Traffic and Monotone Random Walk of Particles: Analytical and Simulation Results

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***Abstract* - An analytical model of random walk of particles on one-dimensional lattices and two-dimensional ones is considered. This model can be used for study of traffic on multi-lane road, traffic on crossroads, etc. We have obtained new exact formulas that allow to calculate the average velocity of particles. The steady state probabilities have been found also. We developed also an appropriate simulation model. The values, obtained with aid of analytical calculations, were compared with simulation results, and the result of this comparison was that the accordance of the models is good correspondence of the values found with aid of the two models.**

Keywords-stochastic models; random walk; multi-lane traffic; optimization.

I. Introduction and formulation of problem

Models of random walk on a lattice, which are used for the traffic study, help to study characteristics of multi-lane traffic, the behaviour of traffic on crossroads, etc. Such models help to solve problems of traffic optimization.

The appearance of cellular automata models in the problem of traffic flows is associated with publications of Nagel et al. [1–2], which were published in mid-nineties. In [1–2], the dependence of the average velocity and intensity of traffic on the model parameters was studied. The following factors, which contribute to the activation of this approach development, can be noted

1) Desire to explain the discrepancy between the solutions of traffic equations and experimental observations, which discovered chaotic behaviour in the so-called unstable mode;

2) Desire to create models discrete in time and space or only by one coordinate, which would be independent of infinite-dimensional models and take into account the presence of individual behaviour of ”particles with motivated behaviour”.

We enumerate papers known to us that contain exact mathematical results and relate to the theme of our paper.

It has been noted that the scheme considered in [1–2] is similar to monotone random walks on a lattice. This theme has its own history. In particular, the papers of Soviet mathematician Belyaev and his students [3, 4] are devoted to traffic flows in the underground and contain exact results for one-dimensional random walk (not only monotone).

There are a lot of experimental results obtained with aid of cellular automata models. That is apparently due to the relative availability of computational tools. A Russian mathematician Blank gave well-defined statements and achieved exact results in a number of important cases, [5–6].

For example, one of problem in this scope is the problem of percolation of fast particles through a lattice with a slow particles on it.

Here we are talking about exact estimates, while simulation results such as the experimental observation about the significant influence of the presence of heavy trucks on traffic intensity certainly exist.

In [7], a model of particles (vehicles) movement on a multi-lane road was considered. In this model the velocity of movement is the sum of determinate and stochastic components, i.e., $v_{model} = v_{determ} + v_{stochast}$. In the model considered in [7] the determinate component of movement corresponds to the background movement on lane and the stochastic component corresponds to individual manoeuvres of particles. Each lane corresponds to the sequence of cells. The size of the cell is determined by the dynamic dimension of the vehicle. The dynamic dimension takes into account the safety requirement and depends on velocity of movement, [8]. Stochastic movement is described by monotone walk on cells of lane and the regular movement is described by uniform movement of the lane, [7], [9].

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In [10], a model of random walk on one-lane ring has been considered. The formula has been found for the average velocity of particles. This formula is a generalization of formula, obtained in [5], for the model of random walk, where randomness occurs only for initial configuration of particles.

In [11–13], models of random walk on a discrete lattice, similar to models introduced in [7, 9], have been used for the solving some problems of traffic optimization.

The present work considers a stochastic model, which describes movement of particles (vehicles) on the multi-lane road. The steady state probabilities and the average velocity of particles have been obtained. The configurations of particles on the lattice determine the model states. The limiting case is studied, in which the length of lane is infinitely large.

Some simulation models have been developed, which are used for the study of characteristics of single-lane and multi-lane movement.

The developed simulation models were realized using Delphi 7 environment.

The volume of PC computers memory was sufficient for the case of nearly 200 cells of the lattice.

A minute of computer program time corresponds to nearly 500 units of discrete model time.

If the simulation time interval is equal nearly 30000 units of discrete model time, the difference between values of average particles velocities found with aid of simulation and analytical models does not usually exceed 1 %.

II. SINGLE-LANE AND MULTI-LANE FLOWS

A. Single-lane flows

Let us describe the stochastic model of movement on the circle, which was considered in [10]. The ring contains n cells and $m < n$ particles. If at the current time there is a particle in the cell and the next cell is empty then the particle moves to the next cell with probability p. The steady states probabilities and the average velocity of particles have been found. States of the model are determined by the configurations of particles on the lattice. As shown in [10], the probabilities of states depend on the number of particle clusters only.

Let us consider a mathematical model of movement on the lane.

This model is the limiting case of the model studied in [10]. In this case the number of cells is big. The formula for average velocity v has been obtained:

$$v = \frac{1 - \sqrt{1 - 4rp(1-r)}}{2r}, \qquad (1)$$

where r is the flow density, $r = m/n$.

In [10], a formula has been obtained for average velocity $v(n,m)$ of the stochastic movement on non-moving lane for number of cells equal n, number of particles equal m, and the probability of realization of attempt equal p. In accordance with this formula

$$v(n,m) = \frac{n}{m} \sum_{k=1}^{\min(m,n-m)} \frac{Cp}{(1-p)^{k-1}} C_{m-1}^{k-1} C_{n-m-1}^{k-1},$$

where

$$C = \left(\sum_{k=1}^{\min(m,n-m)} \frac{n}{k} \cdot C_{m-1}^{k-1} C_{n-m-1}^{k-1} \frac{1}{(1-p)^{k-1}} \right)^{-1},$$

C_m^k is the number of combinations of k elements from m ones.

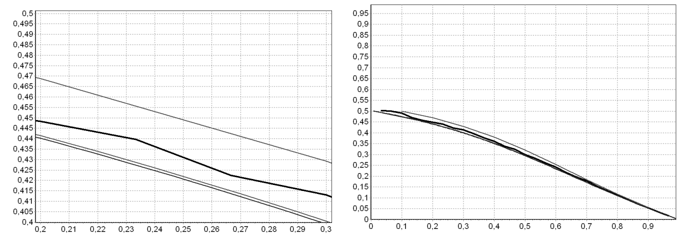

Figure 1. Average velocity (y-axis) of particle movement on the ring for n equal to 10 (the upper curve), equal to 100 (the second curve from the top), 170 (the second curve from the bottom) and ∞ (the lower curve), simulation for $n = 30$ (the second curve from the top). Values of density are indicated on x-axis. The same curves are represented on both the diagrams but the scale of left diagram is larger

Diagrams of $v(n,m)$ for values of n, equal to 10, 100 and 170, and the diagram of function (1), which corresponds to an infinitely great value of n, is showed in Fig. 1. Value of p is supposed to be equal to 0.5. In this figure the corresponding values of the average velocity of particles, which have been obtained by simulation, are also shown. Interval of simulation was supposed to be equal to 10000 simulation steps.

In the case $p \sim 1$ we have, using formula (1),

$$v \sim \begin{cases} 1, \ 0 < r \le \frac{1}{2}, \\ \frac{1}{r} - 1, \ \frac{1}{2} < r < 1, \end{cases}$$

that corresponds to the results of work [5], where a similar model has been considered for $p = 1$.

In case $p \sim 0$ we have

$$v \sim p(1-r).$$

Function (1) is convex on the flow density. Due to this fact the problem of the optimal distribution of particles on the lanes has been solved for canalized movement. This problem has been formulated in [11].

B. Multi-lane flows

A model of stochastic movement on torus was studied (Fig. 2). There are several lanes. Each lane is a circle that consists of some number of cells. The sections of the torus are also circles and every section contains cells of different

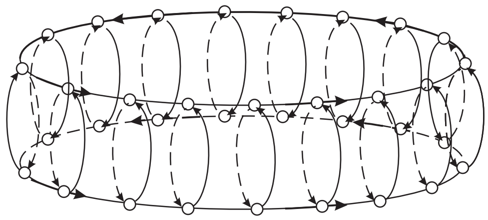

Figure 2. Torus

lanes. The movement of particles is monotone with respect to each coordinate, i.e., the particles can move only in one and the same direction for every coordinate. At the current time with probability $1/2$ the transition of particle in the direction of the first coordinate and with probability $1/2$ the transition of particle in the direction of the second coordinate is planned. The transitions in the given direction are planned simultaneously for all the particles. But every transition is realized with probability that does not depend on behaviour of the other particles. Suppose that at the current time the transition of particle in direction of s-th coordinate is planned, $s = 1, 2$. If the particle is in cell (i, j) and the next cell, in direction of the s-th coordinate, is empty, then with probability $0 < p_s < 1$, which does not depend on results of attempts of the other particles, the particle passes to the next cell in the direction of s-th coordinate, $s = 1, 2$.

The states of the model correspond to configurations of particles. It has been proved that in the case of small values of probabilities p_1 and p_2 probabilities of all the states are asymptotically equal to $1/N$, where N is the number of states. Exact formulas have been found for the average velocity of movement. It is proved that the distribution of particles on the lanes is hypergeometric for the number of lanes equal 2 and a generalization of the hypergeometric distribution if the number of particles is greater than 2.

The behaviour of the model is studied for the case of arbitrary values of p_1 and p_2 and great values of number of cells. It occurs for this case that the particles are distributed on the lanes uniformly. The average velocity can be calculated using formula (1).

III. Simulation model of multi-lane movement. Comparison with the analytical model

A. The case of non-moving lane

A multi-lane traffic simulation model has been developed.

We consider the case of non-moving lane, i.e. $v_{det} = 0$. The model contains a two-dimensional array. There are K lanes. Each lane is a sequence of n cells. There are m particles. The transitions of particles occur at integer time steps. Each cell contains no more than one particle. If at the current time the cell ahead of a particle is empty then it passes to the next cell with probability p. If the cell ahead of the particle is occupied and the neighbouring cell of another lane is empty then the particle passes to the neighbouring cell. If the both neighbouring cells on the other lanes are empty then each of them can be chosen with probability $1/2$. If two particles can come to the same cell the priority is given to the particle that is located on the right.

The results of analytical calculations were compared with the results of simulation.

The differences of the models are the following

1) Movement is monotone in direction of two coordinates in the analytical model and movement is monotone in direction of only coordinate in the simulation model.

2) In the analytical model all the lanes are inner.

3) In the simulation model collisions of particles are prevented by restrictions on changing the lane. This collisions are impossible in the analytical model as at each time the transitions in direction of the only coordinate are allowed.

In the analytical model some simplifications are used and analytical results can be obtained. The simulation model describes the real situation more detailed.

In accordance with the rules of movement in the simulation model a particle moves to another lane if there is another particle in the next cell ahead.

Therefore we should expect that the average velocity of particles in this model will be slightly higher than the average velocity of a particle in the mathematical model (Section II), in which transitions to another lane are independent of whether the cell, neighbouring to the cell to be occupied, is empty or occupied.

In Figure 3 the results of analytical calculations and results of simulation are given for the models of two-lane movement. Values of flow density and average velocity are indicated on x-axis and y-axis respectively.

We suppose that the number of cells on each lane is equal to 100. Let p and r be defined as in Section II. Let $v^*(r)$ be the average velocity of a particle that has been found by simulation and $v^{**}(r)$ be the average velocity calculated using formula (1). As it was expected the maximum value of the relative differences

$$\Delta = \frac{u^*(r) - u^{**}(r)}{u^*(r)}$$

is attained at $r = 0.5$ approximately. This can be explained with the following. By small values of the flow density a situation, where particles pass to the other lane, occurs rarely. By great values of flow densities transitions of particles to the other lane are complicated and therefore occur also rarely.

As the number of cell on the torus tends to infinity, the average velocity converges to values calculated with the aid of formula (1). Therefore data, represented in Fig. 3, give an estimation of difference between values of the average velocities in the analytical model of movement on torus and the simulation model of multi-lane movement.

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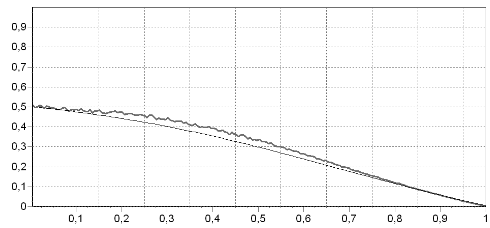

Figure 3. Dependence of average velocity on density for $p = 0.5$. Lower curve corresponds the analytical model. Upper curve corresponds the simulation

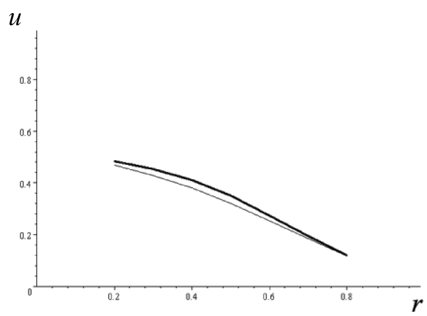

Figure 4. Results of comparing of the two-lane simulation model (the upper curve) and model taking into account a determined component of average velocity (the lower curve). Values of flow density and average velocity are indicated on x-axis and y-axis respectively.

B. Comparison with the analytical model, which takes into account the determinate component of the traffic velocity

In Section IIIa, the comparison of the values of average velocities is represented. We compared also the values of average velocities of particles in simulation and analytical models, developed in [7, 9]. In these models the velocity of movement is represented as sum of the determinate component and the stochastic one. The approach using such representation is mentioned in the Section I.

Results of comparison are represented in Fig. 4. This comparison allows to estimate the differences between the average velocities, which are obtained with the analytical and simulation models. This difference can be explained by that in the simulation model the particles can change lanes and by this reason the average velocity of movement is some higher.

As for models which are considered in Section IIIa the maximum value of the relative differences between values of velocities is attained at $r = 0.5$ approximately.

IV. THE "RING WITH JUNCTIONS" MOVEMENT MODEL.

We consider simulation model of traffic with crossroads, that is "figure-of-eight". This model contains the main ring and k rings of smaller size, each of that has one common cell with the main ring, Fig. 5. The flow passing such a cell (crossroads) divides into two parts.

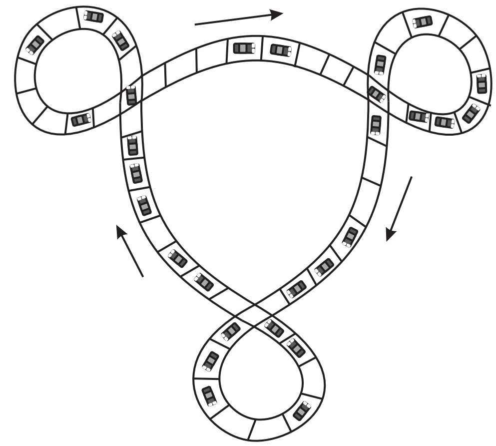

Figure 5. Movement model "ring with junctions"

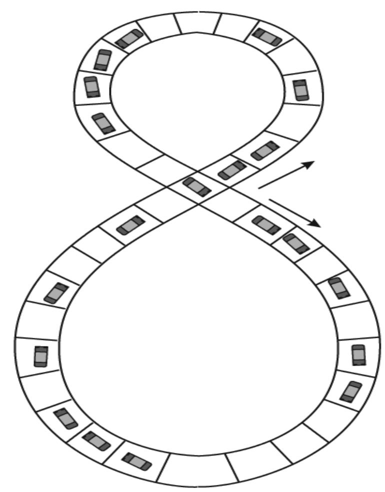

Figure 6. "Figure-of-eight" model

Some of them remain on the main ring and others begin to move on the small ring.

In one of version of the model the particles coming to the crossroads from the main ring have priority and in the second version the particles coming to the crossroads from the small ring have priority.

If $k = 1$ then we have the model of "figure-of-eight". Each ring in the "figure-of-eight" is represented by one-dimensional array, Fig. 6. Two arrays have a common cell, which is a model of cross-road. The common cell is called cell 0. The first ring contains n_1 cells $(1, j)$, $j = 1, 2, \ldots, n_1$. The second cell contains $n_2 < n_1$ cells $(2, j)$, $j = 1, 2, \ldots, n_2$. Coming through the crossroads, particles coming from the first ring have priority over the particles coming from the second ring.

So there are $n = n_1 + n_2 + 1$ cells in the model. There are $m < n_1$ particles. Each of them occupies one cell. There is no more one particle in each cell. Transitions of particles occur at integer time steps. If at integer time k the particle

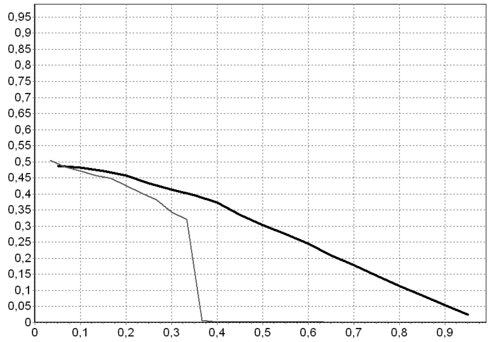

Figure 7. Comparison of diagrams of dependence of average velocity on density for the ”figure-of-eight” (the lower curve) and one-lane ring (the upper curve) models. The values of model parameters are supposed as $n_1 = 20,\ n_2 = 10,\ p = 0,5$. Values of flow density and average velocity are indicated and on x-axis and y-axis respectively.

occupies cell (i, j) and cell $(i, j + 1)$ is empty then with probability $0 < p < 1$ this particle at time $k+1$ will occupy cell $(i, j + 1)$ and with probability $1 - p$ will occupy still cell (i, j), $j = 1, 2, \ldots, n_i - 1$, $i = 1, 2$.

The results of comparison of values of average velocity of particles in the ”figure-of-eight” model and the model of one-lane movement on ring which contains the same number of cells are represented, Fig. 7.

We can see in the figure that for values of m which are sufficiently less than n_1 differences between values, obtained using two models, are small. For values of m, close to n_2, sharp decrease of the average velocity occurs. If $m \geq n_2$ movement stops in some number of simulation steps.

V. Conclusion and future works

An analytical model have been developed where traffic is represented by random walk. Exact results for traffic characteristics have been obtained.

We have developed simulation models taking into account some features of real traffic more detailed. The values, obtained with aid of the analytical calculations, and simulation results was compared. It allows to estimate the difference between average velocities in models with different rules of movement of particles.

We plan to use the simulation model so that it could help to obtain new analytical results.

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