Safety by Construction: Well-behaved Scalable Systems

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***Abstract*—This paper presents a formal framework that provides construction principles for *well-behaved scalable systems*, such that starting with a prototype system satisfying a desired safety property result in a scalable system satisfying a corresponding safety property, called *scalable safety property*. With respect to different aspects of scalability, the focus of this work is on *property preserving structural scalability*. At that, we consider systems composed of a varying set of individual components where individual components of the same type behave in the same manner, which is characteristic for the type. The respective properties can rely on specific component types and a specific number of individual components but not on the specific individuality of the components. Well-behaved scalable systems are characterised by those systems, which fulfil such a kind of property if already one prototype system (depending on the property) fulfils that property. Sufficient conditions to specify a certain kind of basic well-behaved scalable systems are given and it is shown, how to construct more complex systems by the composition of several synchronisation conditions. Scalable safety properties can be used to express privacy policies as well as security and dependability requirements. It is demonstrated, how the parameterised problem of verifying such a property is reduced to a finite state problem for well-behaved scalable systems. The formal framework for well-behaved scalable systems is developed in terms of prefix closed formal languages and alphabetic language homomorphisms.**

Keywords-uniformly parameterised systems, monotonic parameterised systems, behaviour-abstraction, self-similarity of behaviour, privacy policies, scalable safety properties.

I. Introduction

This article is based on [1], where the concept of well-behaved scalable systems has been introduced. It is extended by extensive proofs of the theorems and the definition of *scalable safety properties* as well as their verification for well-behaved scalable systems. This is illustrated by a complex example, where several synchronisation conditions are composed.

Scalability is a desirable property of systems. However, the term scalability is often not clearly defined and thus it is difficult to characterise and understand systems with respect to their scalability properties [2]. In [3], four aspects of scalability are considered, i.e., load scalability, space scalability, space-time scalability, and structural scalability. In this paper, we focus on *structural scalability*, which is "the ability of a system to expand in a chosen dimension without major modifications to its architecture" [3]. Examples of systems that need to be highly scalable comprise grid computing architectures and cloud computing platforms [4], [5]. Usually, such systems consist of few different types of components and for each such type a varying set of individual components exists. Component types can be defined in such a granularity that individual components of the same type behave in the same manner, which is characteristic for the type. For example, a client-server system that is scalable consists of the component types *client* and *server* and several sets of individual clients as well as several sets of individual servers. Let us now call a choice of sets of individual components an *admissible choice of individual component sets*, iff for each component type exactly one set of individual components of that type is chosen. Then, a "scalable system" can be considered as a family of systems, whose elements are systems composed of a specific admissible choice of individual component sets.

For safety critical systems as well as for business critical systems, assuring the correctness is imperative. Formally, the dynamic behaviour of a discrete system can be described by the set of its possible sequences of actions. This way to model the behaviour is important, because it enables the definition of safety requirements as well as the verification of such properties, because for these purposes sequences of actions of the system have to be considered [6], [7], [8]. For short, we often will use the term *system* instead of *systems behaviour* if it does not generate confusions. With this focus, scalable systems are families of system behaviours, which are indexed by admissible choices of individual component sets. We call such families *parameterised systems*. In this paper, we define *well-behaved scalable systems* as a special class of parameterised systems and develop construction principles for such systems. The main goal for this definition is to achieve that well-behaved scalable systems fulfil certain kind of safety properties if already one prototype system (depending on the property) fulfils that property (cf. Section IV). To this end, construction principles for well-behaved scalable systems are *design principles for*

verifiability [9]. We give an example that demonstrates the significance of self-similarity for verification purposes and show that for well-behaved scalable systems scalable safety properties can be verified by finite state methods.

The main content of the paper can basically be divided into three parts. Besides the basic definitions, the first part (Section III and Section IV) comprises a characterisation of the systems under consideration and their fundamental properties. The second part (Section V and Section VI, enriched by the appendix) provides the formal framework for the construction of well-behaved systems. The last part (Section VII) provides a generic verification scheme for scalable safety properties and presents an example for its application. Concluding remarks and further research directions are given in Section VIII.

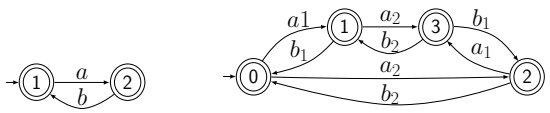
II. Related Work

Considering the behaviour-verification aspect, which is one of our motivations to formally define well-behaved scalable systems, there are some other approaches to be mentioned. An extension to the Murφ verifier to verify systems with replicated identical components through a new data type called RepetitiveID is presented in [10]. The verification is performed by explicit state enumeration in an abstract state space where states do not record the exact numbers of components. A typical application area of this tool are cache coherence protocols. The aim of [11] is an abstraction method through symmetry, which works also when using variables holding references to other processes. In [12], a methodology for constructing abstractions and refining them by analysing counter-examples is presented. The method combines abstraction, model-checking and deductive verification. A technique for automatic verification of parameterised systems based on process algebra CCS [13] and the logic modal mu-calculus [14] is presented in [15]. This technique views processes as property transformers and is based on computing the limit of a sequence of mu-calculus [14] formulas generated by these transformers. The above-mentioned approaches demonstrate that finite state methods combined with deductive methods can be applied to analyse parameterised systems. The approaches differ in varying amounts of user intervention and their range of application. A survey of approaches to combine model checking and theorem proving methods is given in [16]. Far reaching results in verifying parameterised systems by model checking of corresponding abstract systems are given in [17], [18]. It is well known that the general verification problem for parameterised systems is undecidable [19], [20]. To handle that problem, we present (a) a formal framework to specify parameterised systems in a restricted manner, and (b) construction principles for well-behaved scalable systems.

III. Characterisation of Scalable Systems

The behaviour L of a discrete system can be formally described by the set of its possible sequences of actions. Therefore, $L \subset \Sigma^*$ holds where Σ is the set of all actions of the system, and Σ^* (free monoid over Σ) is the set of all finite sequences of elements of Σ, including the empty sequence denoted by ε. This terminology originates from the theory of formal languages [21], where Σ is called the alphabet (not necessarily finite), the elements of Σ are called letters, the elements of Σ^* are referred to as words and the subsets of Σ^* as formal languages. Words can be composed: if u and v are words, then uv is also a word. This operation is called the *concatenation*; especially $\varepsilon u = u\varepsilon = u$. A word u is called a *prefix* of a word v if there is a word x such that $v = ux$. The set of all prefixes of a word u is denoted by $\mathrm{pre}(u)$; $\varepsilon \in \mathrm{pre}(u)$ holds for every word u. Formal languages, which describe system behaviour, have the characteristic that $\mathrm{pre}(u) \subset L$ holds for every word $u \in L$. Such languages are called *prefix closed*. System behaviour is thus described by prefix closed formal languages. Different formal models of the same system are partially ordered with respect to different levels of abstraction. Formally, abstractions are described by alphabetic language homomorphisms. These are mappings $h^* : \Sigma^* \longrightarrow \Sigma'^*$ with $h^*(xy) = h^*(x)h^*(y)$, $h^*(\varepsilon) = \varepsilon$ and $h^*(\Sigma) \subset \Sigma' \cup \{\varepsilon\}$. So, they are uniquely defined by corresponding mappings $h : \Sigma \longrightarrow \Sigma' \cup \{\varepsilon\}$. In the following, we denote both the mapping h and the homomorphism h^* by h. We consider a lot of alphabetic language homomorphisms. So, for simplicity we tacitly assume that a mapping between free monoids is an alphabetic language homomorphism if nothing contrary is stated. We now introduce a guiding example.

Example 1. *A server answers requests of a family of clients. The actions of the server are considered in the following. We assume with respect to each client that a request will be answered before a new request from this client is accepted. If the family of clients consists of only one client, then the automaton in Fig. 1(a) describes the system behaviour $S \subset \Sigma^*$, where $\Sigma = \{a, b\}$, the label a depicts the request, and b depicts the response.*



(a) Actions at a server with respect to a client (b) Two clients served concurrently by one server

Figure 1. Scalable client-server system

Example 2. *Fig. 1(b) now describes the system behaviour $S_{\{1,2\}} \subset \Sigma^*_{\{1,2\}}$ for two clients 1 and 2, under the*

assumption that the server handles the requests of different clients non-restricted concurrently.

For a parameter set I and $i \in I$ let $\Sigma_{\{i\}}$ denote pairwise disjoint copies of Σ. The elements of $\Sigma_{\{i\}}$ are denoted by a_i and $\Sigma_I := \bigcup_{i \in I} \Sigma_{\{i\}}$, where $\Sigma_{\{j\}} \cap \Sigma_{\{k\}} = \emptyset$ for $j \neq k$. The index i describes the bijection $a \leftrightarrow a_i$ for $a \in \Sigma$ and $a_i \in \Sigma_{\{i\}}$.

Example 3. *For $\emptyset \neq I \subset \mathbb{N}$ with finite I, let now $S_I \subset \Sigma_I^*$ denote the system behaviour with respect to the client set I. For each $i \in \mathbb{N}$ $S_{\{i\}}$ is isomorphic to S, and S_I consists of the non-restricted concurrent run of all $S_{\{i\}}$ with $i \in I$.*

It holds $S_{I'} \subset S_I$ for $I' \subset I$.

Let $\mathcal{I}_1$ denote the set of all finite non-empty subsets of $\mathbb{N}$ (the set of all possible clients). Then, the family $(S_I)_{I \in \mathcal{I}_1}$ is an example of a monotonic parameterised system.

If the example is extended to consider several servers, which are depicted by natural numbers, then, e.g.,

$$\mathcal{I}_2 := \{\mathring{I} \times \hat{I} \subset \mathbb{N} \times \mathbb{N} | \mathring{I} \neq \emptyset \neq \hat{I}, \text{ with } \mathring{I}, \hat{I} \text{ finite}\}$$

is a suitable parameter structure.

$\mathcal{I}_2$ used in the example above shows how the component structure of a system can be expressed by a parameter structure using Cartesian products of individual component sets. The following Definition 1 abstracts from the intuition of a component structure.

Definition 1 (parameter structure)**.** *Let N be a countable (infinite) set and $\emptyset \neq \mathcal{I} \subset \mathfrak{P}(N) \setminus \{\emptyset\}$. $\mathcal{I}$ is called a* parameter structure *based on N.*

For scalable systems it is obvious to assume that enlarging the individual component sets does not reduce the corresponding system behaviour. More precisely: let I and K be two arbitrary admissible choices of individual component sets, where each individual component set in I is a subset of the corresponding individual component set in K. If S_I and S_K are the corresponding systems' behaviours, then S_I is a subset of S_K. Families of systems with this property we call *monotonic parameterised systems*. The following definition formalises monotonic parameterised systems.

Definition 2 (monotonic parameterised system)**.** *Let $\mathcal{I}$ be a* parameter structure*. For each $I \in \mathcal{I}$ let $\mathcal{L}_I \subset \Sigma_I^*$ be a prefix closed language. If $\mathcal{L}_{I'} \subset \mathcal{L}_I$ for each $I, I' \in \mathcal{I}$ with $I' \subset I$, then $(\mathcal{L}_I)_{I \in \mathcal{I}}$ is a* monotonic parameterised system*.*

As we assume that individual components of the same type behave in the same manner, S_I and S_K are isomorphic (equal up to the names of the individual components), if I and K have the same cardinality. This property we call *uniform parameterisation*. With these notions we define *scalable systems* as *uniformly monotonic parameterised systems*. Monotonic parameterised systems, in which isomorphic subsets of parameter values describe isomorphic subsystems, we call *uniformly monotonic parameterised systems*.

Definition 3 (isomorphism structure)**.** *Let $\mathcal{I}$ be a parameter structure, $I, K \in \mathcal{I}$, and $\iota : I \to K$ a bijection, then let $\iota_K^I : \Sigma_I^* \to \Sigma_K^*$ the isomorphism defined by*

$$\iota_K^I(a_i) := a_{\iota(i)} \text{ for } a_i \in \Sigma_I.$$

For each $I, K \in \mathcal{I}$ let $\mathcal{B}(I,K) \subset K^I$ a set (possibly empty) of bijections. $\mathcal{B}_\mathcal{I} := (\mathcal{B}(I,K))_{(I,K) \in \mathcal{I} \times \mathcal{I}}$ is called an isomorphism structure *for $\mathcal{I}$.*

Definition 4 (scalable system)**.** *Let $(\mathcal{L}_I)_{I \in \mathcal{I}}$ a monotonic parameterised system and $\mathcal{B}_\mathcal{I} = (\mathcal{B}(I,K))_{(I,K) \in \mathcal{I} \times \mathcal{I}}$ an* isomorphism structure *for $\mathcal{I}$.*
$(\mathcal{L}_I)_{I \in \mathcal{I}}$ is called uniformly monotonic parameterised with respect to $\mathcal{B}_\mathcal{I}$ *iff*

$$\mathcal{L}_K = \iota_K^I(\mathcal{L}_I) \text{ for each } I, K \in \mathcal{I} \text{ and each } \iota \in \mathcal{B}(I,K).$$

Uniformly monotonic parameterised systems for short are called scalable systems*.*

Example 4. *Let $\mathcal{I} = \mathcal{I}_2$.*

$$\mathcal{B}^2(\mathring{I} \times \hat{I}, \mathring{K} \times \hat{K}) := \{\iota \in (\mathring{K} \times \hat{K})^{(\mathring{I} \times \hat{I})} \mid \text{it exist bijections } \mathring{\iota} : \mathring{I} \to \mathring{K} \text{ and } \hat{\iota} : \hat{I} \to \hat{K} \text{ with } \iota((r,s)) = (\mathring{\iota}(r), \hat{\iota}(s)) \text{ for each } (r,s) \in (\mathring{I} \times \hat{I})\}$$

for $\mathring{I} \times \hat{I} \in \mathcal{I}_2$ and $\mathring{K} \times \hat{K} \in \mathcal{I}_2$ defines an isomorphism structure *$\mathcal{B}^2_{\mathcal{I}_2}$.*

IV. Well-behaved Scalable Systems

To motivate our formalisation of *well-behaved*, we consider a typical security requirement of a scalable client-server system: Whenever two different clients cooperate with the same server then certain critical sections of the cooperation of one client with the server must not overlap with critical sections of the cooperation of the other client with the same server. If for example both clients want to use the same resource of the server for confidential purposes, then the allocation of the resource to one of the clients has to be completely separated from the allocation of this resource to the other client. More generally, the concurrent cooperation of one server with several clients has to be restricted by certain *synchronisation conditions* to prevent, for example, undesired race conditions.

According to this example, we focus on properties, which rely on specific component types and a specific number of individual components for these component types but not on the specific individuality of the individual components. Now, we want to achieve that a well

behaved scalable system fulfils such a kind of property if already one *prototype system* (depending on the property) fulfils that property. In our example, a prototype system consists of two specific clients and one specific server.

To formalise this desire, we consider arbitrary I and K as in the definition of monotonic parameterised system. Then we look at S_K from an abstracting point of view, where only actions corresponding to the individual components of I are considered. If the smaller subsystem S_I behaves like the abstracted view of S_K, then we call this property *self-similarity* or more precisely *self-similarity of scalable systems*, to distinguish our notion from geometric oriented notions [22] and organisational aspects [23] of self-similarity. In [7], it is shown that *self-similar uniformly monotonic parameterised systems* have the above desired property. Therefore, we define *well-behaved scalable systems* as self-similar uniformly monotonic parameterised systems. We now formally look at $\mathcal{L}_I$ from an abstracting point of view concerning a subset $I' \subset I$. The corresponding abstractions are formalised by the homomorphisms $\Pi^I_{I'} : \Sigma^*_I \to \Sigma^*_{I'}$.

Definition 5 (self-similar monotonic parameterised system)**.** *For $I' \subset I$ let $\Pi^I_{I'} : \Sigma^*_I \to \Sigma^*_{I'}$ with*

$$\Pi^I_{I'}(a_i) = \left\{ \begin{array}{rl} a_i \mid & a_i \in \Sigma_{I'} \\ \varepsilon \mid & a_i \in \Sigma_I \setminus \Sigma_{I'}. \end{array} \right.$$

A monotonic parameterised system $(\mathcal{L}_I)_{I\in\mathcal{I}}$ is called self-similar *iff $\Pi^I_{I'}(\mathcal{L}_I) = \mathcal{L}_{I'}$ for each $I, I' \in \mathcal{I}$ with $I' \subset I$.*

Definition 6 (well-behaved scalable system)**.** *Self-similar scalable systems for short are called* well-behaved scalable systems*.*

A fundamental construction principle for systems satisfying several constraints is intersection of system behaviours. This emphasises the importance of the following theorem.

Theorem 1 (intersection theorem)**.** *Let $\mathcal{I}$ be a parameter structure, $\mathcal{B}_\mathcal{I}$ an isomorphism structure for $\mathcal{I}$, and $T \neq \emptyset$.*

i) *Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a monotonic parameterised system, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a monotonic parameterised system.*
ii) *Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a scalable system with respect to $\mathcal{B}_\mathcal{I}$, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$.*
iii) *Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ for each $t \in T$ be a self-similar monotonic parameterised system, then $(\bigcap_{t\in T} \mathcal{L}^t_I)_{I\in\mathcal{I}}$ is a self-similar monotonic parameterised system.*

Proof of Theorem 1 (i)–(iii):

Proof of (i): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ a monotonic parameterised system for each $t \in T$, then $\mathcal{L}^t_{I'} \subset \mathcal{L}^t_I$ for $t \in T$, $I, I' \in \mathcal{I}$, and $I' \subset I$. This implies

$$\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \bigcap_{t\in T} \mathcal{L}^t_I,$$

and thus (i).

Proof of (ii): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ an scalable system with respect to $(\mathcal{B}(I,K))_{(I,K)\in\mathcal{I}\times\mathcal{I}}$ for each $t \in T$, then $\iota^I_K(\mathcal{L}^t_I) = \mathcal{L}^t_K$ for $t \in T$, I, $K \in \mathcal{I}$, and $\iota \in \mathcal{B}(I,K)$.

Because all ι^I_K are isomorphisms,

$$\iota^I_K(\bigcap_{t\in T} \mathcal{L}^t_I) = \bigcap_{t\in T} \iota^I_K(\mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_K,$$

which proves (ii).

Proof of (iii): Let $(\mathcal{L}^t_I)_{I\in\mathcal{I}}$ a self-similar monotonic parameterised system for each $t \in T$. For $I, I' \in \mathcal{I}$ with $I' \subset I$ holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I) \subset \bigcap_{t\in T} \Pi^I_{I'}(\mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \bigcap_{t\in T} \mathcal{L}^t_I. \quad (1)$$

Because $\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \Sigma^*_{I'}$ holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_{I'}) = \bigcap_{t\in T} \mathcal{L}^t_{I'}.$$

Together with the second inclusion from (1) it follows

$$\bigcap_{t\in T} \mathcal{L}^t_{I'} \subset \Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I).$$

Because of the first part of (1) now holds

$$\Pi^I_{I'}(\bigcap_{t\in T} \mathcal{L}^t_I) = \bigcap_{t\in T} \mathcal{L}^t_{I'},$$

which proves (iii). ■

Weak additional assumptions for well-behaved scalable systems imply that such systems are characterised by parametrisation of one well-defined minimal prototype system. More precisely:

Definition 7 (minimal prototype system)**.** *Let $\mathcal{I}$ be a parameter structure based on N. For $I \in \mathcal{I}$ and $n \in N$ let $\tau^I_n : \Sigma^*_I \to \Sigma^*$ the homomorphisms given by*

$$\tau^I_n(a_i) = \left\{ \begin{array}{rl} a \mid & a_i \in \Sigma_{I\cap\{n\}} \\ \varepsilon \mid & a_i \in \Sigma_{I\setminus\{n\}} \end{array} \right. .$$

For a singleton index set $\{n\}$, $\tau^{\{n\}}_n : \Sigma^*_{\{n\}} \to \Sigma^*$ is an isomorphism and for each $n \in I \in \mathcal{I}$ holds

$$\Pi^I_{\{n\}} = (\tau^{\{n\}}_n)^{-1} \circ \tau^I_n. \quad (2)$$

If now $(\mathcal{L}_I)_{I\in\mathcal{I}}$ is a well-behaved scalable system with respect to $(\mathcal{B}(I,K))_{(I,K)\in\mathcal{I}\times\mathcal{I}}$ with $\{n\} \in \mathcal{I}$ for $n \in I \in \mathcal{I}$ and $\mathcal{B}(I,K) \neq \emptyset$ for all singleton I and K, then because of (2) holds

$$\mathcal{L}_I \subset \bigcap_{n\in I} (\tau^I_n)^{-1}(L) \text{ for each } I \in \mathcal{I},$$

where $L = \tau_n^{\{n\}}(\mathcal{L}_{\{n\}})$ *for each* $n \in \bigcup_{I\in\mathcal{I}} I$.

L is called the minimal prototype system *of* $(\mathcal{L}_I)_{I\in\mathcal{I}}$.

Definition 8 (behaviour-family $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ generated by the minimal prototype system L and the parameter structure $\mathcal{I}$)**.** *Let $\emptyset \neq L \subset \Sigma^*$ be prefix closed, $\mathcal{I}$ a parameter structure, and*

$$\dot{\mathcal{L}}(L)_I := \bigcap_{i\in I} (\tau_i^I)^{-1}(L) \text{ for } I \in \mathcal{I}.$$

The systems $\dot{\mathcal{L}}(L)_I$ consist of the "non-restricted concurrent run" of all systems $(\tau_i^{\{i\}})^{-1}(L) \subset \Sigma_{\{i\}}^*$ with $i \in I$. Because $\tau_i^{\{i\}} : \Sigma_{\{i\}}^* \to \Sigma^*$ are isomorphisms, $(\tau_i^{\{i\}})^{-1}(L)$ are pairwise disjoint copies of L.

Theorem 2 (simplest well-behaved scalable systems)**.** *$(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ is a well-behaved scalable system with respect to each isomorphism structure for $\mathcal{I}$ based on N and*

$$\dot{\mathcal{L}}(L)_I = \bigcap_{i\in N} (\tau_i^I)^{-1}(L) \text{ for each } I \in \mathcal{I}.$$

The proof of this theorem is given in the appendix.

V. Construction of Well-behaved Systems by Restriction of Concurrency

Now, we show how to construct well-behaved systems by restricting concurrency in the behaviour-family $\dot{\mathcal{L}}$. In Example 3, holds $S_I = \dot{\mathcal{L}}(S)_I$ for $I \in \mathcal{I}_1$. If, in the given example, the server needs specific resources for the processing of a request, then - on account of restricted resources - an non-restricted concurrent processing of requests is not possible. Thus, restrictions of concurrency in terms of synchronisation conditions are necessary. One possible but very strong restriction is the requirement that the server handles the requests of different clients in the same way as it handles the requests of a single client, namely, on the request follows the response and vice versa. This synchronisation condition can be formalised with the help of S and the homomorphisms Θ^I as shown in the following example.

Example 5. *Restriction of concurrency on account of restricted resources: one "task" after another. All behaviours with respect to $i \in I$* influence *each other. Let*

$$\bar{S}_I := S_I \cap (\Theta^I)^{-1}(S) = \bigcap_{i\in I} (\tau_i^I)^{-1}(S) \cap (\Theta^I)^{-1}(S)$$

for $I \in \mathcal{I}_1$, where generally, for each index set I, $\Theta^I : \Sigma_I^* \to \Sigma^*$ is defined by $\Theta^I(a_i) := a$, for $i \in I$ and $a \in \Sigma$.

From the automaton in Fig. 1(b), it is evident that $\bar{S}_{\{1,2\}}$ will be accepted by the automaton in Fig. 2(a).

Given an arbitrary $I \in \mathcal{I}_1$, then $\bar{S}_I$ is accepted by an automaton with state set $\{0\} \cup I$ and state transition relation given by Fig. 2(b) for each $i \in I$.

(a) Automaton accepting $\bar{S}_{\{1,2\}}$ (b) Automaton accepting $\bar{S}_I$

Figure 2. Automata accepting $\bar{S}_{\{1,2\}}$ and $\bar{S}_I$

From this automaton, it is evident that $(\bar{S}_I)_{I\in\mathcal{I}_1}$ is a well-behaved scalable system, with respect to each isomorphism structure $\mathcal{B}_{\mathcal{I}_1}$ for $\mathcal{I}_1$.

Example 6. *A restriction of concurrency in the extended example where a family of servers is involved is more complicated than in the case of $(\bar{S}_I)_{I\in\mathcal{I}_1}$. The reason for that is that in the simple example the restriction of concurrency can be formalised by a restricting influence of the actions with respect to all parameter values (i.e., the entire Σ_I). When considering the restriction of concurrency in the extended example, the actions influence each other only with respect to the parameter values, which are bound to the same server.*

Let the first component of the elements from $\mathbb{N}\times\mathbb{N}$ in the parameter structure $\mathcal{I}_2$ denote the server, then the actions from $\Sigma_{\{r\}\times\hat{I}}$ influence each other for given $r \in \mathring{I}$ with $\mathring{I}\times\hat{I} \in \mathcal{I}$ and thus restrict the concurrency.

For the formalisation of this restriction of concurrency, we now consider the general case of monotonic parameterised systems $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$. As already observed in (2), for each well-behaved scalable system $(\mathcal{L}_I)_{I\in\mathcal{I}}$ there exists (under weak preconditions) a system $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ with $\mathcal{L}_I \subset \dot{\mathcal{L}}(L)_I$ for each $I \in \mathcal{I}$, where $L = \tau_n^{\{n\}}(\mathcal{L}_{\{n\}})$ for each $n \in I \in \mathcal{I}$. Moreover, in context of Definition 8 it was observed that $\dot{\mathcal{L}}(L)_I$ consists of the non-restricted concurrent run of pairwise disjoint copies of L.

In conjunction, this shows that an adequate restriction of concurrency in $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ can lead to the construction of well-behaved scalable systems. Therefore, the restricting influence of actions with respect to specific parameter values described above shall now be formalised.

Definition 9 (influence structure)**.** *Let $T \neq \emptyset$ and $\mathcal{I}$ a parameter structure. For each $I \in \mathcal{I}$ and $t \in T$ a* sphere of influence *is specified by $E(t,I) \subset I$. The family*

$$\mathcal{E}_{\mathcal{I}} = (E(t,I))_{(t,I)\in T\times\mathcal{I}}$$

is called influence structure *for $\mathcal{I}$ indexed by T.*

The non-restricted concurrent run of the pairwise disjoint copies of L will now be restricted in the following way: For each $t \in T$ the runs of all copies k with $k \in E(t,I)$ influence each other independently of the specific values of $k \in E(t,I)$. With respect to our extended example (several servers) with $\mathcal{I}_2$, the spheres of influence $E(t,I)$

are generalisations of the sets $\{r\} \times \hat{I}$, where $I = \mathring{I} \times \hat{I}$ and $t = (r,s) \in \mathring{I} \times \hat{I}$.

Generally, for each $t \in T$ the intersection

$$\dot{\mathcal{L}}(L)_I \cap (\tau^I_{E(t,I)})^{-1}(V) \tag{3}$$

formalises the restriction of the non-restricted concurrent run of the copies of L within $\dot{\mathcal{L}}(L)_I$ by the mutual influence of each element of $E(t,I)$.

Definition 10 (behaviour of influence and influence homomorphisms)**.** *In* (3)*, the* behaviour of influence V *is a prefix closed language $V \subset \Sigma^*$, and for $I, I' \subset N$ the homomorphism $\tau^I_{I'} : \Sigma^*_I \to \Sigma^*$ is defined by:*

$$\tau^I_{I'}(a_i) = \begin{cases} a \mid & a_i \in \Sigma_{I \cap I'} \\ \varepsilon \mid & a_i \in \Sigma_{I \setminus I'} \end{cases} .$$

The homomorphisms $\tau^I_{E(t,I)}$ are called the influence homomorphisms *of $\mathcal{E}_\mathcal{I}$.*

Definition 11 (behaviour-family $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ generated by the minimal prototype system L, the influence structure $\mathcal{E}_\mathcal{I}$, and the behaviour of influence V)**.** *Because the restriction* (3) *shall hold for all $t \in T$, the restricted systems $\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I$ are defined by the prefix closed languages*

$$\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I := \dot{\mathcal{L}}(L)_I \cap \bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V) \text{ for } I \in \mathcal{I}.$$

Definition 11 shows how synchronisation requirements for the systems $\dot{\mathcal{L}}(L)_I$ can be formalised by influence structures and behaviour of influence in a very general manner. Since, similar to the well-behaved scalable systems $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$, in the systems $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ each $\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_{\{i\}}$ shall be isomorphic to L for each $\{i\} \in \mathcal{I}$, $V \supset L$ has to be assumed. Therefore, in general we assume for systems $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ that $V \supset L \neq \emptyset$. Note that $\tau^I_{I'}$ are generalisations of τ^I_n and Θ^I, because

$$\tau^I_n = \tau^I_{\{n\}} \text{ and } \Theta^I = \tau^I_I = \tau^I_N$$

for each $I \subset N$ and $n \in N$.

Further requirements, which assure that $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ are well-behaved scalable systems, will now be given with respect to $\mathcal{E}_\mathcal{I}$, $\mathcal{B}_\mathcal{I}$, L and V. Assuming $T = N$ and $\varepsilon \in V$ the scalability property is assured by the following technical requirements for $\mathcal{E}_\mathcal{I}$ and $\mathcal{B}_\mathcal{I}$:

Theorem 3 (construction condition for scalable systems)**.** *Let $\mathcal{I}$ be a parameter structure based on N, $\mathcal{E}_\mathcal{I} = (E(n,I))_{(n,I)\in N\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$, and let $\mathcal{B}_\mathcal{I} = (\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$. Let $\varepsilon \in V \subset \Sigma^*$, for each $I \in \mathcal{I}$ and $n \in N$ let $E(n,I) = \emptyset$, or it exists an $i_n \in I$ with $E(n,I) = E(i_n,I)$, and for each $(I,I') \in \mathcal{I}\times\mathcal{I}, \iota \in \mathcal{B}(I,I')$ and $i \in I$ holds*

$$\iota(E(i,I)) = E(\iota(i),I').$$

Let $E(t,I') = E(t,I) \cap I'$ for each $t \in T$ and $I, I' \in \mathcal{I}, I' \subset I$. Then $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$ and

$$\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in I} (\tau^I_{E(n,I)})^{-1}(V)).$$

The proof of this theorem is given in the appendix.

Example 7. *Let $\mathcal{I}$ be a parameter structure based on N, and for $I \in \mathcal{I}$ let $\bar{E}(i,I) := I$ for $i \in N$.*

$$\bar{\mathcal{E}}_\mathcal{I} := (\bar{E}(i,I))_{(i,I)\in N\times\mathcal{I}} \text{ satisfies the assumptions of Theorem 3 for each isomorphism structure } \mathcal{B}_\mathcal{I}. \tag{4}$$

It holds $(\Theta^I)^{-1}(V) = (\tau^I_{\bar{E}(i,I)})^{-1}(V)$ for each $i \in N, I \in \mathcal{I}$, and $V \subset \Sigma^*$.
Therefore, $\mathcal{L}(L,\bar{\mathcal{E}}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap (\Theta^I)^{-1}(V)$ for $I \in \mathcal{I}$.
Especially, $\bar{S}_I = \mathcal{L}(S,\bar{\mathcal{E}}_{\mathcal{I}_1},S)_I$ for each $I \in \mathcal{I}_1$.

Example 8. *For the parameter structure $\mathcal{I}_2$, and for $\mathring{I} \times \hat{I} \in \mathcal{I}_2$ let*

$$E^2((\mathring{n},\hat{n}),\mathring{I}\times\hat{I}) := \begin{cases} \{\mathring{n}\}\times\hat{I} \mid & \mathring{n} \in \mathring{I} \\ \emptyset \mid & \mathring{n} \in \mathbb{N} \setminus \mathring{I} \end{cases} .$$

$$\mathcal{E}^2_{\mathcal{I}_2} := (E^2((\mathring{n},\hat{n}),\mathring{I}\times\hat{I}))_{((\mathring{n},\hat{n}),\mathring{I}\times\hat{I})\in(\mathbb{N}\times\mathbb{N})\times\mathcal{I}_2} \tag{5}$$

satisfies the assumptions of Theorem 3 for the isomorphism structure $\mathcal{B}^2_{\mathcal{I}_2}$.
$(\mathcal{L}(S,\mathcal{E}^2_{\mathcal{I}_2},S)_I)_{I\in\mathcal{I}_2}$ is the formalisation of the extended example (several servers) with restricted concurrency.

In order to extend Theorem 3 with respect to self-similarity, an additional assumption is necessary. This is demonstrated by the following counter-example.

Example 9. *Let $G \subset \{\mathrm{a},\mathrm{b},\mathrm{c}\}^*$ the prefix closed language, which is accepted by the automaton Fig. 3(a). Let $H \subset \{\mathrm{a},\mathrm{b},\mathrm{c}\}^*$ the prefix closed language, which is accepted by the automaton in Fig. 3(b). It holds $\emptyset \neq G \subset H$ but $(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_I)_{I\in\mathcal{I}_1}$ is not self-similar, e.g.,*

$$\Pi^{\{1,2,3\}}_{\{2,3\}}(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}}) \neq (\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{2,3\}}$$

because

$$\mathrm{a_1b_1a_2a_3} \in \mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}},$$

and hence

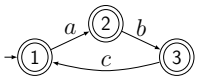
$$\mathrm{a_2a_3} \in \Pi^{\{1,2,3\}}_{\{2,3\}}(\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{1,2,3\}}),$$

but

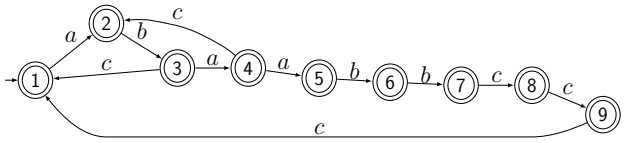
$$\mathrm{a_2a_3} \notin (\mathcal{L}(G,\bar{\mathcal{E}}_{\mathcal{I}_1},H)_{\{2,3\}}.$$

Definition 12 (closed under shuffle projection)**.** *Let $L, V \subset \Sigma^*$. V is closed under shuffle projection with respect to L, iff*

$$\Pi^{\mathbb{N}}_K[(\bigcap_{n\in\mathbb{N}} (\tau^{\mathbb{N}}_n)^{-1}(L)) \cap (\Theta^{\mathbb{N}})^{-1}(V)] \subset (\Theta^{\mathbb{N}})^{-1}(V) \tag{6}$$



(a) Automaton accepting G



(b) Automaton accepting H

Figure 3. Counterexample

for each subset $\emptyset \neq K \subset \mathbb{N}$. *We abbreviate this by* $\mathrm{SP}(L,V)$.

Remark 1. *It can be shown that in* $\mathrm{SP}(L,V)$ $\mathbb{N}$ *can be replaced by each countable infinite set.*

Remark 2. *If* L *and* V *are prefix closed with* $\emptyset \neq L \subset V$, *then it is easy to show that* $\mathrm{SP}(L,V)$ *follows from self-similarity of* $(\mathcal{L}(L,\bar{\mathcal{E}}_{\mathcal{I}_1},V)_I)_{I\in\mathcal{I}_1}$.

With Definition 12 we are now able to formulate our main result for constructing well-behaved scalable systems defined by a single synchronisation condition.

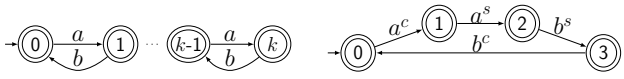
Theorem 4 (construction condition for well-behaved scalable systems)**.** *By the assumptions of Theorem 3 together with* $\mathrm{SP}(L,V)$

$$(\mathcal{L}(L,\mathcal{E}_{\mathcal{I}},V)_I)_{I\in\mathcal{I}}$$

is a well-behaved scalable system.

The proof of this theorem is given in the appendix.

Example 10. *For* $k \in \mathbb{N}$ *let the prefix closed language* $F_k \subset \{a,b\}^*$ *be defined by the automaton in Fig. 4(a).*



(a) Automaton for $F_k \subset \{a,b\}^*$ (b) One client, one server

Figure 4. Automata at different abstraction levels

With respect to Example 1, $F_1 = S$ *holds. It can be shown that* $\mathrm{SP}(S,F_k)$ *holds for each* $k \in \mathbb{N}$. *With Theorem 4 now, by* (4) *and* (5) *especially, the systems* $(\mathcal{L}(L,\bar{\mathcal{E}}_{\mathcal{I}_1},F_k)_I)_{I\in\mathcal{I}_1}$ *and* $(\mathcal{L}(L,\mathcal{E}^2_{\mathcal{I}_2},F_k)_I)_{I\in\mathcal{I}_2}$ *are uniformly monotonic parameterised and self-similar. These are the two cases of the guiding example where the concurrency of the execution of requests is bounded by* k.

Theorem 4 is the main result for constructing well-behaved scalable systems defined by a single synchronisation condition. The following section shows how this result together with the Intersection Theorem can be used for constructing more complex well-behaved scalable systems defined by the combination of several synchronisation conditions, as for example well-behaved scalable systems consisting of several component types.

VI. Well-behaved Scalable Systems Generated by a Family of Influence Structures

Up to now, the examples were considered at an abstraction level, which takes into account only the actions of the server (or the servers, depending on the choice of the parameter structure).

Example 11. *For a finer abstraction level, which additionally takes into account the actions of the clients, a finer alphabet, e.g.,* $\check{\Sigma} = \{a^c,b^c,a^s,b^s\}$ *and a prefix closed language* $\check{S} \subset \check{\Sigma}^*$ *is needed, which, e.g., is defined by the automaton in Fig. 4(b).*

In general, a finer relation for system specifications at different abstraction levels can be defined by alphabetic language homomorphisms.

Definition 13 (abstractions)**.** *In general, let* $\check{L} \subset \check{\Sigma}^*$ *and* $L \subset \Sigma^*$ *be prefix closed languages. We call* $\check{L}$ finer than L *or* L coarser than $\check{L}$ *iff an alphabetic homomorphism* $\nu : \check{\Sigma}^* \to \Sigma^*$ *exists with* $\nu(\check{L}) = L$.

For each parameter structure $\mathcal{I}$ and $I \in \mathcal{I}$ ν defines an homomorphism $\nu^I : \check{\Sigma}_I^* \to \Sigma_I^*$ by $\nu^I(a_i) := (\nu(a))_i$ for $a \in \check{\Sigma}$ and $i \in I$, where $(\varepsilon)_i := \varepsilon$.

Let now $\mathcal{E}_{\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by N, which is the base of $\mathcal{I}$, and let $\emptyset \neq L \subset V \subset \Sigma^*$ be prefix closed. $(\mathcal{L}(L,\mathcal{E}_{\mathcal{I}},V)_I)_{I\in\mathcal{I}}$ induces a restriction of the concurrency in $(\dot{\mathcal{L}}(\check{L})_I)_I$ by the intersections

$$\dot{\mathcal{L}}(\check{L})_I \cap (\nu^I)^{-1}[\bigcap_{t\in N}(\tau^I_{E(t,I)})^{-1}(V)] \text{ for each } I \in \mathcal{I}. \quad (7)$$

If $\check{\tau}^I_{I'} : \check{\Sigma}_I^* \to \check{\Sigma}^*$ is defined analogously to $\tau^I_{I'}$ for $I, I' \subset N$ by

$$\check{\tau}^I_{I'}(a_i) = \begin{cases} a \mid & a \in \check{\Sigma} \text{ and } i \in I \cap I' \\ \varepsilon \mid & a \in \check{\Sigma} \text{ and } i \in I \setminus I' \end{cases},$$

then holds $\tau^I_{I'} \circ \nu^I = \nu \circ \check{\tau}^I_{I'}$. From this it follows that

$$(\nu^I)^{-1}[\bigcap_{t\in N}(\tau^I_{E(t,I)})^{-1}(V)] = \bigcap_{t\in N}(\check{\tau}^I_{E(t,I)})^{-1}(\nu^{-1}(V))$$

and therewith

$$\dot{\mathcal{L}}(\check{L})_I \cap (\nu^I)^{-1}[\bigcap_{t\in N}(\tau^I_{E(t,I)})^{-1}(V)] = \mathcal{L}(\check{L},\mathcal{E}_{\mathcal{I}},\nu^{-1}(V))_I \quad (8)$$

for each $I \in \mathcal{I}$. Notice that $\emptyset \neq \check{L} \subset \nu^{-1}(V) \subset \check{\Sigma}^*$ is prefix closed. So if $(\mathcal{L}(L,\mathcal{E}_{\mathcal{I}},V)_I)_{I\in\mathcal{I}}$ fulfils the assumptions of Theorem 3, then this holds for $(\mathcal{L}(\check{L},\mathcal{E}_{\mathcal{I}},\nu^{-1}(V))_I)_{I\in\mathcal{I}}$ as well and the system

$$(\dot{\mathcal{L}}(\check{L})_I \cap (\nu^I)^{-1}[\bigcap_{t\in N}(\tau^I_{E(t,I)})^{-1}(V)])_{I\in\mathcal{I}}, \quad (9)$$

which is defined by the intersections (7), is a scalable system. The following general theorem can be used to prove self-similarity of such systems.

Theorem 5 (inverse abstraction theorem)**.** *Let $\varphi : \Sigma^* \to \Phi^*$ be an alphabetic homomorphism and $W, X \subset \Phi^*$, then*

$$\mathrm{SP}(W,X) \text{ implies } \mathrm{SP}(\varphi^{-1}(W), \varphi^{-1}(X)).$$

Proof of Theorem 5:
Let K be a non-empty set. Each alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ defines a homomorphism $\varphi^K : \Sigma_K^* \to \Phi_K^*$ by

$$\varphi^K(a_n) := (\varphi(a))_n \text{ for } a_n \in \Sigma_K, \text{ where } (\varepsilon)_n = \varepsilon.$$

If $\bar{\tau}_n^K : \Phi_K^* \to \Phi$ and $\bar{\Theta}^K : \Phi_K^* \to \Phi$ are defined analogous to τ_n^K and Θ^K, then

$$\varphi \circ \tau_n^K = \bar{\tau}_n^K \circ \varphi^K, \text{ and } \varphi \circ \Theta^K = \bar{\Theta}^K \circ \varphi^K. \tag{10}$$

Let now N be an infinite countable set. Because of (10), for $W, X \subset \Phi^*$

$$\begin{aligned}&(\bigcap_{n\in N} (\tau_n^N)^{-1}(\varphi^{-1}(W))) \cap (\Theta^N)^{-1}(\varphi^{-1}(X))\\ &= (\varphi^N)^{-1}[(\bigcap_{n\in N} (\bar{\tau}_n^N)^{-1}(W)) \cap (\bar{\Theta}^N)^{-1}(X)].\end{aligned} \tag{11}$$

Because of $\varphi^K(w) = \varphi^N(w)$ for $w \in \Sigma_K^* \subset \Sigma_N^*$ and $\emptyset \neq K \subset N$

$$(\varphi^K)^{-1}(Z) \subset (\varphi^N)^{-1}(Z) \text{ for } Z \subset \Phi_K^*. \tag{12}$$

If now $\mathrm{SP}(W,X)$, and

$$\Pi_K^N[(\varphi^N)^{-1}(Y)] = (\varphi^K)^{-1}(\bar{\Pi}_K^N[Y]) \tag{13}$$

for $Y \subset \Phi_N^*$ and $\emptyset \neq K \subset N$, where $\bar{\Pi}_K^N : \Phi_N^* \to \Phi_K^*$ is defined analogous to Π_K^N, then follows (with (10) - (13))

$$\begin{aligned}&\Pi_K^N[(\bigcap_{n\in N} (\tau_n^N)^{-1}(\varphi^{-1}(W))) \cap (\Theta^N)^{-1}(\varphi^{-1}(X))]\\ &= (\varphi^K)^{-1}(\bar{\Pi}_K^N[(\bigcap_{n\in N} (\bar{\tau}_n^N)^{-1}(W)) \cap (\bar{\Theta}^N)^{-1}(X)])\\ &\subset (\varphi^K)^{-1}((\bar{\Theta}^N)^{-1}(X)) \subset (\varphi^N)^{-1}((\bar{\Theta}^N)^{-1}(X))\\ &= (\Theta^N)^{-1}(\varphi^{-1}(X)).\end{aligned} \tag{14}$$

With (14)

$$\mathrm{SP}(\varphi^{-1}(W), \varphi^{-1}(X)) \text{ follows from } \mathrm{SP}(W,X), \tag{15}$$

if (13) holds.

It remains to show (13). For the proof of (13) it is sufficient to prove

$$\Pi_K^N((\varphi^N)^{-1}(y)) = (\varphi^K)^{-1}(\bar{\Pi}_K^N(y)) \tag{16}$$

for each $y \in \Phi_N^*$, because of

$$\Pi_K^N((\varphi^N)^{-1}(Y) = \bigcup_{y\in Y} \Pi_K^N((\varphi^N)^{-1}(y))$$

and

$$(\varphi^K)^{-1}(\bar{\Pi}_K^N(Y)) = \bigcup_{y\in Y} (\varphi^K)^{-1}(\bar{\Pi}_K^N(y)).$$

Here, for $f : A \to B$ and $b \in B$ we use the convention

$$f^{-1}(b) = f^{-1}(\{b\}).$$

With $Y = \{y\}$ (16) is also necessary for (13), and so it is equivalent to (13).

Definition 14 ((general) projection)**.** *For arbitrary alphabets Δ and Δ' with $\Delta' \subset \Delta$ general projections $\pi_{\Delta'}^{\Delta} : \Delta^* \to \Delta'^*$ are defined by*

$$\pi_{\Delta'}^{\Delta}(a) := \begin{cases} a \mid & a \in \Delta' \\ \varepsilon \mid & a \in \Delta \setminus \Delta' \end{cases}. \tag{17}$$

In this terminology the projections

$$\Pi_K^N : \Sigma_N^* \to \Sigma_K^* \text{ and } \bar{\Pi}_K^N : \Phi_N^* \to \Phi_K^*$$

considered until now are special cases, which we call *parameter-projections*. It holds

$$\Pi_K^N = \pi_{\Sigma_K}^{\Sigma_N} \text{ and } \bar{\Pi}_K^N = \pi_{\Phi_K}^{\Phi_N}. \tag{18}$$

Because of the different notations, in general we just use the term *projection* for both cases.

We now consider the equation (16) for the special case, where $\varphi : \Sigma^* \to \Phi^*$ is a projection, that is $\varphi = \pi_{\Phi}^{\Sigma}$ with $\Phi \subset \Sigma$. In this case also $\varphi^N : \Sigma_N^* \to \Phi_N^*$ is a projection, with

$$\varphi^N = \pi_{\Phi_N}^{\Sigma_N}. \tag{19}$$

Lemma 1 (projection-lemma)**.**
Let Δ be an alphabet, $\Delta' \subset \Delta$, $\Gamma \subset \Delta$ and $\Gamma' = \Delta' \cap \Gamma$, then

$$\pi_{\Delta'}^{\Delta}((\pi_{\Gamma}^{\Delta})^{-1}(y)) = (\pi_{\Gamma'}^{\Delta'})^{-1}(\pi_{\Delta'}^{\Delta}(y))$$

for each $y \in \Gamma^*$.

Proof: Let $y \in \Gamma^*$. We show

$$\pi_{\Gamma'}^{\Delta'}(\pi_{\Delta'}^{\Delta}(z)) = \pi_{\Delta'}^{\Delta}(y) \text{ for each } z \in (\pi_{\Gamma}^{\Delta})^{-1}(y) \tag{20}$$

and we show that

$$\begin{aligned}&\text{for each } u \in (\pi_{\Gamma'}^{\Delta'})^{-1}(\pi_{\Delta'}^{\Delta}(y)) \text{ there exists a}\\ &v \in (\pi_{\Gamma}^{\Delta})^{-1}(y) \text{ such that } \pi_{\Delta'}^{\Delta}(v) = u.\end{aligned} \tag{21}$$

From (20) it follows that

$$\pi_{\Delta'}^{\Delta}((\pi_{\Gamma}^{\Delta})^{-1}(y)) \subset (\pi_{\Gamma'}^{\Delta'})^{-1}(\pi_{\Delta'}^{\Delta}(y))$$

and from (21) it follows that

$$(\pi_{\Gamma'}^{\Delta'})^{-1}(\pi_{\Delta'}^{\Delta}(y)) \subset \pi_{\Delta'}^{\Delta}((\pi_{\Gamma}^{\Delta})^{-1}(y)),$$

which in turn proves Lemma 1.

Proof of (20): By definition of π_{Γ}^{Δ}, $\pi_{\Gamma'}^{\Delta'}$ and $\pi_{\Delta'}^{\Delta}$ follows

$$\pi_{\Gamma'}^{\Delta'}(\pi_{\Delta'}^{\Delta}(z)) = \pi_{\Delta'}^{\Delta}(\pi_{\Gamma}^{\Delta}(z))$$

for each $z \in \Delta^*$ and therewith (20).

Proof of (21) by induction on $y \in \Gamma^*$:

Induction base. Let $y = \varepsilon$, then $u \in (\Delta' \setminus \Gamma')^*$ for each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$. From this follows

$$\pi^{\Delta}_{\Delta'}(v) = u \text{ with } v := u \in (\pi^{\Delta}_{\Gamma})^{-1}(\varepsilon).$$

Induction step. Let $y = \mathring{y}\hat{y}$ with $\mathring{y} \in \Gamma^*$ and $\hat{y} \in \Gamma$.
Case 1: $\hat{y} \in \Gamma \setminus \Gamma' = \Gamma \cap (\Delta \setminus \Delta')$
Then

$$(\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y)) = (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(\mathring{y})).$$

By induction hypothesis then for each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$ it exists $\mathring{v} \in (\pi^{\Delta}_{\Gamma})^{-1}(\mathring{y})$ such that $\pi^{\Delta}_{\Delta'}(\mathring{v}) = u$.
With $v := \mathring{v}\hat{y}$ holds $\pi^{\Delta}_{\Gamma}(\mathring{v}\hat{y}) = \mathring{y}\hat{y} = y$ and hence

$$v \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \text{ and } \pi^{\Delta}_{\Delta'}(v) = \pi^{\Delta}_{\Delta'}(\mathring{v}) = u.$$

Case 2: $\hat{y} \in \Gamma' \subset \Delta'$
Then $\pi^{\Delta}_{\Delta'}(y) = \pi^{\Delta}_{\Delta'}(\mathring{y})\hat{y}$. Therefore, each $u \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(y))$ can be departed into $u = \mathring{u}\hat{y}\hat{u}$ with $\mathring{u} \in (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Delta}_{\Delta'}(\mathring{y}))$ and $\hat{u} \in (\Delta' \setminus \Gamma')^*$.
By induction hypothesis then exists $\mathring{v} \in (\pi^{\Delta}_{\Gamma})^{-1}(\mathring{y})$ such that $\pi^{\Delta}_{\Delta'}(\mathring{v}) = \mathring{u}$.
With $v := \mathring{v}\hat{y}\hat{u}$ holds $\pi^{\Delta}_{\Gamma}(\mathring{v}\hat{y}\hat{u}) = \mathring{y}\hat{y} = y$ and hence

$$v \in (\pi^{\Delta}_{\Gamma})^{-1}(y) \text{ and } \pi^{\Delta}_{\Delta'}(v) = \pi^{\Delta}_{\Delta'}(\mathring{v})\hat{y}\hat{u} = \mathring{u}\hat{y}\hat{u} = u.$$

This completes the proof of (21). ■

For $y \in \Gamma^*$ holds

$$\pi^{\Delta}_{\Delta'}(y) = \pi^{\Gamma}_{\Delta' \cap \Gamma}(y) = \pi^{\Gamma}_{\Gamma'}(y).$$

Therewith, from Lemma 1 follows

$$\pi^{\Delta}_{\Delta'}((\pi^{\Delta}_{\Gamma})^{-1}(y)) = (\pi^{\Delta'}_{\Gamma'})^{-1}(\pi^{\Gamma}_{\Gamma'}(y)) \text{ for each } y \in \Gamma^*. \tag{22}$$

For $\emptyset \neq K \subset N, \Phi \subset \Sigma, \Delta := \Sigma_N, \Delta' := \Sigma_K$, and $\Gamma := \Phi_N$ holds $\Gamma' = \Delta' \cap \Gamma = \Phi_K$.

Assuming $\varphi = \pi^{\Sigma}_{\Phi}$, which implies $\varphi^K = \pi^{\Sigma_K}_{\Phi_K}$, then from (22) (with (18) and (19)), follows

$$\Pi^N_K((\varphi^N)^{-1}(y)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y))$$

for $y \in \Phi^*_N$, and so (16). With this,

premise (13) is fulfilled for (15), when φ is a projection, (23)

which proves Theorem 5 for projections.

Definition 15 (strictly alphabetic homomorphism)**.** *Let Σ, Φ alphabets, and $\varphi : \Sigma^* \to \Phi^*$ a homomorphism. Then φ is called* alphabetic*, if $\varphi(\Sigma) \subset \Phi \cup \{\varepsilon\}$, and φ is called* strictly alphabetic*, if $\varphi(\Sigma) \subset \Phi$.*

Each alphabetic homomorphism $\varphi : \Sigma^* \to \Phi^*$ is the composition of a projection with a strictly alphabetic homomorphism, more precisely,

$$\varphi = \varphi_S \circ \pi^{\Sigma}_{\varphi^{-1}(\Phi) \cap \Sigma}, \tag{24}$$

where $\varphi_S : (\varphi^{-1}(\Phi) \cap \Sigma)^* \to \Phi^*$ is the strictly alphabetic homomorphism defined by

$$\varphi_S(a) := \varphi(a) \text{ for } a \in \varphi^{-1}(\Phi) \cap \Sigma.$$

For $W, X \subset \Phi^*$ and $\varphi : \Sigma^* \to \Phi^*$ alphabetic (24) implies

$$\begin{aligned} \varphi^{-1}(W) =& (\pi^{\Sigma}_{\varphi^{-1}(\Phi) \cap \Sigma})^{-1}((\varphi_S)^{-1}(W)) \text{ and} \\ \varphi^{-1}(X) =& (\pi^{\Sigma}_{\varphi^{-1}(\Phi) \cap \Sigma})^{-1}((\varphi_S)^{-1}(X)). \end{aligned} \tag{25}$$

Now with (23) and (25) it remains to prove Theorem 5 for strictly alphabetic homomorphisms. This will be done by Lemma 2, which proves (16) for strictly alphabetic homomorphisms.

Lemma 2. *Let $\varphi : \Sigma^* \to \Phi^*$ be a strictly alphabetic homomorphism, then for all $y \in \Phi^*_N$ and $\emptyset \neq K \subset N$ holds*

$$\Pi^N_K((\varphi^N)^{-1}(y)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y)).$$

Proof: Proof by induction on y.
Induction basis: $y = \varepsilon$
Because φ^N is strictly alphabetic

$$(\varphi^N)^{-1}(\varepsilon) = \{\varepsilon\} \text{ and so } \Pi^N_K((\varphi^N)^{-1}(\varepsilon)) = \{\varepsilon\}.$$

For the same reason

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(\varepsilon)) = (\varphi^K)^{-1}(\varepsilon) = \{\varepsilon\}.$$

Induction step: Let $y = y'a_t$ with $a_t \in \Phi_N$, where $a \in \Phi$ and $t \in N$. Because φ^N is alphabetic, it holds

$$(\varphi^N)^{-1}(y'a_t) = ((\varphi^N)^{-1}(y'))((\varphi^N)^{-1}(a_t)),$$

and so

$$\Pi^N_K((\varphi^N)^{-1}(y'a_t)) = \Pi^N_K((\varphi^N)^{-1}(y'))\Pi^N_K((\varphi^N)^{-1}(a_t)).$$

Also holds

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(y'a_t)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y'))(\varphi^K)^{-1}(\bar{\Pi}^N_K(a_t)).$$

According to the induction hypothesis, it holds

$$\Pi^N_K((\varphi^N)^{-1}(y')) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(y')).$$

Therefore, it remains to show

$$\Pi^N_K((\varphi^N)^{-1}(a_t)) = (\varphi^K)^{-1}(\bar{\Pi}^N_K(a_t)).$$

Case 1: $t \notin K$
Because φ^N is strictly alphabetic, it holds $(\varphi^N)^{-1}(a_t) \subset \Sigma_{\{t\}}$, so

$$\Pi^N_K((\varphi^N)^{-1}(a_t)) = \{\varepsilon\}.$$

Additionally holds $\bar{\Pi}^N_K(a_t) = \varepsilon$, and therewith

$$(\varphi^K)^{-1}(\bar{\Pi}^N_K(a_t)) = \{\varepsilon\},$$

because φ^K is strictly alphabetic.

Case 2: $t \in K$
Because φ^N is strictly alphabetic, it holds

$$(\varphi^N)^{-1}(a_t) = \{b_t \in \Sigma_{\{t\}} | \varphi(b) = a\},$$

and therewith

$$\Pi_K^N((\varphi^N)^{-1}(a_t)) = \{b_t \in \Sigma_{\{t\}} | \varphi(b) = a\}.$$

$\bar{\Pi}_K^N(a_t) = a_t$ and therewith

$$(\varphi^K)^{-1}(\bar{\Pi}_K^N(a_t)) = \{b_t \in \Sigma_{\{t\}} | \varphi(b) = a\},$$

because φ^K is strictly alphabetic. This completes the proof of Lemma 2. ∎

This completes the proof of Theorem 5. ∎

Generally, by (6), $\mathrm{SP}(\nu^{-1}(L), \nu^{-1}(V))$ implies $\mathrm{SP}(X, \nu^{-1}(V))$ for each $X \subset \nu^{-1}(L)$. Especially $\mathrm{SP}(\check{L}, \nu^{-1}(V))$ is implied by $\mathrm{SP}(L, V)$ on account of Theorem 5. So, by Theorem 5, if $(\mathcal{L}(L, \mathcal{E}_\mathcal{I}, V)_I)_{I \in \mathcal{I}}$ fulfils the assumptions of Theorem 4, then

$$(\mathcal{L}(\check{L}, \mathcal{E}_\mathcal{I}, \nu^{-1}(V))_I)_{I \in \mathcal{I}} = (\dot{\mathcal{L}}(\check{L})_I \cap (\nu^I)^{-1}[\bigcap_{t \in N} (\tau^I_{E(t,I)})^{-1}(V)])_{I \in \mathcal{I}} \quad (26)$$

is a well-behaved scalable system.

The intersections in (7) formalise restriction of concurrency in $(\dot{\mathcal{L}}(\check{L})_I)_{I \in \mathcal{I}}$ under *one specific aspect* (one specific synchronisation condition), which is given by ν, $\mathcal{E}_\mathcal{I}$, and V. Restriction of concurrency under *several aspects* (several synchronisation conditions) is formalised by the intersections

$$\dot{\mathcal{L}}(\check{L})_I \cap \bigcap_{r \in R} (\nu_r^I)^{-1}[\bigcap_{t \in N} (\tau^I_{E_r(t,I)})^{-1}(V_r)] \quad (27)$$

for each $I \in \mathcal{I}$ based on N, $R \neq \emptyset$ is the index set of the aspects. The family of aspects restricting concurrency is given by

- a family $(\nu_r)_{r \in R}$ of *alphabetic homomorphisms* $\nu_r : \check{\Sigma}^* \to \Sigma^{(r)*}$ for $r \in R$,
- a family $(\mathcal{E}^r_\mathcal{I})_{r \in R}$ of *influence structures* $\mathcal{E}^r_\mathcal{I} = (E_r(t,I))_{(t,I) \in N \times \mathcal{I}}$ indexed by N for $r \in R$, and
- a family $(V_r)_{r \in R}$ of *influence behaviours* $V_r \subset \Sigma^{(r)*}$ for $r \in R$.

From (8) it follows now

$$\dot{\mathcal{L}}(\check{L})_I \cap \bigcap_{r \in R} (\nu_r^I)^{-1}[\bigcap_{t \in N} (\tau^I_{E_r(t,I)})^{-1}(V_r)] = \bigcap_{r \in R} \mathcal{L}(\check{L}, \mathcal{E}^r_\mathcal{I}, \nu_r^{-1}(V_r))_I$$

for each $I \in \mathcal{I}$. Because of the intersection theorem, the uniform monotonic parameterisation and self-similarity of the system

$$(\dot{\mathcal{L}}(\check{L})_I \cap \bigcap_{r \in R} (\nu_r^I)^{-1}[\bigcap_{t \in N} (\tau^I_{E_r(t,I)})^{-1}(V_r)])_{I \in \mathcal{I}}$$

can be inferred from respective properties of the systems

$$(\mathcal{L}(\check{L}, \mathcal{E}^r_\mathcal{I}, \nu_r^{-1}(V_r))_I)_{I \in \mathcal{I}} \text{ for each } r \in R.$$

Using (9) and (26), this requires the verification of the assumptions of Theorem 4 for

$$(\mathcal{L}(\nu_r(\check{L}), \mathcal{E}^r_\mathcal{I}, V_r)_I)_{I \in \mathcal{I}} \text{ for each } r \in R. \quad (28)$$

If $\mathcal{I}$ is based on $N = \bigtimes_{k \in K} N_k$, where K is a finite set and each N_k is countable, then along the lines of $\mathcal{I}_2$, a parameter structure $\mathcal{I}_K$ can be defined for this domain. Such $\mathcal{I}_K$ fit for systems consisting of finitely many component types. Each subset $K' \subset K$ with $\emptyset \neq K' \neq K$ defines a bijection between N and $(\bigtimes_{k \in K'} N_k) \times (\bigtimes_{k \in K \setminus K'} N_k)$. By this bijection, for each of these K' an influence structure $\mathcal{E}^{K'}_{\mathcal{I}_K}$ is defined like $\mathcal{E}^2_{\mathcal{I}_2}$ that satisfies the assumptions of Theorem 3 with respect to an isomorphism structure $\mathcal{B}^K_{\mathcal{I}_K}$ defined like $\mathcal{E}^2_{\mathcal{I}_2}$.

VII. Scalable Safety Properties

We will now give an example that demonstrates the significance of self-similarity for verification purposes and then present a generic verification scheme for scalable safety properties.

Example 12. *We consider a system of servers, each of them managing a resource, and clients, which want to use these resources. We assume that as a means to enforce a given* privacy policy *a server has to manage its resource in such a way that no client may access this resource during it is in use by another client (*privacy requirement*). This may be required to ensure* anonymity *in such a way that clients and their actions on a resource cannot be linked by an observer.*

We formalise this system at an abstract level, where a client may perform the actions $\mathrm{a^c}$ (send a request), $\mathrm{b^c}$ (receive a permission) and $\mathrm{c^c}$ (send a free-message), and a server may perform the corresponding actions $\mathrm{a^s}$ (receive a request), $\mathrm{b^s}$ (send a permission) and $\mathrm{c^s}$ (receive a free-message). The automaton $\mathbb{L}$ depicted in Fig. 5 describes the cooperation of one client and one server.

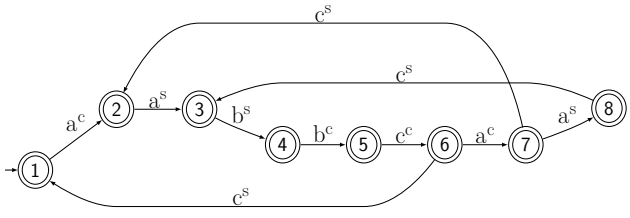


Figure 5. Automaton $\mathbb{L}$

We now formalise the parameterised cooperation $(\mathcal{C}_J)_{J \in \mathcal{I}}$ according to the description in Section VI.

$$\mathcal{C}_J = \dot{\mathcal{L}}(\check{L})_J \cap \bigcap_{r \in R} (\nu_r^J)^{-1}[\bigcap_{t \in N} (\tau^J_{E_r(t,J)})^{-1}(V_r)].$$

Because $(\mathcal{C}_J)_{J\in\mathcal{I}}$ involves several clients as well as several servers, let $\mathcal{I} := \mathcal{I}_2$, $N := \mathbb{N}\times\mathbb{N}$, and $\mathcal{B}_{\mathcal{I}} := \mathcal{B}^2_{\mathcal{I}_2}$, where the first component refers to the client and the second component refers to the server. Now $\check{L}$ is the prefix closed language that is accepted by the automaton $\mathbb{L}$.

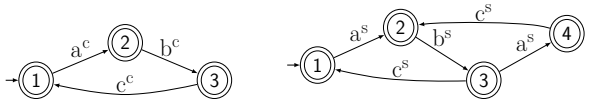
For the examined example we assume that both clients and servers are subject to constaints with respect to processing several cooperations. Thus, two aspects of constaints are considered, therefore: $R := \{c,s\}$, $\Sigma^{(c)} := \{\mathrm{a}^{\mathrm{c}},\mathrm{b}^{\mathrm{c}},\mathrm{c}^{\mathrm{c}}\}$, $\Sigma^{(s)} := \{\mathrm{a}^{\mathrm{s}},\mathrm{b}^{\mathrm{s}},\mathrm{c}^{\mathrm{s}}\}$, $\check{\Sigma} = \Sigma^{(c)}\dot\cup\Sigma^{(s)}$, $\nu_c : \check{\Sigma}^* \to \Sigma^{(c)*}$ with

$$\nu_c(x) := \begin{cases} x \mid & x \in \Sigma^{(c)} \\ \varepsilon \mid & x \in \Sigma^{(s)} \end{cases},$$

and $\nu_s : \check{\Sigma}^* \to \Sigma^{(s)*}$ with

$$\nu_s(x) := \begin{cases} x \mid & x \in \Sigma^{(s)} \\ \varepsilon \mid & x \in \Sigma^{(c)} \end{cases}.$$

$\nu_c(\check{L})$ and $\nu_s(\check{L})$ now describe the behaviour of a client respectively a server in the cooperation of a client with a server. $\nu_c(\check{L})$ and $\nu_s(\check{L})$ are accepted by the automata in Fig. 6(a) and Fig. 6(b).



(a) Automaton accepting $\nu_c(\check{L})$ (b) Automaton accepting $\nu_s(\check{L})$

Figure 6. Client and server behaviour in the cooperation

These automata show that in $\nu_c(\check{L})$ the "phase" $\mathrm{a}^{\mathrm{c}}\mathrm{b}^{\mathrm{c}}\mathrm{c}^{\mathrm{c}}$ can happen repeatedly and in $\nu_s(\check{L})$ two instances of the "phase" $\mathrm{a}^{\mathrm{s}}\mathrm{b}^{\mathrm{s}}\mathrm{c}^{\mathrm{s}}$ can run partly concurrently.

We now assume that this restriction of concurrency shall also hold for the parameterised system. This restriction is then given by the definitions $V_c := \nu_c(\check{L})$ and $V_s := \nu_s(\check{L})$ with an appropriate choice of influence structures.

Because for each client respectively server all cooperations with all servers respectively clients influence each other, let now according to Example 8, for $I\times K \in \mathcal{I}_2$ and $(i,k) \in \mathbb{N}\times\mathbb{N}$:

$$E^c((i,k),I\times K) := \begin{cases} \{i\}\times K \mid & i \in I \\ \emptyset \mid & i \in \mathbb{N}\setminus I \end{cases},$$

$$E^s((i,k),I\times K) := \begin{cases} I\times\{k\} \mid & k \in K \\ \emptyset \mid & k \in \mathbb{N}\setminus K \end{cases},$$

$$\mathcal{E}^c_{\mathcal{I}_2} := (E^c((i,k),I\times K))_{((i,k),I\times K)\in(\mathbb{N}\times\mathbb{N})\times\mathcal{I}_2}, \text{ and}$$

$$\mathcal{E}^s_{\mathcal{I}_2} := (E^s((i,k),I\times K))_{((i,k),I\times K)\in(\mathbb{N}\times\mathbb{N})\times\mathcal{I}_2}.$$

As in Example 8, both influence structures satisfy the assumptions of Theorem 3 for the isomorphism structure $\mathcal{B}^2_{\mathcal{I}_2}$. Therefore, $(\mathcal{L}(\nu_c(\check{L}),\mathcal{E}^c_{\mathcal{I}_2},\nu_c(\check{L}))_J)_{J\in\mathcal{I}_2}$ and $(\mathcal{L}(\nu_s(\check{L}),\mathcal{E}^s_{\mathcal{I}_2},\nu_s(\check{L}))_J)_{J\in\mathcal{I}_2}$ are scalable systems. Because of (28) now $(\mathcal{C}_J)_{J\in\mathcal{I}_2}$ is a well-behaved scalable system if $\mathrm{SP}(\nu_c(\check{L}),\nu_c(\check{L}))$ and $\mathrm{SP}(\nu_s(\check{L}),\nu_s(\check{L}))$ hold.

In [24], sufficient conditions are given for a property equivalent to $\mathrm{SP}(U,V)$. These can be proven for both examples. A comprehensive and more general method for verification of $\mathrm{SP}(U,V)$ is subject of a forthcoming paper.

Considering b^{c} as the begin action and c^{c} as the end action with respect to accessing a resource, the privacy requirement for each $\mathcal{C}_J$ with $J = I\times K \in \mathcal{I}_2$ can be formalised by the following condition (29).

Let $i,i' \in I$, $i \neq i'$, $k \in K$ and

$$\mu^{I\times K}_{<i,i',k>} : \Sigma^*_{I\times K} \to \{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)},\mathrm{b}^{\mathrm{c}}{}_{(i',k)}\}^* \text{ with}$$

$$\mu^{I\times K}_{<i,i',k>}(x) := \begin{cases} x \mid x \in \{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)},\mathrm{b}^{\mathrm{c}}{}_{(i',k)}\} \\ \varepsilon \mid x \in \Sigma_{I\times K}\setminus\{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)},\mathrm{b}^{\mathrm{c}}{}_{(i',k)}\}. \end{cases}$$

Condition: For each $i,i' \in I$, $i \neq i'$ and $k \in K$ holds

$$\mu^{I\times K}_{<i,i',k>}(\mathcal{C}_{I\times K}) \cap \Sigma^*_{\{i,i'\}\times\{k\}}\mathrm{b}^{\mathrm{c}}{}_{(i,k)}\mathrm{b}^{\mathrm{c}}{}_{(i',k)} = \emptyset. \quad (29)$$

For $i,i' \in I$, $i \neq i'$, and $k \in K$ let

$$\rho_{<i,i',k>} : \Sigma^*_{\{i,i'\}\times\{k\}} \to \{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)},\mathrm{b}^{\mathrm{c}}{}_{(i',k)}\}^*$$

be defined by

$$\rho_{<i,i',k>}(x) := \begin{cases} x \mid x \in \{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)},\mathrm{b}^{\mathrm{c}}{}_{(i',k)}\} \\ \varepsilon \mid x \in \Sigma_{\{i,i'\}\times\{k\}}\setminus\{\mathrm{b}^{\mathrm{c}}{}_{(i,k)},\mathrm{c}^{\mathrm{c}}{}_{(i,k)}, \\ \qquad\qquad \mathrm{b}^{\mathrm{c}}{}_{(i',k)}\} \end{cases},$$

then

$$\mu^{I\times K}_{<i,i',k>} = \rho_{<i,i',k>} \circ \Pi^{I\times K}_{\{i,i'\}\times\{k\}}.$$

Hence,

$$\mu^{I\times K}_{<i,i',k>}(\mathcal{C}_{I\times K}) = \rho_{<i,i',k>}(\mathcal{C}_{\{i,i'\}\times\{k\}}) \quad (30)$$

because $(\mathcal{C}_{I\times K})_{I\times K\in\mathcal{I}_2}$ is a well-behaved scalable system.

Let

$$\iota_{<i,i',k>} : \Sigma^*_{\{i,i'\}\times\{k\}} \to \Sigma^*_{\{1,2\}\times\{1\}}$$

be the isomorphism defined by

$$\iota_{<i,i',k>}(x) := \begin{cases} (\tau^{\{1\}\times\{1\}}_{(1,1)})^{-1}(\tau^{\{i\}\times\{k\}}_{(i,k)}(x)) \mid x \in \\ \qquad\qquad \Sigma_{\{i\}\times\{k\}} \\ (\tau^{\{2\}\times\{1\}}_{(2,1)})^{-1}(\tau^{\{i'\}\times\{k\}}_{(i',k)}(x)) \mid x \in \\ \qquad\qquad \Sigma_{\{i'\}\times\{k\}} \end{cases}.$$

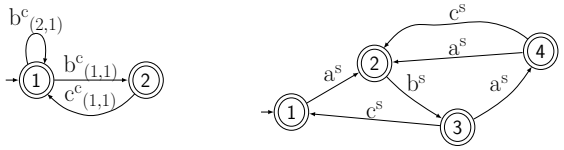
Then

$$\iota_{<i,i',k>} \in \{\iota^{\{i,i'\}\times\{k\}}_{\{1,2\}\times\{k\}} | \iota \in \mathcal{B}^2(\{i,i'\}\times\{k\},\{1,2\}\times\{1\})\}$$

(cf. Example 4), and therefore

$$\iota_{<i,i',k>}(\mathcal{C}_{\{i,i'\}\times\{k\}}) = \mathcal{C}_{\{1,2\}\times\{1\}}, \quad (31)$$

because $(\mathcal{C}_{I\times K})_{I\times K\in\mathcal{I}_2}$ is a scalable system.



(a) Minimal automaton of $\rho_{<1,2,1>}(\mathcal{C}_{\{1,2\}\times\{1\}})$ (b) Automaton accepting $\nu'_s(\check{L})$

Figure 8. Minimal automaton and counter example

Now, by (30), (31), and

$$\rho_{<i,i',k>} = \iota^{-1}_{<i,i',k>} \circ \rho_{<1,2,1>} \circ \iota_{<i,i',k>},$$

$\mathcal{C}_{I\times K}$ fulfils the privacy requirement (29) for each $I \times K \subset \mathcal{I}_2$ iff

$$\rho_{<1,2,1>}(\mathcal{C}_{\{1,2\}\times\{1\}}) \cap \Sigma^*_{\{1,2\}\times\{1\}} \mathrm{b^c}_{(1,1)} \mathrm{b^c}_{(2,1)} = \emptyset. \tag{32}$$

This can be verified by checking the automaton of $\mathcal{C}_{\{1,2\}\times\{1\}}$ that consists of 36 states (see Fig. 7). The actions of interest with regard to the privacy requirement, namely $\mathrm{b^c}$ and $\mathrm{c^c}$, are depicted by solid lines. For example, after the begin action $\mathrm{b^c}_{(1,1)}$ connecting states $7 \to 11$ a respective end action $\mathrm{c^c}_{(1,1)}$ is either directly possible (see $11 \to 15$) or after an intermediate action (see $11 \to 16$) or two intermediate actions (see $11 \to 16 \to 23$).

The minimal automaton of $\rho_{<1,2,1>}(\mathcal{C}_{\{1,2\}\times\{1\}})$ is shown in Fig. 8(a), which implies (32).

On the contrary, let $\mathcal{C}'_{I\times K}$ be defined as $\mathcal{C}_{I\times K}$ but with V'_s instead of V_s, where V'_s is defined by the automaton of Fig. 8(b). Then $(\mathcal{C}'_{I\times K})_{I\times K\in\mathcal{I}_2}$ is not self-similar because

$$\mathrm{a^c}_{(1,1)}\mathrm{a^c}_{(2,1)}\mathrm{a^c}_{(3,1)}\mathrm{a^s}_{(1,1)}\mathrm{b^s}_{(1,1)}\mathrm{a^s}_{(2,1)}\mathrm{a^s}_{(3,1)}\mathrm{b^s}_{(2,1)}\mathrm{b^c}_{(1,1)}\mathrm{b^c}_{(2,1)} \in \mathcal{C}'_{\{1,2,3\}\times\{1\}},$$ and so

$$\mathrm{a^c}_{(1,1)}\mathrm{a^c}_{(2,1)}\mathrm{a^s}_{(1,1)}\mathrm{b^s}_{(1,1)}\mathrm{a^s}_{(2,1)}\mathrm{b^s}_{(2,1)}\mathrm{b^c}_{(1,1)}\mathrm{b^c}_{(2,1)} \in \Pi^{\{1,2,3\}\{1\}}_{\{1,2\}\{1\}}(\mathcal{C}'_{\{1,2,3\}\times\{1\}})$$

but

$$\mathrm{a^c}_{(1,1)}\mathrm{a^c}_{(2,1)}\mathrm{a^s}_{(1,1)}\mathrm{b^s}_{(1,1)}\mathrm{a^s}_{(2,1)}\mathrm{b^s}_{(2,1)}\mathrm{b^c}_{(1,1)}\mathrm{b^c}_{(2,1)} \notin \mathcal{C}'_{\{1,2\}\times\{1\}}.$$

The same action sequence shows that $\mathcal{C}'_{\{1,2,3\}\times\{1\}}$ does not fulfil the privacy requirement.

The privacy requirement of the example is a typical safety property [25]. These properties describe that "nothing forbidden happens". They can be formalised by a set $\mathcal{F}$ of forbidden action sequences. So a system $\mathcal{L}_J \subset \Sigma^*_J$ satisfies a safety property $\mathcal{F}_J \subset \Sigma^*_J$ iff $\mathcal{L}_J \cap \mathcal{F}_J = \emptyset$.

In our example, the privacy requirement (29) is formalised by

$$\begin{aligned}\mathcal{F}^p_{I\times K} &= \bigcup_{\substack{i,i'\in I, i\neq i'\\ k\in K}} (\mu^{I\times K}_{<i,i',k>})^{-1}(\Sigma^*_{\{i,i'\}\times\{k\}} \mathrm{b^c}_{(i,k)} \mathrm{b^c}_{(i',k)}) \\ &= \bigcup_{\substack{i,i'\in I, i\neq i'\\ k\in K}} (\Pi^{I\times K}_{\{i,i'\}\times\{k\}})^{-1}(\iota^{-1}_{<i,i',k>}[\rho^{-1}_{<1,2,1>} \\ &\qquad (\Sigma^*_{\{1,2\}\times\{1\}} \mathrm{b^c}_{(1,1)} \mathrm{b^c}_{(2,1)})])\end{aligned}$$

because of

$$\mu^{I\times K}_{<i,i',k>} = \iota^{-1}_{<i,i',k>} \circ \rho_{<1,2,1>} \circ \iota_{<i,i',k>} \circ \Pi^{I\times K}_{\{i,i'\}\times\{k\}}$$

and

$$\begin{aligned}&\iota_{<i,i',k>}(\Sigma^*_{\{i,i'\}\times\{k\}} \mathrm{b^c}_{(i,k)} \mathrm{b^c}_{(i',k)}) \\ &\qquad = \Sigma^*_{\{1,2\}\times\{1\}} \mathrm{b^c}_{(1,1)} \mathrm{b^c}_{(2,1)}.\end{aligned}$$

As

$$\begin{aligned}&\{(\{i,i'\}\times\{k\}, \iota^{-1}_{<i,i',k>}) \mid i,i'\in I, i\neq i', \text{ and } k\in K\} \\ &= \{(I'\times K', \iota^{\{1,2\}\times\{1\}}_{I'\times K'}) \mid I'\times K' \subset I\times K \text{ and} \\ &\qquad \iota \in \mathcal{B}^2(\{1,2\}\times\{1\}, I'\times K')\}\end{aligned}$$

it follows

$$\mathcal{F}^p_{I\times K} = \bigcup_{\substack{I'\times K'\subset I\times K\\ \iota\in\mathcal{B}^2(\{1,2\}\times\{1\}, I'\times K')}} (\Pi^{I\times K}_{I'\times K'})^{-1}(\iota^{\{1,2\}\times\{1\}}_{I'\times K'}(F^p)) \tag{33}$$

with

$$F^p := \rho^{-1}_{<1,2,1>}(\Sigma^*_{\{1,2\}\times\{1\}} \mathrm{b^c}_{(1,1)} \mathrm{b^c}_{(2,1)}).$$

The representation (33) can be generalised for arbitrary parameter structures $\mathcal{I}$ and corresponding isomorphism structures $\mathcal{B}_\mathcal{I} = (\mathcal{B}(J,J'))_{(J,J')\in\mathcal{I}\times\mathcal{I}}$:

Let $\bar{J} \in \mathcal{I}$ and $\bar{F} \subset \Sigma^*_{\bar{J}}$, then for each $J \in \mathcal{I}$ let

$$F^{\bar{F}}_J := \bigcup_{J'\in\mathcal{I}, J'\subset J, \iota\in\mathcal{B}(\bar{J},J')} (\Pi^J_{J'})^{-1}(\iota^{\bar{J}}_{J'}(\bar{F})). \tag{34}$$

Now by the same argument as in our privacy example, we get

Theorem 6. *Let $(\mathcal{L}_J)_{J\subset\mathcal{I}}$ be a well-behaved scalable system, and let $\bar{F} \subset \Sigma^*_{\bar{J}}$ with $\bar{J} \in \mathcal{I}$, then*

$$\mathcal{L}_J \cap \mathcal{F}^{\bar{F}}_J = \emptyset \text{ for each } J \in \mathcal{I} \text{ iff } \mathcal{L}_{\bar{J}} \cap \mathcal{F}^{\bar{F}}_{\bar{J}} = \emptyset. \tag{35}$$

If $\mathcal{L}_{\bar{J}}$ and $\bar{F}$ are regular subsets of $\Sigma^*_{\bar{J}}$, then (35) can be checked by finite state methods [21].

If $(\mathcal{L}_J)_{J\subset\mathcal{I}}$ is defined as in (27) the regularity of $\check{L}$ and of V_r for each $r \in R$ and finiteness of R and $\bar{J}$ implies regularity of $\mathcal{L}_{\bar{J}}$.

For finite sets $J, \bar{J} \in \mathcal{I}$ with $\#(J) < \#(\bar{J})$, where $\#$ denotes the cardinality of a set, holds $\mathcal{F}^{\bar{F}}_J = \emptyset$, because of $\mathcal{B}(\bar{J}, J') = \emptyset$ for each $J' \in J$ with $J' \in \mathcal{I}$. Therefore,

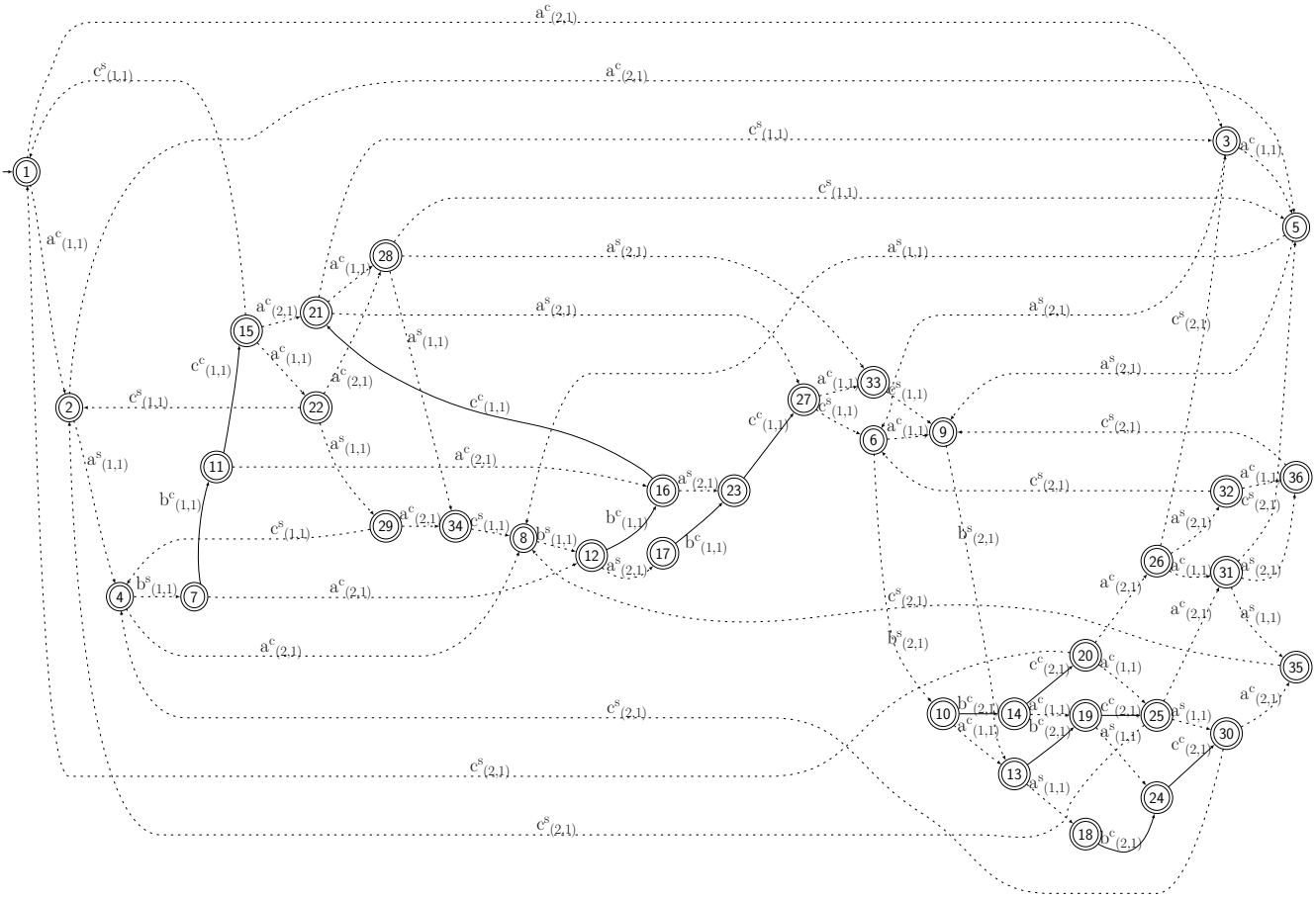


Figure 7. Automaton of $\mathcal{C}_{\{1,2\}\times\{1\}}$

it makes sense to consider safety properties defined by finite unions of sets as defined in (35).

Definition 16 (Scalable safety properties)**.** *Let $\mathcal{I}$ be a parameter structure, $\mathcal{B}_{\mathcal{I}} = (\mathcal{B}(J,J'))_{(J,J')\in\mathcal{I}\times\mathcal{I}}$ a corresponding isomorphism structure, T a finite set, and $\bar{F}_t \subset \Sigma^*_{\bar{J}_t}$ with $\bar{J}_t \in \mathcal{I}$ for each $t \in T$, then $(\mathcal{F}_J)_{J\in\mathcal{I}}$ with $\mathcal{F}_J := \bigcup_{t\in T} \mathcal{F}_J^{\bar{F}_t}$ is called a* scalable *safety property.*

Corollary 1. *For a well-behaved scalable system $(\mathcal{L}_J)_{J\in\mathcal{I}}$ the parameterised problem of verifying a scalable safety property is reduced to finite many finite state problems if the corresponding $\mathcal{L}_{\bar{J}_t}$ and $\bar{F}_t$ are regular languages.*

VIII. Conclusions and Further Work

Structural scalability of a system in terms of the ability to compose a system using a varying number of identical components of a few given types is a desired property that is analysed in this work. For safety critical systems as well as for business critical systems, assuring the correctness of systems composed in such a way is imperative. Thus, the focus of this paper is on property preserving structural scalability.

This motivates the formal definition of well-behaved scalable systems, which starts with a prototype system that fulfils a desired safety property and then "embeds" this prototype system in a scalable system. When this scalable system is constructed according to the methods given in this paper, then corresponding safety properties are fulfilled by any instance of the scalable system. In other words, it is shown that for well-behaved scalable systems a wide class of safety properties can be verified by finite state methods.

For this purpose, a formal framework is presented that can be utilised to construct well-behaved scalable systems in terms of prefix closed formal languages and alphabetic language homomorphisms. The basic parts of that framework are formalisations of parameter structures, influence structures and isomorphisms structures. Together with so-called prototype systems and behaviours of influence these structures formally define scalable systems, if certain conditions are fulfilled. With respect

to such scalable systems, the focus is on properties, which rely on specific component types and a specific number of individual components for these component types but not on the specific individuality a component. Well-behaved scalable systems are characterised by those systems, which fulfil such a kind of property if already one prototype system (depending on the property) fulfils that property. Self-similar scalable systems have this desired property. A sufficient condition for such self-similarity is given in terms of prototype systems and behaviours of influence. A deeper analysis of this condition is subject of a forthcoming paper of the authors.

Usually, behaviour properties of systems are divided into two classes: *safety* and *liveness* properties [25]. Intuitively, a safety property stipulates that "something bad does not happen" and a liveness property stipulates that "something good eventually happens". To extend this verification approach to reliability or general liveness properties, additional assumptions for well-behaved scalable systems have to be established. In [26], such assumptions have been developed for uniformly parametrised two-sided cooperations. To generalise these ideas to a wider class of well-behaved scalable systems is subject of further work.

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Appendix

Theorem 2 (simplest well-behaved scalable systems)**.** *$(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ is a well-behaved scalable system with respect to each isomorphism structure for $\mathcal{I}$ based on N and*

$$\dot{\mathcal{L}}(L)_I = \bigcap_{i\in N} (\tau_i^I)^{-1}(L) \text{ for each } I \in \mathcal{I}.$$

The proof of Theorem 2 will be given in context of influence structures because it consists of special cases of more general results on influence structures (see (59)).

Further requirements, which assure that $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ are well-behaved scalable systems, will be given with respect to $\mathcal{E}_\mathcal{I}$, $\mathcal{B}_\mathcal{I}$, L and V. This will be prepared by some lemmata.

Lemma 3. *Let $\mathcal{E}_\mathcal{I} := (E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T, and let $V \subset \Sigma^*$. If*

$$E(t,I') = E(t,I) \cap I' \tag{36}$$

for each $t \in T$ and $I, I' \in \mathcal{I}$ $I' \subset I$, then

$$((\tau_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a monotonic parameterised system for each $t \in T$, and by the intersection theorem

$$(\bigcap_{t\in T} (\tau_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a monotonic parameterised system.

Proof: Let $I \in \mathcal{I}$ and $t \in T$. From the definitions of influence homomorphisms and influence structures it follows

$$\tau^I_{E(t,I)}(a_i) = \begin{cases} a \mid & a_i \in \Sigma_{E(t,I)} \\ \varepsilon \mid & a_i \in \Sigma_I \setminus \Sigma_{E(t,I)} \end{cases}.$$

For $I' \subset I$, $I' \in \mathcal{I}$ and $a_i \in \Sigma_{I'}$ then because of (36)

$$\begin{aligned} \tau^I_{E(t,I)}(a_i) &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I)} \cap \Sigma_{I'} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \cap \Sigma_I \setminus \Sigma_{E(t,I)} \end{cases} \\ &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I')} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \setminus (\Sigma_{E(t,I)} \cap \Sigma_{I'}) \end{cases} \\ &= \begin{cases} a \mid & a_i \in \Sigma_{E(t,I')} \\ \varepsilon \mid & a_i \in \Sigma_{I'} \setminus \Sigma_{E(t,I')} \end{cases} = \tau^{I'}_{E(t,I')}(a_i), \end{aligned}$$

and therefore

$$(\tau^{I'}_{E(t,I')})^{-1}(V) \subset (\tau^I_{E(t,I)})^{-1}(V) \text{ for } V \subset \Sigma^*.$$

So,

$$((\tau^I_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a monotonic parameterised system for each $t \in T$. ■

Example 13. *Let $\mathcal{I}$ be a parameter structure based on N. For $I \in \mathcal{I}$ and $i \in N$ let:*

$$\dot{E}(i,I) := \begin{cases} \{i\} \mid & i \in I \\ \emptyset \mid & i \in N \setminus I \end{cases}.$$

By the definition of parameter structure $N \neq \emptyset$. So

$$\dot{\mathcal{E}}_\mathcal{I} := (\dot{E}(i,I))_{(i,I)\in N\times\mathcal{I}}$$

defines an influence structure for $\mathcal{I}$ indexed by N. $\dot{\mathcal{E}}_\mathcal{I}$ satisfies (36) and by $\tau^I_i = \tau^I_{\{i\}}$ $\tau^I_i = \tau^I_{\dot{E}(i,I)}$ for $i \in N$ and $I \in \mathcal{I}$.

Now by Lemma 3 for $V \subset \Sigma^*$

$$((\tau^I_i)^{-1}(V))_{I\in\mathcal{I}} \text{ is a monotonic parameterised system} \tag{37}$$

for each $i \in N$.

For this special influence structure $\dot{\mathcal{E}}_\mathcal{I}$ a stronger result can be obtained.

Lemma 4. *Let $\mathcal{I}$ be a parameter structure based on N and $\varepsilon \in L \subset \Sigma^*$. Then*

$$((\tau^I_i)^{-1}(L))_{I\in\mathcal{I}}$$

is a self-similar monotonic parameterised system for each $i \in N$, and by the intersection theorem

$$(\bigcap_{i\in N} (\tau^I_i)^{-1}(L))_{I\in\mathcal{I}}$$

is a self-similar monotonic parameterised system.

Proof: On account of (37)

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) = (\tau^{I'}_i)^{-1}(L)$$

has to be shown for $I, I' \in \mathcal{I}$, $I' \subset I$, and $i \in N$.

(37) implies $(\tau^{I'}_i)^{-1}(L) \subset (\tau^I_i)^{-1}(L)$ and therefore,

$$(\tau^{I'}_i)^{-1}(L) = \Pi^I_{I'}((\tau^{I'}_i)^{-1}(L)) \subset \Pi^I_{I'}((\tau^I_i)^{-1}(L)). \tag{38}$$

It remains to show $\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L)$.

Case 1. $i \notin I'$

Because of $\varepsilon \in L$ and $\tau^{I'}_i(w) = \varepsilon$ for $i \notin I'$ and $w \in \Sigma^*_{I'}$ it holds $(\tau^{I'}_i)^{-1}(L) = \Sigma^*_{I'}$ and so

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L) \text{ for } i \notin I'. \tag{39}$$

Case 2. $i \in I'$

From definitions of $\Pi^I_{I'}, \tau^I_i$ and $\tau^{I'}_i$ follows

$$\tau^I_i = \tau^{I'}_i \circ \Pi^I_{I'} \text{ for } i \in I'. \tag{40}$$

For $x \in \Pi^I_{I'}((\tau^I_i)^{-1}(L))$ exists $y \in \Sigma^*_I$ with $\tau^I_i(y) \in L$ and $x = \Pi^I_{I'}(y)$. Because of (40) holds

$$\tau^{I'}_i(x) = \tau^{I'}_i(\Pi^I_{I'}(y)) = \tau^I_i(y) \in L,$$

hence, $x \in (\tau^{I'}_i)^{-1}(L)$. Therefore,

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) \subset (\tau^{I'}_i)^{-1}(L) \text{ for } i \in I'. \quad (41)$$

Because of (39), (41) and (38) holds

$$\Pi^I_{I'}((\tau^I_i)^{-1}(L)) = (\tau^{I'}_i)^{-1}(L)$$

for $I, I' \in \mathcal{I}$, $I' \subset I$ and $i \in N$. ■

Intersections of system behaviours play an important role concerning uniformity of parameterisation. Therefore, some general properties of intersections of families of sets will be presented.

Let T be a set. A family $f = (f_t)_{t\in T}$ with $f_t \in F$ for each $t \in T$ is formally equivalent to a function $f : T \to F$ with $f_t := f(t)$.

Let M be a set. A family $f = (f_t)_{t\in T}$ with $f_t \in F = \mathcal{P}(M)$ for each $t \in T$ is called a family of subsets of M.

Let now $T \neq \emptyset$ and f a family of subsets of M. The intersection $\bigcap_{t\in T} f_t$ is defined by

$$\bigcap_{t\in T} f_t = \{m \in M | m \in f_t \text{ for each } t \in T\}. \quad (42)$$

If $f = g \circ h$ with $h : T \to H$ and $g : H \to F$ then

$$\bigcap_{t\in T} f(t) = \bigcap_{x\in h(T)} g(x). \quad (43)$$

If especially $f = h$ and g is the identity on F, then from (43) follows

$$\bigcap_{t\in T} f(t) = \bigcap_{x\in f(T)} x.$$

For a second family of sets $f' : T' \to F$ with $f'(T') = f(T)$ follows then

$$\bigcap_{t\in T} f(t) = \bigcap_{t'\in T'} f(t').$$

In the following we will use family and function notations side by side.

Let $f = (f_t)_{t\in T}$ a family of sets with $f : T \to F = \mathcal{P}(M)$. If $T = \mathring{T} \cup \hat{T}$ with $\hat{T} \neq \emptyset$ and $f(\hat{T}) = \{M\}$, then from (42) follows

$$\bigcap_{t\in T} f(t) = \bigcap_{t\in \mathring{T}} f(t). \quad (44)$$

Let $\mathcal{E}_\mathcal{I} = (E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T.

For each $I \in \mathcal{I}$ a family of sets

$$\mathcal{E}_\mathcal{I}(I) := (E(t,I))_{t\in T}$$

with $E(t,I) = \mathcal{E}_\mathcal{I}(I)(t) \in \mathcal{P}(I)$ is defined, and it holds

$$\mathcal{E}_\mathcal{I}(I) : T \to \mathcal{P}(I).$$

From (43) it follows (with $h = \mathcal{E}_\mathcal{I}(I)$)

$$\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V) = \bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T)} (\tau^I_x)^{-1}(V) \quad (45)$$

for each $V \subset \Sigma^*$ and $I \in \mathcal{I}$.

For each $I \in \mathcal{I}$ holds $\tau^I_\emptyset(w) = \varepsilon$ for each $w \in \Sigma^*_I$. It follows,

$$(\tau^I_\emptyset)^{-1}(V) = \Sigma^*_I \text{ if } \varepsilon \in V \subset \Sigma^*. \quad (46)$$

Because of (43), (44), (45), and (46)

$$\begin{aligned}\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V) &= \bigcap_{x\in\mathcal{E}_\mathcal{I}(I)(T_I)} (\tau^I_x)^{-1}(V) \\ &= \bigcap_{t\in T_I} (\tau^I_{E(t,I)})^{-1}(V) \quad (47)\end{aligned}$$

for each T_I with $\emptyset \neq T_I \subset T$ and $\mathcal{E}_\mathcal{I}(I)(T) \setminus \mathcal{E}_\mathcal{I}(I)(T_I) \in \{\emptyset, \{\emptyset\}\}$ and $\varepsilon \in V \subset \Sigma^*$.

Each bijection $\iota : I \to I'$ defines another bijection $\breve{\iota} : \mathcal{P}(I) \to \mathcal{P}(I')$ by

$$\breve{\iota}(x) := \{\iota(y) \in I' | y \in x\} \text{ for each } x \in \mathcal{P}(I).$$

Lemma 5. *Let $\mathcal{E}_\mathcal{I} = (E(t,I))_{(t,I)\in T\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$ indexed by T, and let $\mathcal{B}_\mathcal{I} = (\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$. Let*

$$\begin{aligned}&\varepsilon \in V \subset \Sigma^*, \text{ and let } (T_K)_{K\in\mathcal{I}} \text{ be a family} \\ &\text{with } \emptyset \neq T_K \subset T \text{ and} \\ &\mathcal{E}_\mathcal{I}(K)(T) \setminus \mathcal{E}_\mathcal{I}(K)(T_K) \in \{\emptyset, \{\emptyset\}\} \text{ for each } K \in \mathcal{I}, \\ &\text{such that } \breve{\iota}(\mathcal{E}_\mathcal{I}(I)(T_I)) = \mathcal{E}_\mathcal{I}(I')(T_{I'}) \\ &\text{for each } (I,I') \in \mathcal{I}\times\mathcal{I} \text{ and } \iota \in \mathcal{B}(I,I'), \quad (48)\end{aligned}$$

then

$$\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V) = \bigcap_{t\in T_I} (\tau^I_{E(t,I)})^{-1}(V) \quad (49)$$

for each $I \in \mathcal{I}$, and

$$\iota^I_{I'}[\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V)] = \bigcap_{t\in T} (\tau^{I'}_{E(t,I')})^{-1}(V) \quad (50)$$

for each $(I,I') \in \mathcal{I}\times\mathcal{I}$ and $\iota \in \mathcal{B}(I,I')$.

Proof of (49)*:* Because of (47) from assumption (48) directly follows (49). ■

For the proof of (50) the following property of the homomorphisms τ^I_K is needed:

Let $\iota : I \to I'$ a bijection and $K \subset I$, then $\tau^{I'}_{\iota(K)} \circ \iota^I_{I'} = \tau^I_K$ and so

$$\tau^{I'}_{\iota(K)} = \tau^I_K \circ (\iota^I_{I'})^{-1}. \quad (51)$$

Proof of (51)*:*

The elements of Σ_I are of the form a_i with $i \in I$ and $a \in \Sigma$. For these elements holds

$$\begin{aligned}\tau^I_K(a_i) &= \begin{cases} a \mid & i \in K \\ \varepsilon \mid & i \in I \setminus K \end{cases} \\ &= \begin{cases} a \mid & \iota(i) \in \iota(K) \\ \varepsilon \mid & \iota(i) \in I' \setminus \iota(K) \end{cases} \\ &= \tau^{I'}_{\iota(K)}(a_{\iota(i)}) = \tau^{I'}_{\iota(K)}(\iota^I_{I'}(a_i)),\end{aligned}$$

which proves (51). ∎

Proof of (50)*:* Because of (47) and (51)

$$\begin{aligned}&\iota^I_{I'}[\bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)] \\ &= \iota^I_{I'}[\bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^I_x)^{-1}(V)] \\ &= ((\iota^I_{I'})^{-1})^{-1}[\bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^I_x)^{-1}(V)] \\ &= \bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}((\iota^I_{I'})^{-1})^{-1}[(\tau^I_x)^{-1}(V)] \\ &= \bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^I_x \circ (\iota^I_{I'})^{-1})^{-1}(V) \\ &= \bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^{I'}_{\iota(x)})^{-1}(V) \\ &= \bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^{I'}_{\breve{\iota}(x)})^{-1}(V). \qquad (52)\end{aligned}$$

From (43) (with $h = \breve{\iota}$) and the assumption (48) follows

$$\begin{aligned}\bigcap_{x\in\mathcal{E}_{\mathcal{I}}(I)(T_I)}(\tau^{I'}_{\breve{\iota}(x)})^{-1}(V) &= \bigcap_{x'\in\breve{\iota}(\mathcal{E}_{\mathcal{I}}(I)(T_I))}(\tau^{I'}_{x'})^{-1}(V) \\ &= \bigcap_{x'\in\mathcal{E}_{\mathcal{I}}(I')(T'_I)}(\tau^{I'}_{x'})^{-1}(V).\end{aligned}$$

Furthermore, from (47) follows

$$\bigcap_{x'\in\mathcal{E}_{\mathcal{I}}(I')(T'_I)}(\tau^{I'}_{x'})^{-1}(V) = \bigcap_{t\in T}(\tau^{I'}_{E(t,I')})^{-1}(V). \qquad (53)$$

(52) - (53) prove (50). ∎

The case $T = N$, where $\mathcal{I}$ is based on N, allows a simpler sufficient condition for (49) and (50).

Lemma 6. *Let $\mathcal{I}$ be a parameter structure based on N, $\mathcal{E}_{\mathcal{I}} = (E(n,I))_{(n,I)\in N\times\mathcal{I}}$ be an influence structure for $\mathcal{I}$, and let $\mathcal{B}_{\mathcal{I}} = (\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$.*

Let $\varepsilon \in V \subset \Sigma^*$, (54a)

for each $I \in \mathcal{I}$ and $n \in N$ let $E(n,I) = \emptyset$,
or it exists an $i_n \in I$ with $E(n,I) = E(i_n,I)$, and (54b)

for each $(I,I') \in \mathcal{I}\times\mathcal{I}, \iota \in \mathcal{B}(I,I')$ and $i \in I$ holds
$\iota(E(i,I)) = E(\iota(i),I')$. (54c)

Then

$$\bigcap_{n\in N}(\tau^I_{E(n,I)})^{-1}(V) = \bigcap_{n\in I}(\tau^I_{E(n,I)})^{-1}(V)$$

for each $I \in \mathcal{I}$, and

$$\iota^I_{I'}[\bigcap_{n\in N}(\tau^I_{E(n,I)})^{-1}(V)] = \bigcap_{n\in N}(\tau^{I'}_{E(n,I')})^{-1}(V)$$

for each $(I,I') \in \mathcal{I}\times\mathcal{I}$ and $\iota \in \mathcal{B}(I,I')$.

Proof: From (54b) follows $\mathcal{E}_{\mathcal{I}}(I)(N) = \mathcal{E}_{\mathcal{I}}(I)(I)$ or $\mathcal{E}_{\mathcal{I}}(I)(N) = \mathcal{E}_{\mathcal{I}}(I)(I)\cup\{\emptyset\}$, so

$$\mathcal{E}_{\mathcal{I}}(I)(N) \setminus \mathcal{E}_{\mathcal{I}}(I)(I) \in \{\emptyset, \{\emptyset\}\} \text{ for each } I \in \mathcal{I}. \qquad (55)$$

From (54c) follows

$$\breve{\iota}(\mathcal{E}_{\mathcal{I}}(I)(I)) \subset \mathcal{E}_{\mathcal{I}}(I')(I'). \qquad (56)$$

Because $\iota : I \to I'$ is a bijection, for each $i' \in I'$ exists an $i \in I$ with $\iota(i) = i'$. Because of (54c) holds $\breve{\iota}(E(i,I)) = E(i',I')$, where $E(i,I) \in \mathcal{E}_{\mathcal{I}}(I)(I)$. From this follows

$$\mathcal{E}_{\mathcal{I}}(I')(I') \subset \breve{\iota}(\mathcal{E}_{\mathcal{I}}(I)(I)). \qquad (57)$$

Because of (55) - (57), with $T = N$ and $(T_I)_{I\in\mathcal{I}} = (I)_{I\in\mathcal{I}}$,

(54a) – (54c) implies (48).

∎

Example 14 (Example 13 (continued))**.** *Let $\mathcal{I}$ be a parameter structure based on N and $\mathcal{B}_{\mathcal{I}} = (\mathcal{B}(I,I'))_{(I,I')\in\mathcal{I}\times\mathcal{I}}$ be an isomorphism structure for $\mathcal{I}$. Then $\dot{\mathcal{E}}_{\mathcal{I}}$ satisfies* (54b) *and* (54c)*.*

So for $\varepsilon \in L \subset \Sigma^*$ Lemma 6 implies

$$\begin{aligned}&\bigcap_{n\in N}(\tau^I_n)^{-1}(L) = \bigcap_{n\in I}(\tau^I_n)^{-1}(L) \text{ for each } I \in \mathcal{I} \text{ and} \\ &\iota^I_{I'}[\bigcap_{n\in N}(\tau^I_n)^{-1}(L)] = \bigcap_{n\in N}(\tau^{I'}_n)^{-1}(L) \qquad (58)\end{aligned}$$

for each $(I,I') \in \mathcal{I}\times\mathcal{I}$ and $\iota \in \mathcal{B}(I,I')$.

Now Lemma 4 together with (58) proves Theorem 2. (59)

Because of $\tau^I_n = \tau^I_{\dot{E}(n,I)}$ for $I \in \mathcal{I}$ and $n \in N$, (58) and the definitions of $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ and $(\mathcal{L}(L,\mathcal{E}_{\mathcal{I}},V)_I)_{I\in\mathcal{I}}$ imply

$$\begin{aligned}\dot{\mathcal{L}}(L)_I &= \bigcap_{n\in I}(\tau^I_n)^{-1}(L) = \bigcap_{n\in I}(\tau^I_n)^{-1}(L) \cap \bigcap_{n\in I}(\tau^I_n)^{-1}(V) \\ &= \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in N}(\tau^I_n)^{-1}(V) \\ &= \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in N}(\tau^I_{\dot{E}(n,I)})^{-1}(V) \\ &= \mathcal{L}(L,\dot{\mathcal{E}}_I,V)_I \qquad (60)\end{aligned}$$

for $I \in \mathcal{I}$ and $V \supset L$.

(60) gives a representation of $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ in terms of $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$.

For the following theorems please remember that by the general definition of $\mathcal{L}(L,\mathcal{E}_I,V)_I$ it is assumed that $\emptyset \neq L \subset V$ and L,V are prefix closed. This implies $\varepsilon \in L \subset V$.

Lemma 7. *Let $\mathcal{I}$ be a parameter structure, $\mathcal{E}_\mathcal{I}$ an influence structure for $\mathcal{I}$ indexed by T and $\mathcal{B}_\mathcal{I}$ an isomorphism structure for $\mathcal{I}$.*

Assuming (36) *and* (48)*, then*

$$(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$$

is a scalable systems with respect to $\mathcal{B}_\mathcal{I}$. It holds

$$\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in T_I} (\tau^I_{E(n,I)})^{-1}(V)$$

for each $I \in \mathcal{I}$.

Proof: By Theorem 2, $(\dot{\mathcal{L}}(L)_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$. By Lemma 3 and 5 (50)

$$(\bigcap_{t\in T} (\tau^I_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$$

is a scalable system with respect to $\mathcal{B}_\mathcal{I}$ too. Now part (ii) of the intersection theorem proves $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ to be a scalable system with respect to $\mathcal{B}_\mathcal{I}$. Lemma 5 (49) completes the proof of Lemma 7. ■

Using Lemma 6 instead of Lemma 5 proves the following.

Theorem 3 (construction condition for scalable systems)**.** *By the assumptions of Lemma 6 and* (36) *with $T = N$, $(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$ is a scalable system with respect to $\mathcal{B}_\mathcal{I}$. It holds*

$$\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I = \dot{\mathcal{L}}(L)_I \cap \bigcap_{n\in I} (\tau^I_{E(n,I)})^{-1}(V)).$$

Remark 3. *It can be shown that in* $\mathrm{SP}(L,V)$ $\mathbb{N}$ *can be replaced by each countable infinite set.*

More precisely, let N' be another set and $\iota : \mathbb{N} \to N'$ a bijection. $\iota^\mathbb{N}_{N'} : \Sigma^*_\mathbb{N} \to \Sigma^*_{N'}$ is the isomorphism defined as in the definition of isomorphism structure. It now holds

$$\Theta^\mathbb{N} = \Theta^{N'} \circ \iota^\mathbb{N}_{N'} \text{ and } \tau^\mathbb{N}_n = \tau^{N'}_{\iota(n)} \circ \iota^\mathbb{N}_{N'} \tag{61}$$

for each $n \in \mathbb{N}$. Furthermore,

$$\iota^\mathbb{N}_{N'} \circ \Pi^\mathbb{N}_K = \Pi^{N'}_{\iota(K)} \circ \iota^\mathbb{N}_{N'} \tag{62}$$

for each $K \subset \mathbb{N}$. From (61) and commutativity of intersection now

$$\begin{aligned}(\bigcap_{n\in\mathbb{N}} (\tau^\mathbb{N}_n)^{-1}(L)) \cap (\Theta^\mathbb{N})^{-1}(V) = \\ = (\iota^\mathbb{N}_{N'})^{-1}[(\bigcap_{n\in\mathbb{N}} (\tau^{N'}_{\iota(n)})^{-1}(L)) \cap (\Theta^{N'})^{-1}(V)] \\ = (\iota^\mathbb{N}_{N'})^{-1}[(\bigcap_{n'\in N'} (\tau^{N'}_{n'})^{-1}(L)) \cap (\Theta^{N'})^{-1}(V)].\end{aligned} \tag{63}$$

By (62),

$$\Pi^\mathbb{N}_K \circ (\iota^\mathbb{N}_{N'})^{-1} = (\iota^\mathbb{N}_{N'})^{-1} \circ \Pi^{N'}_{\iota(K)}. \tag{64}$$

Because of (63) and (64)

$$\begin{aligned}\Pi^\mathbb{N}_K[(\bigcap_{n\in\mathbb{N}} (\tau^\mathbb{N}_n)^{-1}(L)) \cap (\Theta^\mathbb{N})^{-1}(V)] = \\ = (\iota^\mathbb{N}_{N'})^{-1}(\Pi^{N'}_{\iota(K)}[(\bigcap_{n'\in N'} (\tau^{N'}_{n'})^{-1}(L)) \cap (\Theta^{N'})^{-1}(V)]).\end{aligned}$$

From

$$\Pi^\mathbb{N}_K[(\bigcap_{n\in\mathbb{N}} (\tau^\mathbb{N}_n)^{-1}(L)) \cap (\Theta^\mathbb{N})^{-1}(V)] \subset (\Theta^\mathbb{N})^{-1}(V)$$

now follows

$$\begin{aligned}\Pi^{N'}_{\iota(K)}[(\bigcap_{n'\in N'} (\tau^{N'}_{n'})^{-1}(L)) \cap (\Theta^{N'})^{-1}(V)] \\ \subset \iota^\mathbb{N}_{N'}((\Theta^\mathbb{N})^{-1}(V)).\end{aligned} \tag{65}$$

Because of (61) $\Theta^\mathbb{N} \circ (\iota^\mathbb{N}_{N'})^{-1} = \Theta^{N'}$ and so

$$(\Theta^{N'})^{-1}(V) = \iota^\mathbb{N}_{N'}((\Theta^\mathbb{N})^{-1}(V)).$$

Therefore, from (65) follows

$$\Pi^{N'}_{\iota(K)}[(\bigcap_{n'\in N'} (\tau^{N'}_{n'})^{-1}(L)) \cap (\Theta^{N'})^{-1}(V)] \subset (\Theta^{N'})^{-1}(V).$$

Because for each $\emptyset \neq K' \subset N'$ it exists an $\emptyset \neq K \subset \mathbb{N}$ with $K' = \iota(K)$, by $\mathrm{SP}(L,V)$, we get for each $\emptyset \neq K \subset \mathbb{N}$ a corresponding inclusion with N' replacing $\mathbb{N}$ and K' for K.

Lemma 8. *The assumptions of Lemma 3 and Lemma 4 together with* $\mathrm{SP}(L,V)$ *imply that $(X(L,V,t)_I)_{I\in\mathcal{I}}$ with*

$$X(L,V,t)_I := \bigcap_{n\in N} (\tau^I_n)^{-1}(L) \cap (\tau^I_{E(t,I)})^{-1}(V)$$

is a self-similar monotonic parameterised system for each $t \in T$.

Proof: By Lemma 3 and Lemma 4, $((\tau^I_{E(t,I)})^{-1}(V))_{I\in\mathcal{I}}$ and $(\bigcap_{n\in N} (\tau^I_n)^{-1}(L))_{I\in\mathcal{I}}$ are monotonic parameterised systems. So by the intersection

theorem $(X(L,V,t)_I)_{I\in\mathcal{I}}$ is a monotonic parameterised system for each $t \in T$. Therefore,

$$X(L,V,t)_{I'} = \Pi^I_{I'}(X(L,V,t)_{I'}) \subset \Pi^I_{I'}(X(L,V,t)_I)$$

for each $I, I' \in \mathcal{I}$ with $I' \subset I$. So the proof of self-similarity can be reduced to the proof of

$$\Pi^I_{I'}(X(L,V,t)_I) \subset X(L,V,t)_{I'} \tag{66}$$

for each $t \in T$ and $I, I' \in \mathcal{I}$ with $I' \subset I$.

Because by Lemma 4

$$(\bigcap_{n\in N}(\tau^I_n)^{-1}(L))_{I\in\mathcal{I}}$$

is self-similar, it holds

$$\Pi^I_{I'}(X(L,V,t)_I) \subset \Pi^I_{I'}(\bigcap_{n\in N}(\tau^I_n)^{-1}(L)) = \bigcap_{n\in N}(\tau^I_n)^{-1}(L).$$

So the proof of (66) can be reduced to the proof of

$$\Pi^I_{I'}[\bigcap_{n\in N}(\tau^I_n)^{-1}(L) \cap (\tau^I_{E(t,I)})^{-1}(V)] \subset (\tau^{I'}_{E(t,I')})^{-1}(V) \tag{67}$$

for each $t \in T$ and $I, I' \in \mathcal{I}$ with $I' \subset I$.

For each

$$w \in (\bigcap_{n\in N}(\tau^I_n)^{-1}(L)) \cap (\tau^I_{E(t,I)})^{-1}(V)$$

exists a $r \in \mathbb{N}$ and $u_i \in \Sigma^*_{E(t,I)}$ for $1 \le i \le r$ and $v_i \in \Sigma^*_{I\setminus E(t,I)}$ for $1 \le i \le r$ with $w = u_1v_1u_2v_2\dots u_rv_r$. Note that $\Sigma_\emptyset := \emptyset$ and $\emptyset^* = \{\varepsilon\}$. Because $u_1u_2\dots u_r \in \Sigma^*_{E(t,I)}$ and $v_1v_2\dots v_r \in \Sigma^*_{I\setminus E(t,I)}$ holds

$$\begin{aligned}\Theta^N(u_1u_2\dots u_r) &= \tau^I_{E(t,I)}(u_1u_2\dots u_r)\\ &= \tau^I_{E(t,I)}(w) \in V.\end{aligned} \tag{68}$$

With the same argumentation holds

$$\tau^N_n(u_1u_2\dots u_r) = \tau^I_n(u_1u_2\dots u_r) = \tau^I_n(w) \in L \tag{69}$$

for $n \in E(t,I)$ and

$$\tau^N_n(u_1u_2\dots u_r) = \varepsilon \in L \tag{70}$$

for $n \in N \setminus E(t,I)$. With (68) - (70) now

$$u_1u_2\dots u_r \in (\bigcap_{n\in N}(\tau^N_n)^{-1}(L)) \cap (\Theta^N)^{-1}(V),$$

and on behalf of precondition $\mathrm{SP}(L,V)$ holds

$$\begin{aligned}\Pi^N_{I'}(u_1u_2\dots u_r) =& \Pi^{E(t,I)}_{I'\cap E(t,I)}(u_1u_2\dots u_r)\\ &\in \Sigma^*_{I'\cap E(t,I)} \cap (\Theta^N)^{-1}(V).\end{aligned} \tag{71}$$

Furthermore,

$$\begin{aligned}\Pi^I_{I'}(w) =& \Pi^I_{I'}(u_1v_1u_2v_2\dots u_rv_r)\\ =& \Pi^{E(t,I)}_{I'\cap E(t,I)}(u_1)\Pi^{I\setminus E(t,I)}_{I'\setminus E(t,I)}(v_1)\dots\\ & \Pi^{E(t,I)}_{I'\cap E(t,I)}(u_r)\Pi^{I\setminus E(t,I)}_{I'\setminus E(t,I)}(v_r).\end{aligned} \tag{72}$$

Because of (36), $E(t,I') \subset E(t,I)$ and so $I' \setminus E(t,I) \subset I' \setminus E(t,I')$ and thus

$$\tau^{I'}_{E(t,I')}(\Pi^{I\setminus E(t,I)}_{I'\setminus E(t,I)})(v_i) = \varepsilon$$

for $1 \le i \le r$. With (36) and (72) it follows

$$\tau^{I'}_{E(t,I')}(\Pi^I_{I'}(w)) = \tau^{I'}_{E(t,I')}(\Pi^{E(t,I)}_{E(t,I')}(u_1\dots u_r)). \tag{73}$$

Because $\tau^{I'}_{E(t,I')}(x) = \Theta^N(x)$ for each $x \in \Sigma^*_{E(t,I')}$ now on behalf of (73), (36), and (71)

$$\tau^{I'}_{E(t,I')}(\Pi^I_{I'}(w)) = \Theta^N(\Pi^{E(t,I)}_{E(t,I')}(u_1\dots u_r)) \in V,$$

and thus

$$\Pi^I_{I'}(w) \in (\tau^{I'}_{E(t,I')})^{-1}(V).$$

This proves (67) and completes the proof of Lemma 8. ■

Because of the idempotence of intersection

$$\begin{aligned}&\bigcap_{n\in N}(\tau^I_n)^{-1}(L) \cap \bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)\\ &\quad= \bigcap_{t\in T}[\bigcap_{n\in N}(\tau^I_n)^{-1}(L) \cap (\tau^I_{E(t,I)})^{-1}(V)].\end{aligned}$$

Now the intersection theorem and Lemma 8 imply

Lemma 9. *If* $\mathrm{SP}(L,V)$*, then by the assumptions of Lemma 3 and 4*

$$[\bigcap_{n\in N}(\tau^I_n)^{-1}(L) \cap \bigcap_{t\in T}(\tau^I_{E(t,I)})^{-1}(V)]_{I\in\mathcal{I}}$$

is a self-similar monotonic parameterised system.

Combining Lemma 9 with Lemma 7 or Theorem 3 imply

Theorem 4 (construction condition for well-behaved scalable systems)**.** *By the assumptions of Lemma 7 or Theorem 3 together with* $\mathrm{SP}(L,V)$

$$(\mathcal{L}(L,\mathcal{E}_\mathcal{I},V)_I)_{I\in\mathcal{I}}$$

is a well-behaved scalable system.