the point,  $S$  is the number of the modelling trajectories,  $\tilde{u}(r)$  is the numerical solution,  $\sigma^2$  is the variance of the random estimate  $u(r) - \tilde{u}(r)$ .

##### B. Metaharmonic equation solving

Let us consider the following problem

$$\Delta^2 u + cu = -g, \quad u|_{\Gamma} = \varphi_0, \quad \Delta u|_{\Gamma} = \varphi_1. \quad (14)$$

in a domain  $D \subset R^2$ .

Suppose  $c$  is a random variable such that

$$Ec = 0, \quad Dc \ll 1, \quad c < c^*,$$

$g$  is a random field,  $\varphi_0$  and  $\varphi_1$  are random functions.

The aim of this subsection is to estimate covariance function  $cov(r_1, r_2) = Eu(r_1)u(r_2)$ . Using series expansion of  $u(r, c)$  at the point  $c = 0$  we change

$$u(r, c, g, \varphi_0, \varphi_1) \approx u(r, 0, g, \varphi_0, \varphi_1) + cu^{(1)}(r, 0, g, \varphi_0, \varphi_1). \quad (15)$$

We may assume that corresponding error  $\delta$  is equal to

$$\frac{1}{2}u^{(2)}(r, 0, g, \varphi_0, \varphi_1)c^2.$$

Then

$$\begin{aligned} cov(u(r_1, c, g, \varphi_0, \varphi_1), u(r_2, c, g, \varphi_0, \varphi_1)) &\approx \\ &\approx Eu(r_1, 0, g, \varphi_0, \varphi_1), u(r_2, 0, g, \varphi_0, \varphi_1) + \\ &+ E[u^{(1)}(r_1, 0, g, \varphi_0, \varphi_1)u^{(1)}(r_2, 0, g, \varphi_0, \varphi_1)] Dc, \\ 2\delta &= E[u^{(2)}(r, 0, g, \varphi_0, \varphi_1)u(r, 0, g, \varphi_0, \varphi_1)] Dc + \\ &+ E[u^{(2)}(r, 0, g, \varphi_0, \varphi_1)u^{(1)}(r, 0, g, \varphi_0, \varphi_1)] E[c^3] \\ &+ E[u^{(2)}(r, 0, g, \varphi_0, \varphi_1)u^{(2)}(r, 0, g, \varphi_0, \varphi_1)] E[c^4] \end{aligned}$$

The parametric derivative  $u^{(1)}(r, 0, g, \varphi_0, \varphi_1)$  is a solution to the following problem

$$\begin{cases} \Delta^4 u = -g, & u|_{\Gamma} = 0, & \Delta u|_{\Gamma} = 0, \\ \Delta^2 u|_{\Gamma} = -\varphi_0, & \Delta^3 u|_{\Gamma} = -\varphi_1. \end{cases} \quad (16)$$

The parametric derivative  $u^{(2)}(r, 0, g, \varphi_0, \varphi_1)$  is a solution to the following problem

$$\begin{cases} \Delta^6 u = g, & \Delta^k u|_{\Gamma} = 0, & k = 0, \dots, 3, \\ \Delta^4 u|_{\Gamma} = \varphi_0, & \Delta^5 u|_{\Gamma} = \varphi_1 \end{cases} \quad (17)$$

The corresponding estimates of the solutions to the problems (16), (17) has form

$$\begin{aligned} \tilde{\eta}_{1,\varepsilon}^{(3)} &= \sum_{i=0}^N \left\{ \sum_{k=0}^3 C_3^k S_i^{(3-k)}(0) \frac{G^{(k)}(\rho; 0, d_i)}{G(\rho; 0, d_i)} \right\} \frac{[-d_i^2 g(\rho_i)]}{24} + \\ &\quad + S_N^{(3)}(0) \frac{\varphi_1(r_N)}{6} - S_N^{(2)}(0) \frac{\varphi_0(r_N)}{2}, \\ \tilde{\eta}_{1,\varepsilon}^{(5)} &= \left\{ \sum_{i=0}^N \sum_{k=0}^5 C_5^k S_i^{(5-k)}(0) \frac{G^{(k)}(\rho; 0, d_i)}{G(\rho; 0, d_i)} \right\} \frac{d_i^2 g(\rho_i)}{480} - \\ &\quad - S_N^{(5)}(0) \frac{\varphi_1(r_N)}{120} + S_N^{(4)}(0) \frac{\varphi_0(r_N)}{24}, \end{aligned}$$

where  $S_i(c) = \prod_{j=0}^{i-1} s(c, d_j)$ .

1) *Numerical results:* Here we consider the following problem

$$\Delta^2 u + cu = g, \quad u|_{\Gamma} = 0, \quad \Delta u|_{\Gamma} = 0,$$

in the  $D = \{x_1, x_2 : 0 \leq x_1, x_2 \leq 1\}$ .

Suppose  $c$  is uniformly distributed in the  $(-1/2; 1/2)$ ,  $g$  is a homogeneous, isotropic Gaussian field with the spectral density

$$\rho(\lambda) = \frac{1}{2\pi\alpha^2} (1 + |\lambda|^2/\alpha^2)^{-3/2}, \quad \mathbb{E}g = 0,$$

Corresponding covariance function of  $g$  is equal to  $e^{-\alpha|x|}$ , where  $|x| = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}$ . In the Table 2, we

TABLE II  
CALCULATIONS FOR THE COVARIANCE ESTIMATOR

$\delta$	$\varepsilon$	S	$(v \pm \sqrt{\frac{\sigma_v^2}{N}})*$ * $10^{-5}$	$(\Delta \pm \sqrt{\frac{\sigma_{\Delta}^2}{N}})*$ * $10^{-11}$
0.0	$10^{-2}$	$10^3$	1.078±0.041	1.08±0.12
0.0	$10^{-2}$	$10^5$	1.085 ±0.005	1.26±0.03
0.1	$10^{-2}$	$10^3$	1.026±0.071	1.26±0.03
0.1	$10^{-4}$	$10^5$	1.010±0.007	1.40±0.00
0.2	$10^{-2}$	$10^5$	0.885 ±0.006	1.04±0.03
0.3	$10^{-2}$	$10^5$	0.654 ±0.006	0.81±0.04
0.4	$10^{-2}$	$10^5$	0.343 ±0.004	0.41±0.18

assume:  $v(r, r')$  is a covariance function of solution at the point  $r = (0.5; 0.5)$  and  $r' = (0.5 + \delta; 0.5)$ ,  $\sigma_v^2$  is the variance of the random estimate for  $v$ , S is the number of the modelling trajectories,  $\Delta$  is the error of approximation,  $\sigma_{\Delta}^2$  is the variance of the random estimate for  $\Delta$ .

## V. CONCLUSION AND FUTURE WORK

In this paper the parametric differentiation approach has been considered as an efficient method for constructing scalar 'random walk by spheres' estimates. This method is based on parameter differentiation standard estimates of the solution to the special constructed boundary value problem. Using this approach and partial averaging method, 'random walk by spheres' estimates of covariance function were obtained for the Dirichlet problems of the biharmonic equation. For testing efficiency of this method, two examples for different types of equations have been considered. The constructed algorithms are particularly useful for estimating the covariance function in the local domain. Another improvement is the need to store only one trajectory of a random walk.

The developed algorithms find practical application in the theory of elasticity, discussed in Section 2. Problems with random functional parameters are suitable in the study of vibrations of plates under the action of random forces such as the wind or earthquakes.

Future work will focus on the development of similar algorithms for boundary value problems of the second and third type. Additionally, it is supposed to develop 'random walk by spheres' algorithms for biharmonic equations where spectral parameter is a random value with high dispersion and non-zero expectation.

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