Design of Half-Band FIR Filters for Signal Compression

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***Abstract*—An efficient design of equiripple half-band FIR filters for signal compression is presented. Solution of the approximation problem in terms of generating function and zero phase transfer function for the equiripple half-band FIR filter is shown. The equiripple half-band FIR filters are optimal in the Chebyshev sense. The closed form solution provides an efficient computation of the impulse response of the filter. Two examples are included. The robustness of the design is emphasized. The Matlab code of the design procedure is included.**

***Keywords*-*FIR filter; half-band filter; equiripple approximation; signal compression.*

I. Introduction

Half-band (HB) filter is a fundamental building block in multirate signal processing [2]. HB filters are used among others in filter banks and in image compression techniques, where the signal is iteratively decomposed using filtering and downsampling into its lower and higher subbands. This procedure is found, e.g., in the JPEG2000 compression [3]. Finite impulse response (FIR) filters are preferred because of their linear phase which is essential in the digital image processing. The equiripple (ER) filters are attractive because of their optimality in terms of the filter degree for the specified filter selectivity. Hence, the ER HB FIR filters are appreciated in these tasks. There is a numerical method for designing of ER HB FIR filters available. It is based on the numerical McClellan - Parks program [4]. It is usually combined with a clever "Trick" [5]. The analytical design procedure [6] for trivial lowest order $(n \leq 2)$ ER HB FIR filters has limited practical value. Besides this, some non-numerical design methods are available for almost ER HB FIR filters, e.g., [7] and [8]. In [1] and [9], we have presented a general non-numerical method for the design of ER HB FIR filters. Here we are focused on this method in more detailed manner. We are primarily concerned with the ER approximation of HB FIR filters and with the related non-numerical design procedure suitable for practical design of ER HB FIR filters. We present the generating function and the zero phase transfer function of the ER HB FIR filter. These functions give an insight into the nature of this approximation problem. Our design procedure is based on Chebyshev polynomials of the second kind [10]. Based on the differential equation for the Chebyshev polynomials of the second kind, we have derived formulas for an effective evaluation of the coefficients of the impulse response. We present an approximating degree equation which is useful in practical filter design. The advantage of the proposed approach over the numerical design procedures consists in the fact that the coefficients of the impulse response are evaluated by formulas. Hence, the design procedure and its speed is deterministic.

The structure of the paper is as follows. After an introduction of the basic terminology in Section II, we present the generating polynomial and the zero phase transfer function of an ER HB FIR filter in Section III. The differential equation for the generating polynomial and the impulse response of an ER HB FIR filter are presented in Section IV. Sections V and VI introduce the degree equation of an ER HB FIR filter and its secondary values. The design procedure is summarizes step by step in Section VII. It is followed by two examples in Section VIII. Section IX emphasizes the robustness of the presented design procedure. Appendix I summarizes the derivation of the algebraic procedure for the evaluation of the impulse response of the filter. In Appendix II, the Matlab code of the design procedure is presented.

II. Impulse Response, Transfer Function and Zero Phase Transfer Function

A HB filter is specified by the minimal passband frequency $\omega_p T$ (or maximal stopband frequency $\omega_s T$) and by the minimal attenuation in the stopband a_s [dB] (or maximal attenuation in the passband a_p [dB]). The antisymmetric behavior of its frequency response implies the relations $\omega_s T = \pi - \omega_p T$ and $10^{0.05 a_p} + 10^{0.05 a_s} = 1$. The goal in the filter design is to get the minimum filter length N satisfying the filter specification and to evaluate the coefficients of the impulse response of the filter. We assume the impulse response $h(k)$ with odd length $N = 2(2n+1)+1$ coefficients and with even symmetry $h(k) = h(N-1-k)$. The impulse response of the HB FIR filter with the length $N = 2(2n+1)+1$ contains $2n$ zero coefficients as follows

$$\begin{aligned} h(2n+1) &= a(0) = 0.5 \\ 2h(2n+1 \pm 2k) &= a(2k) = 0 \ , \ k = 1 \ldots n \end{aligned} \tag{1}$$

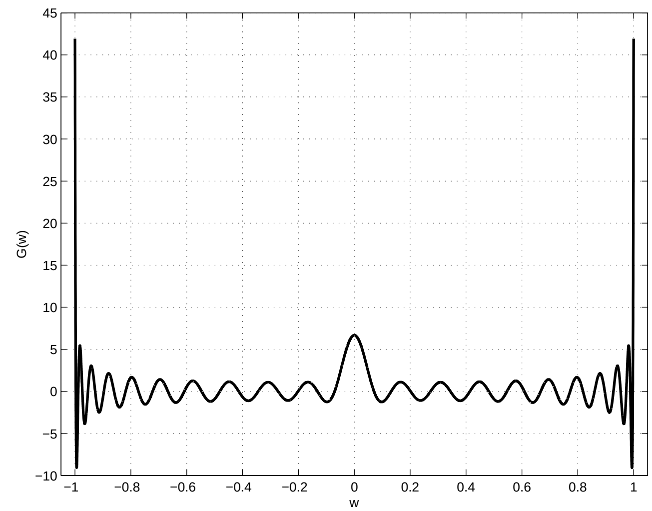

Fig. 1. Generating polynomial $G(w)$ for $n = 20$, $\kappa' = 0.03922835$, $A = 1.08532371$ and $B = 0.95360863$.

$$2h(2n+1 \pm (2k+1)) \quad = \quad a(2k+1) \ , \ k = 0 \ldots n \ .$$

The transfer function of the HB FIR filter is

$$H(z) = z^{-(2n+1)} \left[\frac{1}{2} + \sum_{k=0}^{n} a(2k+1) \, T_{2k+1}(w) \right] \tag{2}$$

where

$$T_n(w) = \cos(n \arccos(w)) \tag{3}$$

is Chebyshev polynomial of the first kind. The frequency response $H(e^{j\omega T})$ of the HB FIR filter is

$$H(e^{j\omega T}) = e^{-j(2n+1)\omega T} \ Q(\cos \omega T) \tag{4}$$

where $Q(w)$ is a polynomial in the variable $w = (z + z^{-1})/2$ which on the unit circle reduces to a real valued zero phase transfer function $Q(w)$ of the real argument $w = \cos(\omega T)$.

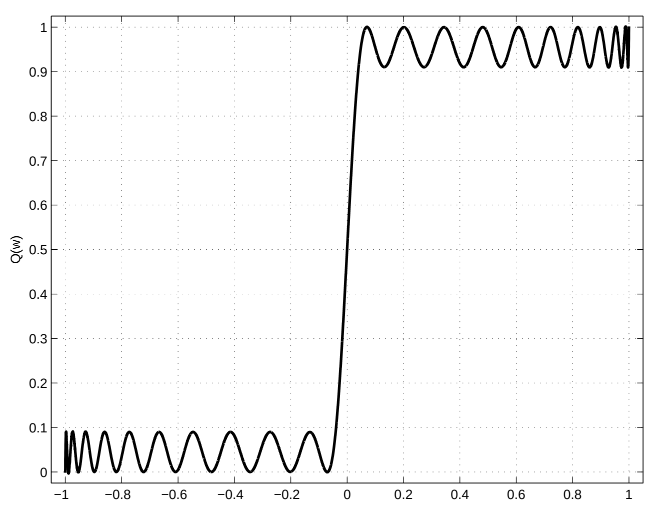

Fig. 2. Zero phase transfer function $Q(w)$ for $n = 20$, $\kappa' = 0.03922835$, $A = 1.08532371$, $B = 0.95360863$ (cf. Fig. 1) and $\mathcal{N} = 0.55091994$.

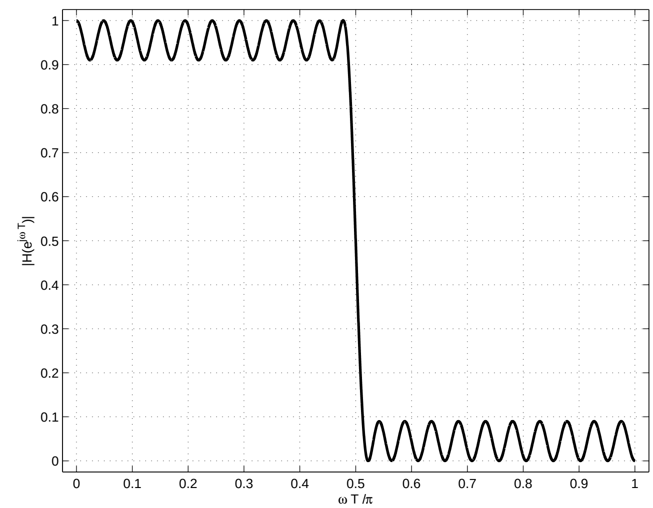

Fig. 3. Amplitude frequency response $|H(e^{j\omega T})|$ corresponding to the zero phase transfer function $Q(w)$ from Fig. 2.

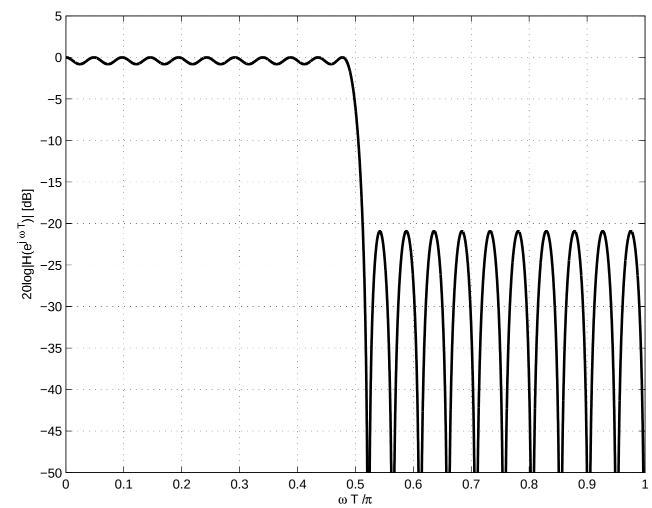

Fig. 4. Amplitude frequency response $20 \log |H(e^{j\omega T})|$ corresponding to the zero phase transfer function $Q(w)$ from Fig. 2.

III. Generating Polynomial and Zero Phase Transfer Function of an ER HB FIR Filter

A straightforward theory for the generating polynomial of an ER HB FIR filter is currently not available. The generating polynomial of an ER HB FIR filter is related to the generating polynomial of the almost ER HB FIR filter presented in [8]. Based on our experiments conducted in [8], we have found that the generating polynomial $G(w)$ (Fig. 1) of the ER HB FIR filter is obtained by weighting of Chebyshev polynomials in the generating polynomial of the AER HB FIR filter, namely

$$G(w) = AU_n\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) + BU_{n-1}\left(\frac{2w^2 - 1 - \kappa'^2}{1 - \kappa'^2}\right) \tag{5}$$

where

$$U_n(x) = \frac{\sin\left[(n+1)\arccos(x)\right]}{\sin\left[\arccos(x)\right]} \tag{6}$$

is Chebyshev polynomial of the second kind and A, B, κ' are real numbers. The zero phase transfer function $Q(w)$ (Fig. 2) of the ER HB FIR is related to the generating polynomial

$$Q(w) = \frac{1}{2} + \frac{1}{\mathcal{N}} \int G(w)dw \qquad (7)$$

where the norming factor $\mathcal{N}$ is given by (19). Both the generating polynomial $G(w)$ and the zero phase transfer function $Q(w)$ show the nature of the approximation of an ER HB FIR filter.

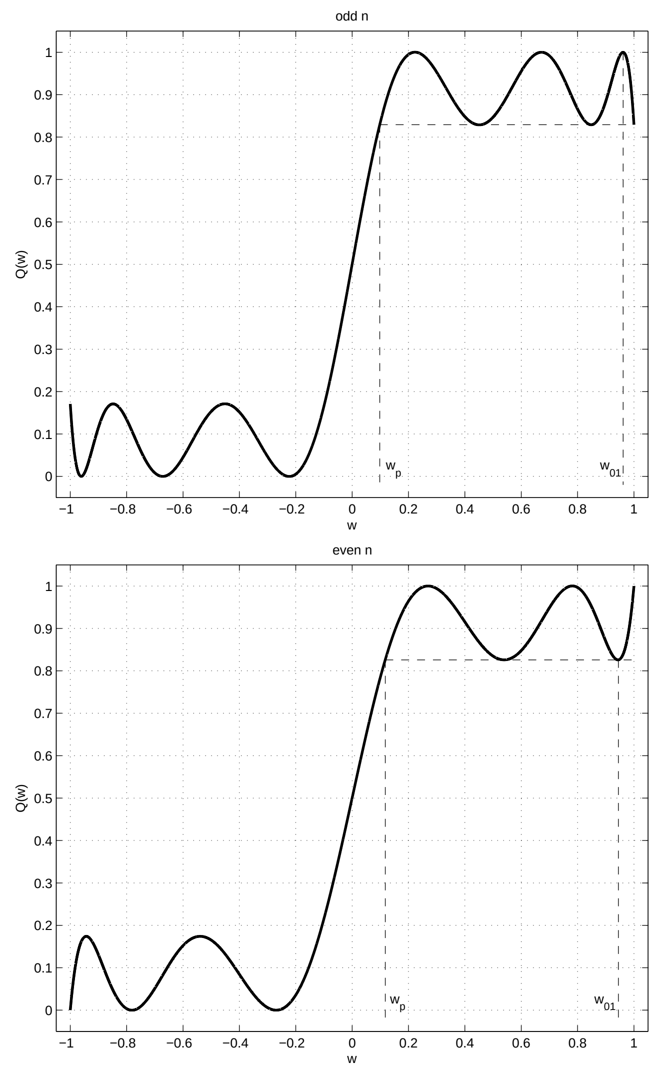

Fig. 5. $Q(w)$ for odd and even n.

IV. Differential Equation and Impulse Response of an ER HB FIR Filter

The Chebyshev polynomial of the second kind $U_x(w)$ fulfils the differential equation

$$(1-x^2)\frac{d^2U_n(x)}{dx^2} - 3x\frac{dU_n(x)}{dx} + n(n+2)U_n(x) = 0 \ . \qquad (8)$$

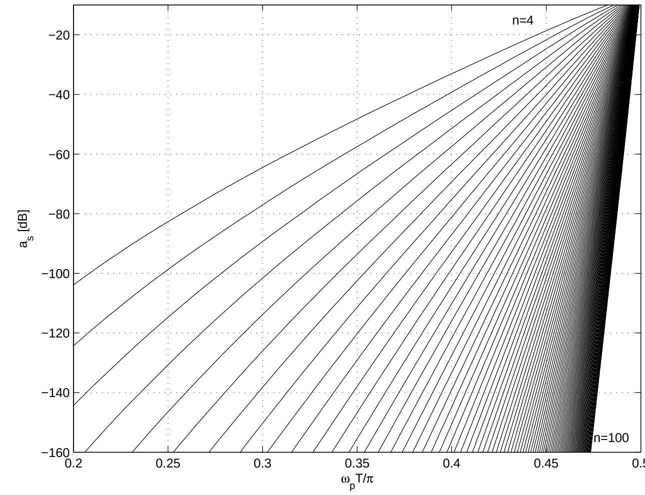

Fig. 6. Empirical dependence of a_s[dB] on $\omega_p T/\pi$ and n.

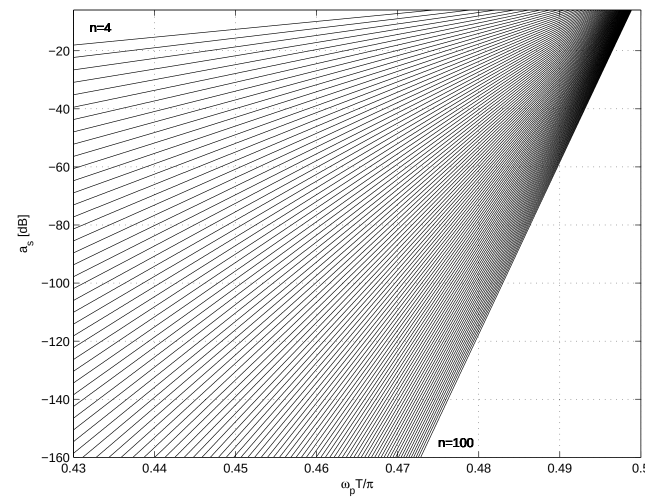

Fig. 7. Detailed view of Fig. 6 near $\omega_p T = 0.5\pi$.

Using substitution

$$x = \left(\frac{2w^2 - 1 - \kappa'^2}{1-\kappa'^2}\right) \qquad (9)$$

we get the differential equation (8) in the form

$$\begin{aligned} w(w^2-\kappa'^2)&\left[(1-w^2)\frac{d^2U_n(w)}{dw^2} - 3w\frac{dU_n(w)}{dw}\right] \\ &+ \left[\kappa'^2(1-w^2) + 2w^2(1-w^2)\right]\frac{dU_n(w)}{dw} \\ &+ 4w^3 n(n+2)U_n(w) = 0 \ . \end{aligned} \qquad (10)$$

Based on the differential equation (10), we have derived the non-numerical procedure for the evaluation of the impulse response $h_n(k)$ corresponding to polynomial $\mathcal{U}_n(w)$

$$\mathcal{U}_n(w) = \int U_n\left(\frac{2w^2-1-\kappa'^2}{1-\kappa'^2}\right)dw \ . \qquad (11)$$

This procedure is summarized in Tab. I. The principle of its derivation is shown in the Appendix I. The impulse response

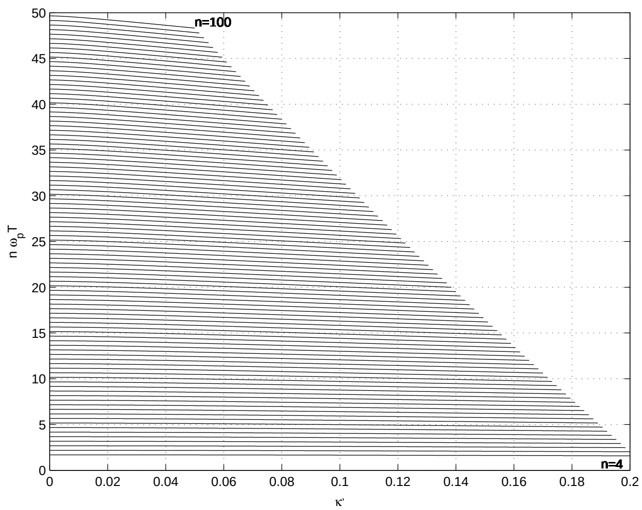

Fig. 8. Empirical dependence of $n\,\omega_p T$ on κ'.

$h(k)$ of the ER HB FIR filter is

$$h(k) = \frac{1}{2} + \frac{A}{\mathcal{N}} h_n(k) + \frac{B}{\mathcal{N}} h_{n-1}(k) \tag{12}$$

where the real norming factor $\mathcal{N}$ is given by (19). The non-numerical evaluation of the impulse response $h(k)$ is essential in the practical filter design because of its determinism.

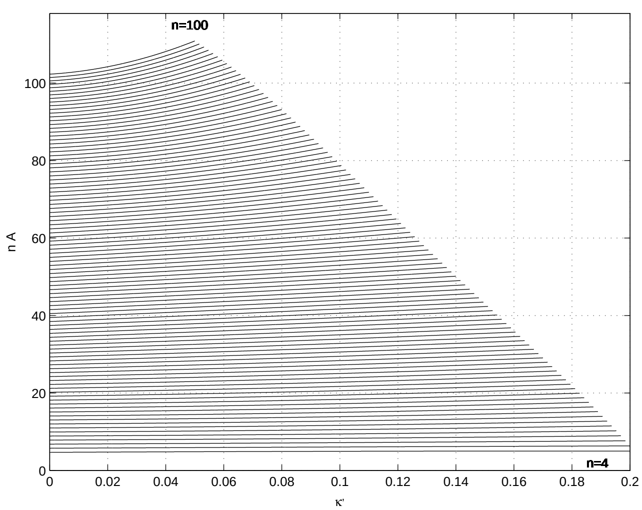

Fig. 9. Empirical dependence of $n\,A$ on κ'.

V. Degree of an ER HB FIR Filter

The exact degree formula is not available. In the practical filter design, the degree n can be obtained with excellent accuracy from the specified minimal passband frequency $\omega_p T$ and from the minimal attenuation in the stopband a_s [dB] using the approximating degree formula

$$n \doteq \frac{a_s[dB] - 18.18840664\,\omega_p T + 33.64775300}{18.54155181\,\omega_p T - 29.13196871} \quad . \tag{13}$$

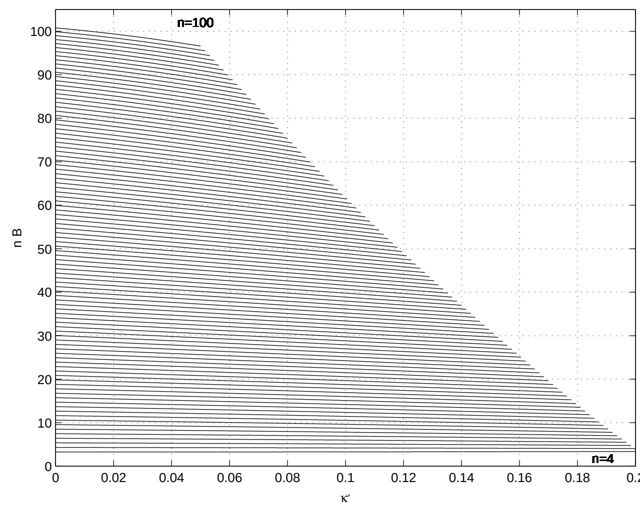

Fig. 10. Empirical dependence of $n\,B$ on κ'.

The exact relation between the minimal attenuation in the stopband a_s [dB], the minimal passband frequency $\omega_p T$ and the degree n were obtained experimentally. It is shown in Fig. 6 and Fig. 7. Equation (13) was obtained by the approximation of exact experimental values in Fig. 7. The approximating degree formula (13) is very precise. However, for very low values $\omega_p T$ its precision slightly decreases. In order to demonstrate the negligible inaccuracy for very low values $\omega_p T$, let us assume an ER HB FIR filter specified by by the minimal passband frequency $\omega_p T = 0.25\pi$ and by the minimal attenuation in the stopband $a_s = -80$ dB. The approximating degree formula (13) results in $n = 4.16194938$ while the filter specification is met for $n = 4$, cf. Fig. 6.

VI. Secondary Values of the ER HB FIR Filter

The secondary real values κ', A and B can be obtained from the specified passband frequency $\omega_p T$ and from the degree n of the generating polynomial. The approximating formulas

$$\kappa' = \frac{n\omega_p T - 1.57111377\,n + 0.00665857}{-1.01927560\,n + 0.37221484} \tag{14}$$

$$A = \left(0.01525753\,n + 0.03682344 + \frac{9.24760314}{n}\right)\kappa' + 1.01701407 + \frac{0.73512298}{n} \tag{15}$$

and

$$B = \left(0.00233667\,n - 1.35418408 + \frac{5.75145813}{n}\right)\kappa' + 1.02999650 - \frac{0.72759508}{n} \tag{16}$$

are obtained by the approximation of experimental values summarized in graphs in Fig. 8 - Fig. 10. The approximating formulas provide a very good accuracy useful in practical filter design. If desired, the exact values κ', A and B can be

TABLE I
ALGORITHM FOR THE EVALUATION OF THE COEFFICIENTS $h_n(k)$.

given $\quad n$ (integer value), κ' (real value)

initialization

$$\alpha(2n) = \frac{1}{(1-\kappa'^2)^n}$$

$$\alpha(2n-2) = -(2n\kappa'^2+1)\,\alpha(2n)$$

$$\alpha(2n-4) = -\frac{4n+1+(n-1)(2n-1)\kappa'^2}{2n}\,\alpha(2n-2) - \frac{(2n+1)(n+1)\kappa'^2}{2n}\,\alpha(2n)$$

body
(*for* $k = n$ *down to* 3)

$$\alpha(2k-6) =$$

$$\Big\{ -\Big[3(n(n+2)-k(k-2))+2k-3+2(k-2)(2k-3)\kappa'^2\Big]\,\alpha(2k-4) - \Big[3(n(n+2)-(k-1)(k+1))+2(2k-1)+2k(2k-1)\kappa'^2\Big]\,\alpha(2k-2) - [n(n+2)-(k-1)(k+1)]\,\alpha(2k) \Big\} \;/\; [n(n+2)-(k-3)(k-1)]$$

(*end loop on* k)

integration

(*for* $k = 0$ *to* n) $\quad a(2k+1) = \frac{\alpha(2k)}{2k+1}$ $\quad$ (*end loop on* k)

impulse response $h_n(k)$

$$h_n(2n+1) = 0$$

(*for* $k = 0$ *to* n) $\quad h_n(2n+1\pm(2k+1)) = \frac{a(2k+1)}{2}$ $\quad$ (*end loop on* k)

obtained from (7) numerically (e.g. using the Matlab function $fminsearch$) by satisfying the equality (see Fig. 5)

$$Q(w_p) = \begin{cases} Q(1) & \text{if n is odd} \\ Q(w_{01}) & \text{if n is even} \end{cases} . \qquad (17)$$

The position of the local extremal value w_{01} (Fig. 5)

$$w_{01} = \sqrt{\kappa'^2 + (1-\kappa'^2)\cos^2\frac{\pi}{2n+1}} \qquad (18)$$

was introduced in [8]. The relation (17) guarantees the equiriple behaviour of the zero phase transfer function $Q(w)$.

VII. DESIGN OF THE ER HB FIR FILTER

The design procedure is as follows:

- Specify the ER HB FIR filter by the minimal passband frequency $\omega_p T$ and by the minimal attenuation in the stopband a_s [dB].
- Calculate the integer degree n of the generating polynomial (13).
- Calculate the real values κ' (14), A (15) and B (16).
- Evaluate the partial impulse responses $h_n(k)$ and $h_{n-1}(k)$ (Tab. I).
- Evaluate the final impulse response $h(k)$ (12) where the real norming factor $\mathcal{N}$ is

$$\mathcal{N} = \begin{cases} 2\,[\,A\,\mathcal{U}_n(1) + \mathrm{B}\,\mathcal{U}_{n-1}(1)\,] & \text{if n is even} \\ 2\,[\,A\,\mathcal{U}_n(w_{01}) + \mathrm{B}\,\mathcal{U}_{n-1}(w_{01})\,] & \text{if n is odd} \end{cases} . \qquad (19)$$

The Matlab source code of the design procedure is summarized in Appendix II.

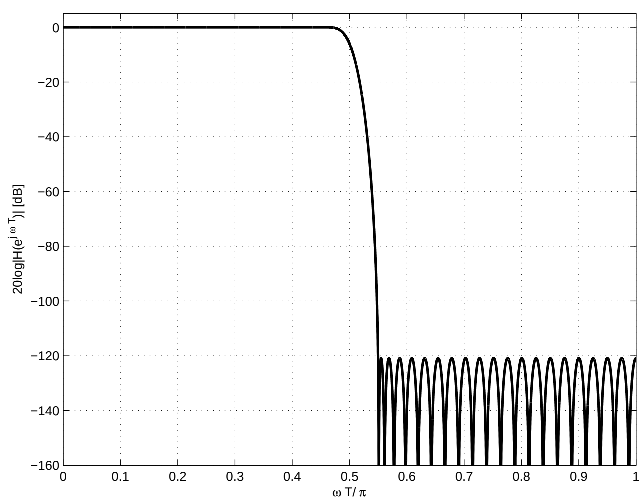

Fig. 11. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

VIII. EXAMPLES OF THE DESIGN

Example 1.
Design an ER HB FIR filter specified by the minimal passband frequency $\omega_p T = 0.45\pi$ *and by the minimal attenuation in the stopband* $a_s = -120$ dB.
Using formulas we get $n = 38.3856 \rightarrow 39$ (13), $\kappa' = 0.15571103$ (14), $A = 1.17117396$ (15), $B = 0.83763199$ (16) and $\mathcal{N} = -2747.96038544$ (19). The impulse response $h(k)$ (Tab. II) with the length $N = 159$ coefficients is evaluated using algorithm summarized in Tab. I and eq. (12). The actual values $\omega_{p\ act} T = 0.4502\pi$ and $a_{act} = -120.91$ dB satisfy the filter specification. The

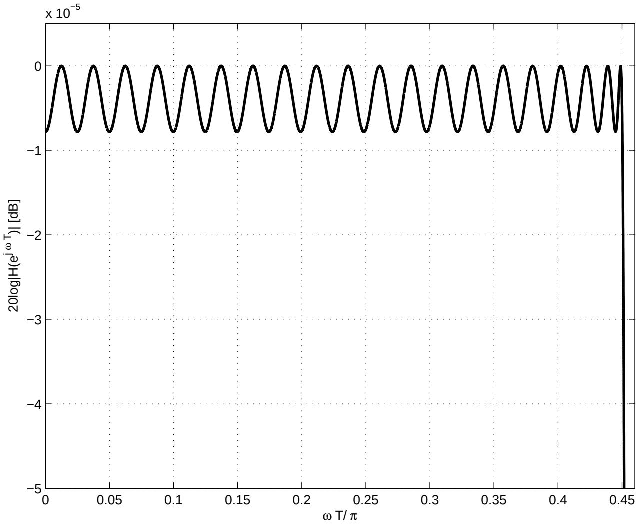

Fig. 12. Passband of the filter.

TABLE II
COEFFICIENTS OF THE IMPULSE RESPONSE.

k	h(k)	k	h(k)
0 , 158	-0.00000070	42 , 116	0.00231877
2 , 156	0.00000158	44 , 114	-0.00283354
4 , 154	-0.00000331	46 , 112	0.00344038
6 , 152	0.00000622	48 , 110	-0.00415347
8 , 150	-0.00001087	50 , 108	0.00498985
10 , 148	0.00001799	52 , 106	-0.00597048
12 , 146	-0.00002852	54 , 104	0.00712193
14 , 144	0.00004363	56 , 102	-0.00847897
16 , 142	-0.00006481	58 , 100	0.01008867
18 , 140	0.00009384	60 , 98	-0.01201717
20 , 138	-0.00013287	62 , 96	0.01436125
22 , 136	0.00018446	64 , 94	-0.01726924
24 , 134	-0.00025161	66 , 92	0.02098117
26 , 132	0.00033779	68 , 90	-0.02591284
28 , 130	-0.00044697	70 , 88	0.03285186
30 , 128	0.00058370	72 , 86	-0.04348979
32 , 126	-0.00075311	74 , 84	0.06223123
34 , 124	0.00096097	76 , 82	-0.10523903
36 , 122	-0.00121375	78 , 80	0.31802058
38 , 120	0.00151871	79	0.50000000
40 , 118	-0.00188398		

amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 11. The detailed view of its passband is shown in Fig. 12.

Example 2.
Design an ER HB FIR filter specified by the minimal passband frequency $\omega_p T = 0.475\pi$ and by the minimal attenuation in the stopband $a_s = -80$ dB.
We get $n = 50.2277 \rightarrow 51$ (13), $\kappa' = 0.07779493$ (14), $A = 1.10893402$ (15), $B = 0.92842531$ (16) and $\mathcal{N} = -226.43793850$ (19). The impulse response $h(k)$ (Tab. III) with the length $N = 207$ coefficients is evaluated using algorithm summarized in Tab. I and eq. (12). The actual values $\omega_{p\ act}T = 0.4752\pi$ and $a_{act} = -81.23$ dB satisfy the filter specification. The amplitude frequency response 20log$|H(e^{j\omega T})|$ [dB] of the filter is shown in Fig. 13. The detailed view of its passband is shown in Fig. 14.

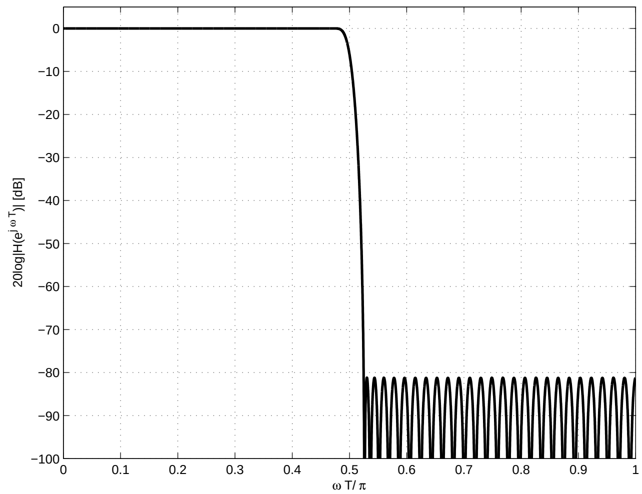

Fig. 13. Amplitude frequency response $20\log|H(e^{j\omega T})|$ [dB].

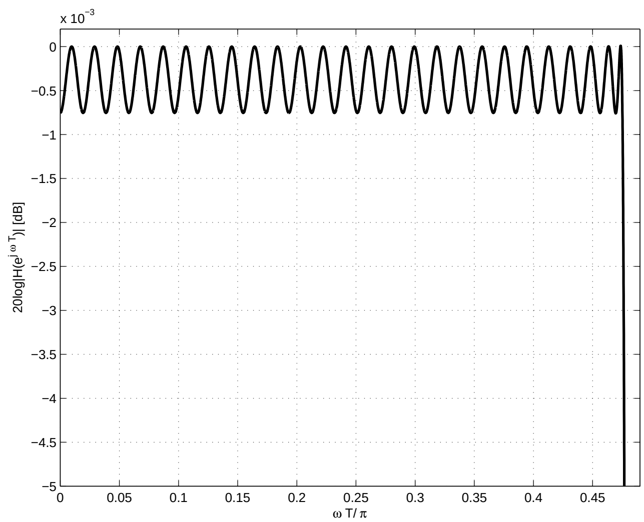

Fig. 14. Passband of the filter.

IX. ROBUSTNESS OF THE DESIGN

The recursive evaluation of the impulse response summarized in Tab. I is extremely fast and robust. Using the proposed procedure it is possible to evaluate the impulse response of the ER HB FIR filter with the length of thousands of coefficients within a fraction of a second. In order to demonstrate this robustness, let us design an ER HB FIR filter with strict specification, namely with the specified minimal passband frequency $\omega_p T = 0.495\pi$ and minimal attenuation in the stopband $a_s = -180$ dB. The length of the designed filter is $N = 2347$ coefficients. The amplitude frequency response $|H(e^{j\omega T})|$ of the filter is shown in Fig. 15.

X. CONCLUSIONS

The presented design procedure is useful in the design of equiripple halfband FIR filters. The designed filters are optimal in Chebyshev sense. The generating polynomial and

TABLE III
COEFFICIENTS OF THE IMPULSE RESPONSE.

k	h(k)	k	h(k)
0 , 206	-0.00003240	54 , 152	0.00279585
2 , 204	0.00002594	56 , 150	-0.00312880
4 , 202	-0.00003613	58 , 148	0.00349527
6 , 200	0.00004879	60 , 146	-0.00389883
8 , 198	-0.00006433	62 , 144	0.00434368
10 , 196	0.00008318	64 , 142	-0.00483480
12 , 194	-0.00010580	66 , 140	0.00537821
14 , 192	0.00013270	68 , 138	-0.00598122
16 , 190	-0.00016445	70 , 136	0.00665292
18 , 188	0.00020162	72 , 134	-0.00740467
20 , 186	-0.00024486	74 , 132	0.00825098
22 , 184	0.00029484	76 , 130	-0.00921066
24 , 182	-0.00035231	78 , 128	0.01030857
26 , 180	0.00041804	80 , 126	-0.01157828
28 , 178	-0.00049287	82 , 124	0.01306624
30 , 176	0.00057766	84 , 122	-0.01483857
32 , 174	-0.00067338	86 , 120	0.01699269
34 , 172	0.00078102	88 , 118	-0.01967820
36 , 170	-0.00090166	90 , 116	0.02313707
38 , 168	0.00103644	92 , 114	-0.02778764
40 , 166	-0.00118660	94 , 112	0.03442063
42 , 164	0.00135345	96 , 110	-0.04473116
44 , 162	-0.00153844	98 , 108	0.06312764
46 , 160	0.00174312	100 , 106	-0.10577606
48 , 158	-0.00196920	102 , 104	0.31817614
50 , 156	0.00221857	103	0.50000000
52 , 154	-0.00249333		

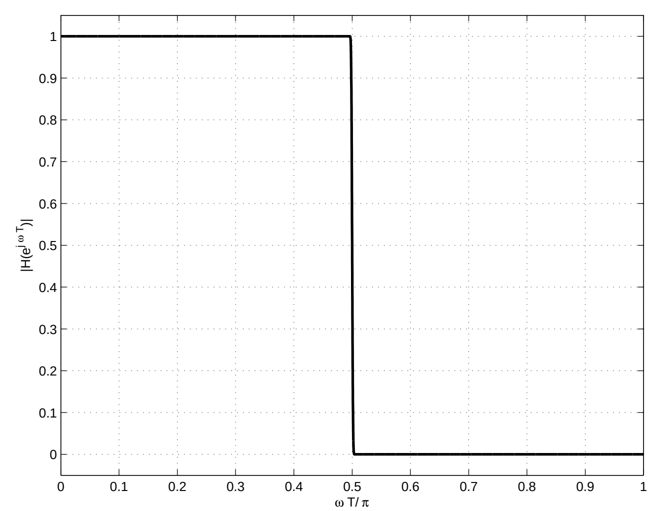

Fig. 15. Amplitude frequency response $|H(e^{j\omega T})|$.

the zero phase transfer function based on the Chebyshev polynomials of second kind illustrate the nature of the approximation problem. The degree formula indispensable for the filter design is presented. The strength of the proposed design method consists in the fact that the coefficients of the impulse response of the filter are straightforwardly evaluated using formulas from the filter specification. Because of its inherent determinism resulting from the non-numerical approach, the presented design procedure is useful in the adaptive digital signal processing as well. The enclosed Matlab source code is useful in the filter design.

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APPENDIX I

In the following considerations, the identities

$$2nU_{n-1}(w) = \frac{dU_n(w)}{dw} - \frac{dU_{n-2}(w)}{dw} \tag{20}$$

$$\frac{dU_{2n-1}(w)}{dw} = \sum_{k=0}^{n-1} 2(2k+1)U_{2k}(w) \tag{21}$$

$$\frac{dU_{2n}(w)}{dw} = \sum_{k=1}^{n} 2(2k)U_{2k-1}(w) \tag{22}$$

$$w\frac{dU_{2n-1}(w)}{dw} = \sum_{k=0}^{n-1}(2k+1)\left[U_{2k+1}(w) + U_{2k-1}(w)\right] \tag{23}$$

$$w\frac{dU_{2n}(w)}{dw} = \sum_{k=1}^{n} 2k\left[U_{2k}(w) + U_{2k-2}(w)\right] \tag{24}$$

are useful. For the three terms in the differential equation (10) we get the relations

$$\begin{aligned} w(w^2-\kappa'^2)\left[(1-w^2)\frac{d^2U_n(w)}{dw^2} - 3w\frac{dU_n(w)}{dw}\right] = \\ \sum_{k=0}^{n} -\alpha(2k)k(k+1)\frac{1}{2}\left[U_{2k+3}(w) + 3U_{2k+1}(w)\right. \\ \left. + 3U_{2k-1}(w) + U_{2k-3}(w)\right] \\ +\kappa'^2\alpha(2k)4k(k+1)\frac{1}{2}\left[U_{2k+1}(w) + U_{2k-1}(w)\right] \end{aligned} \tag{25}$$

$$\left[\kappa'^2(1-w^2)+2w^2(1-w^2)\right]\frac{dU_n(w)}{dw} = \sum_{k=0}^{n}\alpha(2k)\kappa'^2\left[(k+1)U_{2k-1}(w)-kU_{2k+1}(w)\right] + \alpha(2k)\frac{1}{2}\left[(k+1)(U_{2k+1}(w)+2U_{2k-1}(w)+U_{2k-3}(w)) - k(U_{2k+3}(w)+2U_{2k+1}(w)+U_{2k-1}(w))\right] \quad (26)$$

and

$$4w^3 n(n+2)U_n(w) = \sum_{k=0}^{n} n(n+2)\alpha(2k)\frac{1}{2}\left[U_{2k+3}(w)+3U_{2k+1} + 3U_{2k-1}+U_{2k-3}(w)\right]. \quad (27)$$

By collecting and summing of the coefficients belonging to the particular degree of the Chebyshev polynomial we get

$$U_{2k+3}(w): \frac{1}{2}\left[n(n+2)-k(k+2)\right]\alpha(2k) \quad (28)$$

$$U_{2k+1}(w): \frac{1}{2}\left[3(n(n+2)-k(k+2)) + 2k+1+2k(2k+1)\kappa'^2\right]\alpha(2k) \quad (29)$$

$$U_{2k-1}(w): \frac{1}{2}\left[3(n(n+2)-k(k+2))+2(2k+1) + 2(k+1)(2k+1)\kappa'^2\right]\alpha(2k) \quad (30)$$

$$U_{2k-3}(w): \frac{1}{2}\left[n(n+2)-(k-1)(k+1)\right]\alpha(2k)\ . \quad (31)$$

By manipulation of k in (28)-(30) we get

$$\frac{1}{2}\left[n(n+2)-(k-3)(k-1)\right]\alpha(2k-6) \quad (32)$$

$$\frac{1}{2}\left[3(n(n+2)-(k-2)k)+2k-3 + 2(k-2)(2k-3)\kappa'^2\right]\alpha(2k-4) \quad (33)$$

$$\frac{1}{2}\left[3(n(n+2)-(k-1)(k+1))+2(2k-1) + 2k(2k-1)\kappa'^2\right]\alpha(2k-2) \quad (34)$$

$$\frac{1}{2}\left[n(n+2)-(k-1)(k+1)\right]\alpha(2k)\ . \quad (35)$$

The initial values $\alpha(2n)$, $\alpha(2n-2)$ and $\alpha(2n-4)$ follow from (28)-(31). For $k=n$ we get from (28) the relation

$$\left[n(n+2)-n(n+2)\right]\alpha(2n)=0 \quad (36)$$

which is fulfilled for arbitrary value $\alpha(2n)$. Let us choose the value (Tab. I)

$$\alpha(2n)=\frac{1}{(1-\kappa'^2)^n}\ . \quad (37)$$

For $k=n-1$ we get from (28) and (29) the relation

$$(n(n+2)-(n-1)(n+1))\alpha(2n-2) + (2n+1+2n(2n+1)\kappa'^2)\alpha(2n)=0 \quad (38)$$

yielding the initial value $\alpha(2n-2)$ (Tab. I).

$$\alpha(2n-2)=-(2n\kappa'^2+1)\,\alpha(2n) \quad (39)$$

Finally, from (28)-(30) for $k=n-2$ we get

$$2n\alpha(2n-4) + (4n+1+(n-1)(2n-1)\kappa'^2)\alpha(2n-2) + (2n+1)((n+1)\kappa'^2+1)\alpha(2n)=0 \quad (40)$$

which yields the initial value $\alpha(2n-4)$ (Tab. I). The formula for the evaluation of the remaining coefficients $\alpha(2k-6)$ (body in Tab. I) follows from (32)-(35).

APPENDIX II

```
% Design of Equiripple Half-Band FIR Filter
%
clear, clf reset,
adB=-80;                                % Specifications for Example No. 2
omp=0.475*pi;
A00=-33.64775299940740; A01=-29.13196870512581;
A10= 18.18840663850262; A11 =18.54155180910656;
N=(adB-A10*omp-A00)/(A11*omp+A01); N=ceil(N);
k00=-0.00665856769717; k01= 1.57111377495119;
k10= 0.37221483652163; k11=-1.01927559802890;
k=(N*omp-k01*N-k00)/(k11*N+k10);
a12=0.01525753184125;   a11=0.03682344002622;   a10=9.24760314166335;
a01=1.01701406973534;   a00=0.73512297663750;
b12=0.00233666682716;   b11=-1.35418408482371;  b10=5.75145813400838;
b01=1.02999650170704;   b00=-0.72759508233144;
AA=(a12*N*N+a11*N+a10)*k+a01*N+a00; BB=(b12*N*N+b11*N+b10)*k+b01*N+b00;
h1=h_uarg(N,k); h2=h_uarg(N-1,k); h = h1_plus_h2(AA*h1,BB*h2);
w01=sqrt(k*k+(1-k*k)*(cos(pi/(2*N+1)))^2); om01=acos(-w01);
vals=abs(freqz(h,1,[om01 pi]));
if 2*floor(N/2)==N  extr=vals(2);  else   extr=-vals(1);  end
NN=1+2*(2*N+1);   N2=floor(NN/2);
hleft=h(1,1:N2);  hright=h(1,N2+2:NN);
h=[hleft/(2*extr) 0.5 hright/(2*extr)]; % Impulse Response
[H,om]=freqz(h,1,10000);
figure(1),
plot(om/pi,20*log10(abs(H))), grid on, axis([0 1 -100 5]),
xlabel('\omega T/ \pi'), ylabel('20log|H(e^{j \omega T})| [dB]'),

function [alfa,kc]=uarg(n,kc)
%
if n==0
alfa(2*n+1)=1/(1-kc*kc)^n;
elseif n==1
alfa(2*n+1)=1/(1-kc*kc)^n; alfa(2*n-1)=-(2*n*kc*kc+1)*alfa(2*n+1);
else
alfa=zeros(1,2*n+3);
%initialization
alfa(2*n+1)=1/(1-kc*kc)^n; alfa(2*n-1)=-(2*n*kc*kc+1)*alfa(2*n+1);
alfa(2*n-3)=-(4*n+1+kc*kc*(n-1)*(2*n-1))/2/n*alfa(2*n-1)-(2*n+1)*((n+1)*kc*kc+1)/2/n*alfa(2*n+1);
m=n;
% recursion
for j=m:-1:3
  c7=m*(m+2)-(j-3)*(j-1); c5=3*(m*(m+2)-(j-2)*j)+2*j-3 + 2*(j-2)*(2*j-3)*kc*kc;
  c3=3*(m*(m+2)-(j-1)*(j+1))+2*(2*j-1)+ 2*j*(2*j-1)*kc*kc; c1=m*(m+2)-(j-1)*(j+1);
  alfa(2*j-5)= - (c5*alfa(2*j-3) + c3*alfa(2*j-1)  + c1*alfa(2*j+1))/c7;
end
end

function h=h_uarg(m,k)
%
a=uarg(m,k);
ai=0;
for j=0:1:m,  ai(1+2*j)=a(1+2*j)/(1+2*j);  end
ai=[0 ai];
h=a2h(ai);
```