Distributionally Robust Chance-Constrained Zero-Sum Games with Moments Based and Statistical Based Uncertainty Sets

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***Abstract* - We consider a two-player zero-sum game with random linear chance constraints whose distributions are known to belong to moments based uncertainty sets or statistical distance based uncertainty sets. The game with chance constraints can be used in various applications, e.g., risk constraints in portfolio optimization, resource constraints in stochastic shortest path problem, renewable energy aggregators in the local market. We propose a reformulation of the chance constraints using distributionally chance-constrained optimization framework. We show that there exists a saddle point equilibrium of the game, which is the optimal solution of a primal-dual pair of second-order cone programs. As an application, we present a competition of two firms in financial market to simulate our theoretical results.**

Keywords-Distributionally robust chance constraints; Zero-sum game; Saddle point equilibrium; Second-order cone program.

I. Introduction

This paper is an extended version of [1], presented at the Seventeenth International Conference on Internet and Web Applications and Services (ICIW), from June 26 to June 30, 2022 in Porto, Portugal.

Equilibrium is an important notion in game theory, in which there is no incentive for any player to deviate unilaterally. The researches in the literature usually focus on sufficient conditions for the existence of an equilibrium point and its characterization. The first notion of equilibrium was introduced in the book Researches into the Mathematical Principles of the Theory of Wealth by Cournot in 1838 [2]. In 1951, Nash [3] showed that there exists an equilibrium point in a finite strategic game, which is known as a Nash equilibrium nowadays. The theory of Nash equilibrium is especially hard when it deals with practical applications with random payoffs and strategy sets. In order to deal with random payoffs, the most common way is using the expectation function, which is equivalent to study deterministic payoffs. In many real life applications, the strategy sets are restricted by random linear constraints, which are called chance constraints. The distribution of random factors in chance constraints can be known exactly or unknown, which leads to different approaches to define a game. In known distribution case, the true distribution of random factors is usually assumed to be elliptically distributed, which includes many known distributions, e.g., Gaussian distribution, Laplace distribution, Kotz distribution or Pearson distribution. Otherwise, in unknown distribution case, the true distribution of random factors is assumed to belong to an uncertainty set, where only a partial information of the distribution is known due to historical data and we call these games as distributionally robust chance-constrained games. A two-player zero-sum game is modeled using continuous strategy sets, where the sum of two players' payoffs is zero. Consequently, it is defined using a single payoff function, where one player plays the role of maximizer and another player plays the role of minimizer. More commonly, a zero-sum game is introduced with a payoff matrix, where the rows and the columns are the actions of player 1 and player 2, respectively. A Saddle Point Equilibrium (SPE) is the solution concept to study the zero-sum games and it exists in the mixed strategies [4].

In the conference paper [1], we considered a two player zero-sum game with continuous strategy set, where the payoff function has a special form and the strategies of each player are modeled using random linear constraints reformulated as distributionally robust chance constraints. We proposed an SOCP reformulation of distributionally robust chance constraints under two uncertainty sets based on the partial information about the mean vectors and covariance matrices of the random constraint vectors. We showed the existence of an SPE and characterized it as the optimal solution of a primal-dual pair of SOCPs. The conference paper has some shortcomings, e.g., the payoff function has a quadratic form, the uncertainty sets are mainly constructed based on moments from historical data and it lacks of numerical results which allow us to compare different uncertainty sets. As an extended version of [1], our contribution of this paper is as follows:

- We study a more general framework as compared to [1] by considering two types of uncertainty sets based on either the partial information on the mean vectors and covariance matrices of the random constraint vectors (moments based uncertainty sets) or the statistical distance between their true distribution and a nominal distribution (statistical based uncertainty sets). We show that in both cases, there exists an SPE of the game and an SPE problem is equivalent to a primal-dual pair of SOCPs.
- As an application, we present a competition problem of two firms in financial market and we show our numerical results using randomly generated data to compare different uncertainty sets considered in the paper.

We keep the same form of payoff function as considered in the conference paper, since we need a different game model for different form of payoff function, which would break the uniformity of our paper. We might consider this point in future works.

The rest of the paper is organized as follows. We present related works in Section II. The definition of a distributionally robust zero-sum game is given in Section III. Section IV presents the reformulation of distributionally robust chance constraints as second order cone constraints under different uncertainty sets. Section V outlines a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game. Section VI presents a competition of two firms in financial market as and shows numerical results. Conclusion and future works are given in Section VII.

II. Related work

In this section, we introduce previous studies on chance-constrained games. Dantzig and later Adler showed the equivalence between linear programming problems and two-player zero-sum games [5] [6]. Charnes [7] generalized the zero-sum game considered in [4] by introducing linear inequality constraints on the mixed strategies of both the players and called it a constrained zero-sum game. An SPE of a constrained zero-sum game can be obtained from the optimal solutions of a primal-dual pair of linear programs [7]. Singh and Lisser [8] considered a stochastic version of constrained zero-sum game considered by Charnes [7], where the mixed strategies of each player are restricted by random linear inequality constraints, which are modelled using chance constraints. When the random constraint vectors follow a multivariate elliptically symmetric distribution, the zero-sum game problem is equivalent to a primal-dual pair of Second-Order Cone Programs (SOCPs) [8]. Nash equilibrium is the generalization of SPE and it is used as a solution concept for the general-sum games [3] [9]. Under certain conditions on payoff functions and strategy sets, there always exists a Nash equilibrium [10]. The general-sum games under uncertainties are considered in the literature [11]–[15], which capture both risk neutral and risk averse situations. To the best of our knowledge, the distributionally robust chance-constrained approach has been widely studied in the literature but still not completed in game setup. In this paper, we want to apply different approaches in the literature to define uncertainty sets in a distributionally robust chance-constrained game and compare the performance of these approaches by simulation using randomly generated data models.

III. The model

We consider a two player zero-sum game, where each player has continuous strategy set. Let $C^1 \in \mathbb{R}^{K_1 \times m}$, $C^2 \in \mathbb{R}^{K_2 \times n}$, $d^1 \in \mathbb{R}^{K_1}$ and $d^2 \in \mathbb{R}^{K_2}$. We consider $X = \{x \in \mathbb{R}^m \mid C^1 x = d^1,\, x \geq 0\}$ and $Y = \{y \in \mathbb{R}^n \mid C^2 y = d^2,\, y \geq 0\}$ as the strategy sets of player 1 and player 2, respectively. We assume that X and Y are compact sets. Let $u : X \times Y \to \mathbb{R}$ be a payoff function associated to the zero-sum game and we assume that player 1 (resp. player 2) is interested in maximizing (resp. minimizing) $u(x, y)$ for a fixed strategy y (resp. x) of player 2 (resp. player 1). For a given strategy pair $(x, y) \in X \times Y$, the payoff function $u(x, y)$ is given by

$$u(x, y) = x^{\mathrm{T}} G y + g^{\mathrm{T}} x + h^{\mathrm{T}} y, \tag{1}$$

where $G \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$ and $h \in \mathbb{R}^n$. The first term of (1) results from the interaction between both the players whereas the second and third term represents the individual impact of player 1 and player 2 on the game, respectively. The strategy sets are often restricted by random linear constraints, which are modeled using chance constraints. The chance constraint based strategy sets appear in many practical problems, e.g., risk constraints in portfolio optimization [16]. In this paper, we consider the case, where the strategies of player 1 satisfy the following random linear constraints,

$$(a_k^1)^{\mathrm{T}} x \leq b_k^1,\ k = 1, 2, \ldots, p, \tag{2}$$

whilst the strategies of player 2 satisfy the following random linear constraints

$$(a_l^2)^{\mathrm{T}} y \geq b_l^2,\ l = 1, 2, \ldots, q. \tag{3}$$

Let $\mathcal{I}_1 = \{1, 2, \ldots, p\}$ and $\mathcal{I}_2 = \{1, 2, \ldots, q\}$ be the index sets for the constraints of player 1 and player 2, respectively. For each $k \in \mathcal{I}_1$ and $l \in \mathcal{I}_2$, the vectors a_k^1 and a_l^2 are random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the case, where the only information we have about the distributions of a_k^1 and a_l^2 is that they belong to some uncertainty sets $\mathcal{D}_k^1$ and $\mathcal{D}_l^2$, respectively. The uncertainty sets $\mathcal{D}_k^1$ and $\mathcal{D}_l^2$, are constructed based on the partially available information on the distributions of a_k^1 and a_l^2, respectively. Using the worst case approach, the random linear constraints (2) and (3) can be formulated as distributionally robust chance constraints given by

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\left((a_k^1)^{\mathrm{T}} x \leq b_k^1\right) \geq \alpha_k^1,\ \forall\ k \in \mathcal{I}_1, \tag{4}$$

and

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\left((-a_l^2)^{\mathrm{T}} y \leq -b_l^2\right) \geq \alpha_l^2,\ \forall\ l \in \mathcal{I}_2, \tag{5}$$

where α_k^1 and α_l^2 are the confidence levels of player 1 and player 2 for kth and lth constraints, respectively. Therefore, for a given $\alpha^1 = (\alpha_k^1)_{k\in\mathcal{I}_1}$ and $\alpha^2 = (\alpha_l^2)_{l\in\mathcal{I}_2}$, the feasible strategy sets of player 1 and player 2 are given by

$$S_{\alpha^1}^1 = \Big\{x \in X| \inf_{F_k^1\in\mathcal{D}_k^1} \mathbb{P}\{(a_k^1)^{\mathrm{T}}x \le b_k^1\} \ge \alpha_k^1, \ \forall\ k \in \mathcal{I}_1\Big\}, \tag{6}$$

and

$$S_{\alpha^2}^2 = \Big\{y \in Y| \inf_{F_l^2\in\mathcal{D}_l^2} \mathbb{P}\{(-a_l^2)^{\mathrm{T}}y \le -b_l^2\} \ge \alpha_l^2, \ \forall\ l \in \mathcal{I}_2\Big\}. \tag{7}$$

We call the zero-sum game with the strategy set $S_{\alpha^1}^1$ for player 1 and the strategy set $S_{\alpha^2}^2$ for player 2 as a distributionally robust zero-sum game. We denote this game by Z_α. A strategy pair $(x^*, y^*) \in S_{\alpha^1}^1 \times S_{\alpha^2}^2$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0,1]^p \times [0,1]^q$, if

$$u(x, y^*) \le u(x^*, y^*) \le u(x^*, y), \ \forall\ x \in S_{\alpha^1}^1, \ y \in S_{\alpha^2}^2. \tag{8}$$

IV. Reformulation of distributionally robust chance constraints

We consider five different uncertainty sets based on the partial information about the mean vectors and covariance matrices of the random constraint vectors a_k^i, $i = 1, 2$, $k \in \mathcal{I}_i$ and four different uncertainty sets based on the statistical distance between the distribution of a_k^i and a nominal distribution. For each uncertainty set, the distributionally robust chance constraints (4) and (5) are reformulated as second-order cone (SOC) constraints.

A. Moments Based Uncertainty Sets

We consider five moments based uncertainty sets defined as follows.

1) Uncertainty set with known mean and known covariance matrix: In some situations, we do not know exactly the true distribution of the random constraint vectors a_k^i, for all $k \in \mathcal{I}_i$, $i = 1, 2$. We can only obtain some information of the underlying distribution from historical data. For example, by observing a sufficiently large number of data, we deduce the values of mean vector and covariance matrix of a_k^i approximated by the sample mean μ_k^i and the sample covariance matrix Σ_k^i. We consider an uncertainty set, which includes all distributions F_k^i with mean vector μ_k^i and covariance matrix Σ_k^i defined as follows

$$\mathcal{D}_k^{1,i}\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} \text{The distribution of } x \text{ is } F_k^i \\ E\left[x\right] = \mu_k^i \\ Cov[x] = \Sigma_k^i \end{array} \right\}, \tag{9}$$

We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{1,i}\left(\mu_k^i, \Sigma_k^i\right)$ and the matrix Σ_k^i is a postive definite matrix. This uncertainty set has been widely considered in the literature, e.g., [17]. We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 1. *The constraints* (4) *and* (5) *are equivalent to* (10) *and* (11)*, respectively, given by*

$$(\mu_k^1)^T x + \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}||(\Sigma_k^1)^{\frac{1}{2}}x||_2 \le b_k^1, \quad \forall\ k \in \mathcal{I}_1, \tag{10}$$

$$-(\mu_k^2)^T y + \sqrt{\frac{\alpha_k^2}{1-\alpha_k^2}}||(\Sigma_k^2)^{\frac{1}{2}}y||_2 \le -b_k^2, \quad \forall\ k \in \mathcal{I}_2. \tag{11}$$

Remark 1. *An SOC constraint is the set of points $x \in \mathbb{R}^n$ such that the following inequality holds*

$$\|Ax + b\|_2 \le c^T x + d,$$

where $A \in \mathbb{R}^{m\times n}$ is an $m \times n$ real matrix, $b \in \mathbb{R}^m$ is an $m \times 1$ real vector, $c \in \mathbb{R}^n$ is an $n \times 1$ real vector and $d \in \mathbb{R}$ is a real number, $\|\cdot\|_2$ denotes the Euclidean norm. It is clear that (10) *and* (11) *are equivalent to SOC constraints. An SOC reformulation is useful since optimization problems with SOC constraints can be solved efficiently by known algorithms in polynomial time.*

Proof. Using the one-sided Chebyshev inequality, we have

$$\inf_{F_k^1\in\mathcal{D}_k^{1,i}(\mu,\Sigma)} \mathbb{P}\big\{(a_k^1)^{\mathrm{T}}x \le b_k^1\big\} = \begin{cases} 1 - \frac{1}{1+\frac{((\mu_k^1)^{\mathrm{T}}x - b_k^1)^2}{\left(x^{\mathrm{T}}\Sigma_k^1 x\right)}}, \\ \text{if } \ (\mu_k^1)^{\mathrm{T}}x \le b_k^1, \\ \\ 0, \ \text{otherwise}. \end{cases}$$

The bound of one-sided Chebyshev inequality can be achieved by a two-point distribution given by equation (2) of [18]. For the case $(\mu_k^1)^{\mathrm{T}}x > b_k^1$,

$$\inf_{F_k^1\in\mathcal{D}_k^{1,i}(\mu,\Sigma)} \mathbb{P}\big\{a_k^1 x \le b_k^1\big\} = 0,$$

which makes constraint (4) infeasible for any $\alpha_1 > 0$. Therefore, for $x \in S_{\alpha_1}^1$, the condition $(\mu_k^1)^{\mathrm{T}}x \le b_k^1$ always holds and the constraint (4) is equivalent to

$$1 - \frac{1}{1 + ((\mu_k^1)^{\mathrm{T}}x - b_k^1)^2/(x^{\mathrm{T}}\Sigma_k^1 x)} \ge \alpha_k^1.$$

The above inequality can be reformulated as (10). Similarly, we can show that (5) is equivalent to (11). □

2) Uncertainty set with known mean and unknown covariance matrix: For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where the mean vector of the random vector a_k^i is known exactly (approximated by the sample mean μ_k^i) but the covariance matrix is unknown due to several reasons, e.g., the lack of data. We assume that it is only known to belong to a positive semidefinite cone defined with a linear matrix inequality as follows

$$Cov[a_k^i] \preceq \gamma_k^i \Sigma_k^i,$$

where $\gamma_k^i > 0$ is a strictly positive real number, Σ_k^i is a positive definite matrix, for the given matrices B_1 and B_2, $B_1 \preceq B_2$ implies that $B_2 - B_1$ is a positive semidefinite matrix. In practical applications, we usually approximate the matrix Σ_k^i by the sample covariance matrix. The parameter γ_k^i is used in controlling the uncertainty level, i.e., high value of γ_k^i implies a large number of distributions in the uncertainty set, which deals uncertain factors in a more secure way. We consider un uncertainty set, which includes all distributions F_k^i with mean vector μ_k^i and covariance matrix satisfied the above constraint as follows

$$\mathcal{D}_k^{2,i}\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} \text{The distribution of } x \text{ is } F_k^i \\ E\left[x\right] = \mu_k^i \\ Cov[x] \preceq \gamma_k^i \Sigma_k^i \end{array} \right\}. \tag{12}$$

This uncertainty set is considered in [19]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{2,i}\left(\mu_k^i, \Sigma_k^i\right)$. We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 2. *The constraints* (4) *and* (5) *are equivalent to* (13) *and* (14)*, respectively, given by*

$$(\mu_k^1)^T x + \sqrt{\gamma_k^1}\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}||(\Sigma_k^1)^{\frac{1}{2}} x||_2 \leq b_k^1, \quad \forall\ k \in \mathcal{I}_1, \tag{13}$$

$$-(\mu_k^2)^T y + \sqrt{\gamma_k^2}\sqrt{\frac{\alpha_k^2}{1-\alpha_k^2}}||(\Sigma_k^2)^{\frac{1}{2}} y||_2 \leq -b_k^2, \quad \forall\ k \in \mathcal{I}_2. \tag{14}$$

Proof. Based on the structure of uncertainty set (12), the constraint (4) can be written as

$$\inf_{(\mu,\Sigma)\in\mathcal{U}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu,\Sigma)} \mathbb{P}\left\{(a_k^1)^{\mathrm{T}} x \leq b_k^1\right\} \geq \alpha_k^1,$$

where

$$\mathcal{U}_k^1 = \left\{(\mu, \Sigma)\ \middle| \mu = \mu_k^1, \Sigma \preceq \gamma_k^i \Sigma_k^i \right\}.$$

Here, the inner infimum is taken over all distributions with same value of mean vector and covariance matrix. The outer infimum is taken over all couples (μ, Σ) satisfying the conditions in (12). Using the similar arguments as in the Lemma 1, the constraint (4) is equivalent to

$$\frac{b_k^1 - (\mu_k^1)^{\mathrm{T}} x}{\max\limits_{\Sigma \preceq \gamma_k^1 \Sigma_k^1} \sqrt{x^{\mathrm{T}} \Sigma x}} \geq \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}. \tag{15}$$

The above inequality (15) can be reformulated as (13). Similarly, we can show that (5) is equivalent to (14). □

3) Uncertainty set with unknown mean and unknown covariance matrix: For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where both mean vector and covariance matrix of a_k^i are unknown. From historical data, we obtain the sample mean μ_k^i and the sample covariance matrix Σ_k^i. We deal the uncertainty level in a secure way by assuming that the mean vector and the covariance matrix of a_k^i are not exactly the same as its sample mean and sample covariance matrix. The mean vector lies in an ellipsoid of size $\gamma_{k1}^i \geq 0$ centered at μ_k^i defined by the following constraint

$$\left(\mathbb{E}[a_k^i] - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \left(\mathbb{E}[a_k^i] - \mu_k^i\right) \leq \gamma_{k1}^i,$$

and the covariance matrix of a_k^i lies in a positive semidefinite cone defined as follows

$$Cov[a_k^i] \preceq \gamma_{k2}^i \Sigma_k^i.$$

where $\gamma_{k2}^i > 0$ and Σ_k^i is a positive definite matrix. The parameters γ_{k1}^i and γ_{k2}^i are used in controlling the uncertainty level. If $\gamma_{k1}^i = 0$, the mean vector is exactly the same as its sample mean. We consider un uncertainty set, which includes all distributions F_k^i with mean vector and covariance matrix satisfied the above constraints as follows

$$\mathcal{D}_k^{3,i}(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \;\middle|\; \begin{array}{l} \text{The distribution of } x \text{ is } F_k^i \\ \\ \left(\mathbb{E}[x] - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \\ \times \left(\mathbb{E}[x] - \mu_k^i\right) \leq \gamma_{k1}^i, \\ \\ Cov[x] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right\}, \tag{16}$$

The uncertainty set (16) is considered in [20]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{3,i}\left(\mu_k^i, \Sigma_k^i\right)$. We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 3. *The constraints* (4) *and* (5) *are equivalent to* (17) *and* (18)*, respectively, given by*

$$(\mu_k^1)^T x + \left(\sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}\sqrt{\gamma_{k2}^1} + \sqrt{\gamma_{k1}^1}\right) \left\|\left(\Sigma_k^1\right)^{\frac{1}{2}} x\right\|_2 \leq b_k^1, \quad \forall\ k \in \mathcal{I}_1, \tag{17}$$

$$-(\mu_k^2)^T y + \left(\sqrt{\frac{\alpha_k^2}{1-\alpha_k^2}}\sqrt{\gamma_{k2}^2} + \sqrt{\gamma_{k1}^2}\right) \left\|\left(\Sigma_k^2\right)^{\frac{1}{2}} y\right\|_2 \leq -b_k^2, \quad \forall\ k \in \mathcal{I}_2. \tag{18}$$

Proof. Based on the structure of the uncertainty set (16), the constraint (4) can be written as

$$\inf_{(\mu,\Sigma)\in\tilde{\mathcal{U}}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu,\Sigma)} \mathbb{P}\left\{a_k^1 x \leq b_k^1\right\} \geq \alpha_k^1,$$

where

$$\tilde{\mathcal{U}_k^1} = \left\{(\mu, \Sigma) \;\middle|\; \begin{array}{l} \left(\mu - \mu_k^1\right)^\top \left(\Sigma_k^1\right)^{-1} \left(\mu - \mu_k^1\right) \leq \gamma_{k1}^1, \\ \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{array} \right\}.$$

Using the similar arguments as in the Lemma 1, the constraint (4) is equivalent to

$$\frac{b_k^1 + v_1(x)}{\sqrt{v_2(x)}} \geq \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}, \tag{19}$$

where

$$v_1(x) = \begin{cases} \min\limits_{\mu} -\mu^{\mathrm{T}} x \\ \text{s.t. } \ \left(\mu - \mu_k^1\right)^{\top} \left(\Sigma_k^1\right)^{-1} \left(\mu - \mu_k^1\right) \leq \gamma_{k1}^1, \end{cases} \tag{20}$$

$$v_2(x) = \begin{cases} \max\limits_{\Sigma} \ x^{\mathrm{T}} \Sigma x \\ \text{s.t. } \Sigma \preceq \gamma_{k2}^1 \Sigma_k^1. \end{cases}$$

Let $\beta \geq 0$ be a Lagrange multiplier associated with the constraint of optimization problem (20). By applying the KKT conditions, the optimal solution of (20) is given by $\mu = \mu_k^1 + \frac{\sqrt{\gamma_{k1}^1}\Sigma_k^1 x}{\sqrt{x^{\mathrm{T}}\Sigma_k^1 x}}$ and the associated Lagrange multiplier is given by $\beta = \sqrt{\frac{x^{\mathrm{T}}\Sigma_k^1 x}{4\gamma_{k1}^1}}$. Therefore, the corresponding optimal value $v_1(x) = -(\mu_k^1)^{\mathrm{T}} x - \sqrt{\gamma_{k1}^1}\sqrt{x^{\mathrm{T}}\Sigma_k^1 x}$. Since, $u^{\mathrm{T}}\Sigma u \leq u^{\mathrm{T}}\gamma_{k2}^1\Sigma_k^1 u$, then, $v_2(x) = \gamma_{k2}^1 x^{\mathrm{T}}\Sigma_k^1 x$. Therefore, using (19), (4) is equivalent to (17). Similarly, we can show that (5) is equivalent to (18). ☐

4) Polytopic uncertainty set: For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where both mean vector and covariance matrix of the random vector a_k^i are unknown. From historical data, we consider M samples i.i.d of the random vector a_k^i. We obtain M sample means $\mu_{k1}^i, \ldots, \mu_{kM}^i$ and M sample covariance matrix $\Sigma_{k1}^i, \ldots, \Sigma_{kM}^i$, where Σ_{kj}^i is positive definite, for any $j = 1, \ldots, M$. We consider polytopes $U_{\mu_k^i} = \text{Conv}(\mu_{k1}^i, \mu_{k2}^i, \ldots, \mu_{kM}^i)$ and $U_{\Sigma_k^i} = \text{Conv}(\Sigma_{k1}^i, \Sigma_{k2}^i, \ldots, \Sigma_{kM}^i)$, where Conv denotes the convex hull. We assume that the mean vector and the covariance matrix of a_k^i are known to belong to polytopes $U_{\mu_k^i}$ and $U_{\Sigma_k^i}$, respectively. We consider an uncertainty set, which includes all distributions F_k^i defined as follows

$$\mathcal{D}_k^{4,i}\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} \text{The distribution of } x \text{ is } F_k^i \\ E\left[x\right] \in U_{\mu_k^i} \\ Cov[x] \in U_{\Sigma_k^i} \end{array} \right\}. \tag{21}$$

The uncertainty set (21) is considered in [17]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a_k^i belongs to the uncertainty set $\mathcal{D}_k^{4,i}\left(\mu_k^i, \Sigma_k^i\right)$. We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 4. *The constraints* (4) *and* (5) *are equivalent to* (22) *and* (23)*, respectively, given by*

$$(\mu_{kj}^1)^T x + \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}} ||(\Sigma_{kw}^1)^{\frac{1}{2}} x||_2 \leq b_k^1,$$
$$\forall\ j = 1, \ldots, M,\ w = 1, \ldots, M,\ k \in \mathcal{I}_1, \tag{22}$$

$$-(\mu_{kj}^2)^T y + \sqrt{\frac{\alpha_k^2}{1-\alpha_k^2}} ||(\Sigma_{kw}^2)^{\frac{1}{2}} y||_2 \leq -b_k^2,$$
$$\forall\ j = 1, \ldots, M,\ w = 1, \ldots, M,\ k \in \mathcal{I}_2. \tag{23}$$

Remark 2. *Lemma 4 shows that the constraint* (4) *(resp.* (5)*) is equivalent to a system of M^2 constraints in* (22) *(resp.* (23)*).*

Proof. Based on the structure of uncertainty set (21), the constraint (4) can be written as

$$\inf_{(\mu,\Sigma)\in\hat{\mathcal{U}}_k^1} \inf_{F_k^1 \in \mathcal{D}_k^{1,i}(\mu,\Sigma)} \mathbb{P}\left\{(a_k^1)^{\mathrm{T}} x \leq b_k^1\right\} \geq \alpha_k^1,$$

where

$$\hat{\mathcal{U}}_k^1 = \left\{(\mu, \Sigma) \ \middle| \mu \in U_{\mu_k^1}, \Sigma \in U_{\Sigma_k^1} \right\}.$$

Using the similar arguments as in the Lemma 1, the constraint (4) can be reformulated as

$$\frac{\min\limits_{\mu\in U_{\mu_k^1}} \left(b_k^1 - \mu^{\mathrm{T}} x\right)}{\max\limits_{\Sigma\in U_{\Sigma_k^1}} \sqrt{x^{\mathrm{T}}\Sigma x}} \geq \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}}. \tag{24}$$

The above inequality (24) can be reformulated as

$$\frac{b_k^1 - (\mu_{kj}^1)^{\mathrm{T}} x}{\sqrt{x^{\mathrm{T}}\Sigma_{kw}^1 x}} \geq \sqrt{\frac{\alpha_k^1}{1-\alpha_k^1}},$$
$$\forall\, j = 1, \ldots, M,\, w = 1, \ldots, M,\, k \in \mathcal{I}_1,$$

which is equivalent to (22). Similarly, we can show that (5) is equivalent to (23). ☐

5) Uncertainty set with componentwise bounds on mean vector and covariance matrix: For all $i = 1, 2$ and $k \in \mathcal{I}_i$, we consider the case, where the mean vector and the covariance matrix of a_k^i are unknown. We obtain from historical data, a sample mean vector μ_k^i and a sample covariance matrix Σ_k^i. We do not approximate the mean vector and the covariance matrix of a_k^i by its sample mean vector and sample covariance matrix, but we deal the uncertainty level by a more secure way. For each $j = 1, \ldots, m$, we assume that the j^{th}-component of the mean vector of a_k^i lies in a ball of radius $\epsilon_{\mu,k}^i(j) \geq 0$, centered at the j^{th}-component of the sample mean vector μ_k^i, which can be reformulated as follows

$$\mu_k^i - \epsilon_{\mu,k}^i \leq \mathbb{E}[a_k^i] \leq \mu_k^i + \epsilon_{\mu,k}^i,$$

where $\epsilon_{\mu,k}^i = \left(\epsilon_{\mu,k}^i(1), \ldots, \epsilon_{\mu,k}^i(m)\right)$ is an $m \times 1$ vector and the above inequalities are understood componentwise. Similarly, for each $j = 1, \ldots, m$ and $w = 1, \ldots, m$, we assume that the $(j, w)-$ entry of the covariance matrix of a_k^i lies in a ball of radius $\epsilon_{\Sigma,k}^i(j, w) \geq 0$, centered at the $(j, w)-$ entry of the sample covariance matrix Σ_k^i, which can be reformulated as follows

$$\Sigma_k^i - \epsilon_{\Sigma,k}^i \leq \text{Cov}[a_k^i] \leq \mu_k^i + \epsilon_{\Sigma,k}^i,$$

where $\epsilon_{\Sigma,k}^i = \epsilon_{\Sigma,k}^i(j, w)_{1\leq j,w\leq m}$ is an $m \times m$ matrix. Let $\mu_{k-}^i = \mu_k^i - \epsilon_{\mu,k}^i, \mu_{k+}^i = \mu_k^i + \epsilon_{\mu,k}^i, \quad \Sigma_{k-}^i = \Sigma_k^i - \epsilon_{\Sigma,k}^i,$

and $\Sigma^i_{k+} = \Sigma^i_k + \epsilon^i_{\Sigma,k}$. We consider an uncertainty set, which includes all distributions F^i_k defined as follows

$$\mathcal{D}^{5,i}_k(\mu^i_k, \Sigma^i_k) = \left\{ F^i_k \;\middle|\; \begin{array}{l} \text{The distribution of } x \text{ is } F^i_k \\ \mu^i_{k-} \le \mathbb{E}[x] \le \mu^i_{k+}, \\ \Sigma^i_{k-} \le \text{Cov}[x] \le \Sigma^i_{k+} \end{array} \right\}, \tag{25}$$

Since Σ^i_k is a positive definite matrix, we can take $\epsilon^i_{\Sigma,k} > 0$ such that for any matrix H, if $\Sigma^i_{k-} \le H \le \Sigma^i_{k+}$, then H is a positive definite matrix. We define a set of vectors S^1_k as follows

$$S^1_k = \left\{ \mu \in \mathbb{R}^m \mid \mu(j) = \mu^1_{k-}(j) \,\text{or}\, \mu^1_{k+}(j),\, \forall\, j = 1, \dots, m \right\},$$

where $\mu(j)$ is the $j^{\text{th}}-$ component of μ, $\mu^1_{k-}(j)$ is the $j^{\text{th}}-$ component of μ^1_{k-}, and $\mu^1_{k+}(j)$ is the $j^{\text{th}}-$ component of μ^1_{k+}. For example, if $\mu^1_{k-} = (1,2)^{\text{T}}, \mu^1_{k+} = (5,6)^{\text{T}}$, then S^1_k is a set of 4 vectors $\left\{(1,5)^{\text{T}}, (1,6)^{\text{T}}, (2,5)^{\text{T}}, (2,6)^{\text{T}}\right\}$. We define a set of covariance matrix T^1_k as follows

$$T^1_k = \left\{ \Sigma \mid \Sigma(j,w) = \Sigma^1_{k-}(j,w) \,\text{or}\, \Sigma^1_{k+}(j,w),\, 1 \le j, w \le m \right\},$$

Similarly, we define a set of vectors S^2_k and a set of covariance matrix T^2_k. The uncertainty set (25) is considered in [17]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a^i_k belongs to the uncertainty set $\mathcal{D}^{5,i}_k\left(\mu^i_k, \Sigma^i_k\right)$. We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 5. *The constraints* (4) *and* (5) *are equivalent to* (26) *and* (27)*, respectively, given by*

$$(\mu^1)^T x + \sqrt{\frac{\alpha^1_k}{1-\alpha^1_k}} ||(\Sigma^1)^{\frac{1}{2}} x||_2 \le b^1_k,$$
$$\forall\ \mu^1 \in S^1_k,\ \Sigma^1 \in T^1_k,\ k \in \mathcal{I}_1, \tag{26}$$

$$-(\mu^2)^T y + \sqrt{\frac{\alpha^2_k}{1-\alpha^2_k}} ||(\Sigma^2)^{\frac{1}{2}} y||_2 \le -b^2_k,$$
$$\forall\ \mu^2 \in S^2_k,\ \Sigma^2 \in T^2_k,\ k \in \mathcal{I}_2. \tag{27}$$

Remark 3. *Note that S^1_k is a set of 2^m vectors and T^1_k is a set of 2^{m^2} matrix. Then, Lemma 5 shows that the constraint* (4) *is equivalent to a system of $2^m \times 2^{m^2}$ constraints in* (26)*, for any $k \in \mathcal{I}_1$ and the constraint* (5) *is equivalent to a system of $2^n \times 2^{n^2}$ constraints in* (27)*, for any $k \in \mathcal{I}_2$.*

Proof. Based on the structure of the uncertainty set (25), the constraint (4) can be written as

$$\inf_{(\mu,\Sigma)\in\bar{\mathcal{U}}^1_k} \inf_{F^1_k \in \mathcal{D}^{1,i}_k(\mu,\Sigma)} \mathbb{P}\left\{a^1_k x \le b^1_k\right\} \ge \alpha^1_k,$$

where

$$\bar{\mathcal{U}}^1_k = \left\{ (\mu, \Sigma) \;\middle|\; \begin{array}{l} \mu^1_{k-} \le \mu \le \mu^1_{k+}, \\ \Sigma^1_{k-} \le \mu \le \Sigma^i_{k+}. \end{array} \right\}.$$

Using the similar arguments as in the Lemma 1, the constraint (4) is equivalent to

$$\frac{b^1_k + v_1(x)}{\sqrt{v_2(x)}} \ge \sqrt{\frac{\alpha^1_k}{1-\alpha^1_k}}, \tag{28}$$

where

$$v_1(x) = \begin{cases} \min\limits_{\mu} -\mu^{\text{T}} x \\ \text{s.t. } \ \mu^1_{k-} \le \mu \le \mu^1_{k+}, \end{cases}$$

$$v_2(x) = \begin{cases} \max\limits_{\Sigma} \ x^{\text{T}} \Sigma x \\ \text{s.t. } \Sigma^1_{k-} \le \Sigma \le \Sigma^i_{k+}. \end{cases}$$

Note that the objective functions $-\mu^{\text{T}} x$ and $x^{\text{T}} \Sigma x$ are linear functions w.r.t μ (resp. Σ). Then, it is clear that the optimal values $v_1(x)$ and $v_2(x)$ hold only when $\mu \in S^1_k$ and $\Sigma \in T^1_k$. Then, the constraint (4) can be reformulated as (26). Similarly, we can show that (5) is equivalent to (27). ☐

B. Statistical Distance Based Uncertainty Sets

In this section, we define uncertainty sets using a metric called $\phi-$divergence. For any $i = 1, 2$ and $k \in \mathcal{I}_i$, the decision makers (the two players in the game) believe that the true distribution of a^i_k oscillates around a Normal distribution of mean vector μ^i_k and covariance matrix Σ^i_k, where μ^i_k and Σ^i_k are sample mean vector and sample covariance matrix obtained from historical data. We assume that the true distribution of a^i_k lies in a ball of radius θ^i_k, centered at a nominal distribution ν^i_k and the distance between these two distributions is given by $\phi-$divergence metric. The nominal distribution ν^i_k is assumed to be Normal distributed of mean vector μ^i_k and covariance matrix Σ^i_k.

Definition 1. *The $\phi-$divergence distance between two measures μ and ν with densities f_μ and f_ν, respectively, with support in $\mathbb{R}^{r_i}$ is defined as follows:*

$$I_\phi(\mu, \nu) = \int_{\mathbb{R}^{r_i}} \phi\left(\frac{f_\mu(\xi)}{f_\nu(\xi)}\right) f_\nu(\xi) d\xi.$$

where $r_1 = m$ and $r_2 = n$.

There are different types of $\phi-$divergences distance, we refer to [21] and [22] for different choices of function ϕ. We consider an uncertainty set $\mathcal{D}^{\phi,i}_k$ defined as follows

$$\mathcal{D}^{\phi,i}_k = \left\{ F^i_k \in \mathcal{M}+ \mid I_\phi(F^i_k, \nu^i_k) \le \theta^i_k \right\}, \tag{29}$$

where $\mathcal{M}^i+$ is the set of all probability measures on $\mathbb{R}^{r_i}$, with $r_1 = m$, $r_2 = n$, and $\theta^i_k > 0$. This uncertainty set is considered in [23]. We assume that for each $i = 1, 2$ and $k \in \mathcal{I}_i$, the true distribution of a^i_k belongs to the uncertainty set $\mathcal{D}^{\phi,i}_k\left(\mu^i_k, \Sigma^i_k\right)$.

Definition 2. *The conjugate of the function ϕ is a function $\phi^* : \mathbb{R} \to \mathbb{R} \cup +\infty$ such that*

$$\phi^*(s) = \sup_{t \ge 0} \left\{ st - \phi(t) \right\}.$$

We study some special cases of $\phi-$divergences, which are summarized in Table I. The data of Table I are taken from [21]. The following lemma provides the first reformulation of the constraints (4) and (5).

TABLE I
LIST OF SELECTED $\phi-$DIVERGENCES WITH THEIR CONJUGATE RESPECTIVELY

Divergence	$\phi(t), t \geq 0$	$\phi^*(s)$
Kullback-Leibler	$t\log(t) - t + 1.$	$\mathrm{e}^s - 1$
Variation distance	$\lvert t-1 \rvert.$	$\begin{cases} -1, & s \leq -1, \\ s, & -1 \leq s \leq 1, \\ +\infty, & s > 1. \end{cases}$
Modified χ^2 - distance	$(t-1)^2.$	$\begin{cases} -1, & s \leq -2, \\ s + \frac{s^2}{4}, & s > -2. \end{cases}$
Hellinger distance	$(\sqrt{t}-1)^2.$	$\begin{cases} \frac{s}{1-s}, & s < 1, \\ +\infty, & s \geq 1. \end{cases}$

Lemma 6. *The constraint* (4) *is equivalent to*

$$\sup_{\lambda > 0, \beta \in \mathbb{R}} \left\{ f_k^1(\lambda, \beta) \right\} \geq \alpha_k^1, \tag{30}$$

where $f_k^1(\lambda, \beta) = \beta - \lambda\theta_k^1 - \lambda\phi^*\left(\frac{-1+\beta}{\lambda}\right)\mathbb{P}_{\nu_k^1}(M_k^1) - \lambda\phi^*\left(\frac{\beta}{\lambda}\right)\left[1 - \mathbb{P}_{\nu_k^1}(M_k^1)\right]$, *and* $M_k^1 = \left\{\xi \in \mathbb{R}^m \mid \xi^T x \leq b_k^1\right\}$. *The constraint* (5) *is equivalent to*

$$\sup_{\lambda > 0, \beta \in \mathbb{R}} \left\{ f_k^2(\lambda, \beta) \right\} \geq \alpha_k^2,$$

where $f_k^2(\lambda, \beta) = \beta - \lambda\theta_k^2 - \lambda\phi^*\left(\frac{-1+\beta}{\lambda}\right)\mathbb{P}_{\nu_k^2}(M_k^2) - \lambda\phi^*\left(\frac{\beta}{\lambda}\right)\left[1 - \mathbb{P}_{\nu_k^2}(M_k^2)\right]$, *and* $M_k^2 = \left\{\xi \in \mathbb{R}^n \mid \xi^T x \leq b_k^2\right\}$.

Proof. For $k \in \mathcal{I}_1$, consider the following optimization problem

$$v_k^{\mathrm{P}} = \inf_{F_k^1 \in \mathcal{D}_k^{\phi,1}} \mathbb{P}\left((a_k^1)^{\mathrm{T}} x \leq b_k^1\right).$$

The above problem is rewritten as

$$\begin{aligned} v_{\mathrm{P}}^k = \inf_{F \geq 0} & \int_{\mathbb{R}^m} \mathbf{1}_{M_k^1}(\xi) F(\xi) \mathrm{d}\xi \\ \text{s.t.} \quad (i) & \quad \int_{\mathbb{R}^m} f_{\nu_k^1}(\xi) \phi\left(\frac{F(\xi)}{f_{\nu_k^1}(\xi)}\right) \mathrm{d}\xi \leq \theta_k^1, \\ (ii) & \quad \int_{\mathbb{R}^m} F(\xi) \mathrm{d}\xi = 1, \end{aligned} \tag{31}$$

where the infimum value is taken over all positive measures on $\mathbb{R}^m$. The Lagrangian dual of (31) can be written as follows

$$v_{\mathrm{D}}^k = \sup_{\lambda \geq 0, \beta \in \mathbb{R}} \left\{ \beta - \lambda\theta_k^1 + \inf_{F(\xi) \geq 0} \int_{\mathbb{R}^m} g_k^1(\lambda, \beta) \right\},$$

where $g_k^1(\lambda, \beta) = \mathbf{1}_{M_k^1}(\xi) F(\xi) - \beta F(\xi) + \lambda f_{\nu_k^1}(\xi) \phi\left(\frac{F(\xi)}{f_{\nu_k^1}(\xi)}\right) \mathrm{d}\xi$, λ is the dual variable of the constraint (i) and β is the dual variable of the constraint (ii). Since $\theta_k^1 > 0$, the Slater's condition holds, then the strong duality holds, i.e., $v_{\mathrm{P}}^k = v_{\mathrm{D}}^k$. The rest of the proof follows from Theorem 1 [23]. □

We present an SOC reformulation of the constraints (4) and (5) by the following lemma.

Lemma 7. *The constraints* (4) *and* (5) *are equivalent to* (32) *and* (33), *respectively, given by:*

$$(\mu_k^1)^T x + \Phi^{(-1)}\left[H(\theta_k^1, 1 - \alpha_k^1)\right] \left\| \left(\Sigma_k^1\right)^{\frac{1}{2}} x \right\|_2 \leq b_k^1, \quad \forall\ k \in \mathcal{I}_1, \tag{32}$$

$$-(\mu_k^2)^T y + \Phi^{(-1)}\left[H(\theta_k^2, 1 - \alpha_k^2)\right] \left\| \left(\Sigma_k^2\right)^{\frac{1}{2}} y \right\|_2 \leq -b_k^2, \quad \forall\ k \in \mathcal{I}_2. \tag{33}$$

where $\Phi^{(-1)}$ *is the quantile of the standard Normal distribution and* H *is a function whose value is given in Table II.*

TABLE II
LIST OF SELECTED $\phi-$DIVERGENCES WITH THE FUNCTION f RESPECTIVELY

Divergence	$H(\theta, \epsilon) =$	θ, ϵ
Kullback-Leibler	$\inf_{x \in (0,1)} \frac{e^{-\theta} x^{1-\epsilon} - 1}{x - 1}$	$\theta > 0$ $0 < \epsilon < 1$
Variation distance	$1 - \epsilon + \frac{\theta}{2}$	$\theta > 0$ $0 < \epsilon < 1$
Modified χ^2 - distance	$1 - \epsilon + \frac{\sqrt{\theta^2 + 4\theta(\epsilon - \epsilon^2)} - (1 - 2\epsilon)\theta}{2\theta + 2},$	$\theta > 0$ $0 < \epsilon < \frac{1}{2}$
Hellinger distance	$\frac{-B + \sqrt{\Delta}}{2},$ where $B = -(2 - (2-\theta)^2)\epsilon - \frac{(2-\theta)^2}{2},$ $C = \left(\frac{(2-\theta)^2}{4} - \epsilon\right)^2,$ $\Delta = B^2 - 4C = (2-\theta)^2\left[4 - (2-\theta)^2\right]\epsilon(1-\epsilon),$	$0 < \theta < 2 - \sqrt{2}$ $0 < \epsilon < 1$

Proof. Using Lemma 6, we prove that the constraint (4) is equivalent to

$$\mathbb{P}_{\nu_k^1}(M_k^1) \geq H(\theta_k^1, 1 - \alpha_k^1). \tag{34}$$

Since ν_k^1 follows a Normal distribution with mean vector μ_k^1 and covariance matrix Σ_k^1, it is well known that (34) is equivalent to the SOC constraint (32). We refer to Propositions 2, 3, and 4, [23] for the proof of the cases Kullback-Leibler, Variation distance and Modified χ^2 - distance. The proof of the case Hellinger distance is given in Appendix A. □

C. Second Order Cone Reformulation

In this section, we summarize our SOC reformulation results from Lemmas 1, 2, 3, 4, 5, and 7. They show that in all cases of uncertainty sets defined in Sections IV-A and IV-B, the feasible strategy sets (6) and (7) can be written as

$$S_{\alpha^1}^1 = \Big\{ x \in X \mid (\mu_{kj}^1)^{\mathrm{T}} x + \kappa_{\alpha_k^1} ||(\Sigma_{kw}^1)^{\frac{1}{2}} x||_2 \leq b_k^1, \ \forall\ j = 1, 2, \ldots, N_1,\ w = 1, 2, \ldots, P_1,\ k \in \mathcal{I}_1 \Big\}, \tag{35}$$

and

$$S_{\alpha^2}^2 = \Big\{ y \in Y \mid -(\mu_{lj}^2)^{\mathrm{T}} y + \kappa_{\alpha_l^2} ||(\Sigma_{lw}^2)^{\frac{1}{2}} y||_2 \leq -b_l^2, \ \forall\ j = 1, 2, \ldots, N_2,\ w = 1, 2, \ldots, P_2,\ l \in \mathcal{I}_2. \Big\}. \tag{36}$$

- If the uncertainty set is defined by (9), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1 - \alpha_k^i}}$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2$, $k \in \mathcal{I}_i$.

- If the uncertainty set is defined by (12), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}\sqrt{\gamma_k^i}$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2, \quad k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (16), then $\kappa_{\alpha_k^i} = \left(\sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}\sqrt{\gamma_{k2}^i} + \sqrt{\gamma_{k1}^i}\right)$ and $N_1 = P_1 = N_2 = P_2 = 1$, for all $i = 1, 2, \quad k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (21), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$ and $N_1 = P_1 = N_2 = P_2 = M$, for all $i = 1, 2$, $k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (25), then $\kappa_{\alpha_k^i} = \sqrt{\frac{\alpha_k^i}{1-\alpha_k^i}}$ and $N_1 = 2^m; P_1 = 2^{(m^2)}, N_2 = 2^n, P_2 = 2^{(n^2)}$, for all $i = 1, 2, \quad k \in \mathcal{I}_i$.
- If the uncertainty set is defined by (29), then $\kappa_{\alpha_k^i} = \Phi^{(-1)}\left[H(\theta_k^i, 1-\alpha_k^i)\right]$ and $N_1 = P_1 = N_2 = P_2 = 1$, where H and $\Phi^{(-1)}$ are defined in Lemma 7.

We assume that the strategy sets (35) and (36) satisfy the strict feasibility condition given by Assumption 1.

Assumption 1. 1) *There exists an $x \in S^1_{\alpha^1}$ such that the inequality constraints of $S^1_{\alpha^1}$ defined by* (35) *are strictly satisfied.*
2) *There exists an $y \in S^2_{\alpha^2}$ such that the inequality constraints of $S^2_{\alpha^2}$ defined by* (36) *are strictly satisfied.*

The conditions given in Assumption 1 are Slater's condition, which are sufficient for strong duality in a convex optimization problem. We use these conditions in order to derive equivalent SOCPs for the zero-sum game Z_α.

V. Existence and Characterization of Saddle Point Equilibrium

In this section, we show that there exists an SPE of the game Z_α if the distributions of the random constraint vectors of both the players belong to the uncertainty sets defined in Sections IV-A and IV-B. We further propose a primal-dual pair of SOCPs whose optimal solutions constitute an SPE of the game Z_α.

Theorem 1. *Consider the game Z_α, where the distributions of the random constraint vectors a_k^i, $k \in \mathcal{I}_i$, $i = 1, 2$, belong to the uncertainty sets described in Sections IV-A and IV-B. Then, there exists an SPE of the game for all $\alpha \in (0,1)^p \times (0,1)^q$.*

Proof. Let $\alpha \in (0,1)^p \times (0,1)^q$. For uncertainty sets described in Sections IV-A and IV-B, the strategy sets $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are given by (35) and (36), respectively. It is easy to see that $S^1_{\alpha^1}$ and $S^2_{\alpha^2}$ are convex and compact sets. The function $u(x, y)$ is a bilinear and continuous function. Hence, there exists an SPE from the minimax theorem [4]. □

A. Equivalent Primal-Dual Pair of Second-Order Cone Programs

From the minimax theorem [4], (x^*, y^*) is an SPE for the game Z_α if and only if

$$x^* \in \arg\max_{x \in S^1_{\alpha^1}} \min_{y \in S^2_{\alpha^2}} u(x, y), \tag{37}$$

$$y^* \in \arg\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x, y). \tag{38}$$

We start with the optimization problem

$$\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x, y).$$

By introducing auxiliary variables t^1_{kjw}, the inner optimization problem $\max_{x \in S^1_{\alpha^1}} u(x, y)$ can be equivalently written as

$$\begin{aligned}
&\max_{x,\, t^1_{kjw}} \; x^{\mathrm{T}} G y + g^{\mathrm{T}} x + h^{\mathrm{T}} y \\
&\text{s.t.} \\
&(i) \quad -x^{\mathrm{T}} \mu^1_{kj} - \kappa_{\alpha^1_k} \left\| t^1_{kjw} \right\|_2 + b^1_k \geq 0, \\
&\quad \forall\; j = 1, 2 \ldots, N_1,\; w = 1, 2 \ldots, P_1,\; k \in \mathcal{I}_1, \\
&(ii) \quad t^1_{kjw} - \left(\Sigma^1_{kw}\right)^{\frac{1}{2}} x = 0, \\
&\quad \forall\; j = 1, 2 \ldots, N_1,\; w = 1, 2 \ldots, P_1,\; k \in \mathcal{I}_1, \\
&(iii) \quad C^1 x = d^1,\; x_r \geq 0,\; \forall\; r = 1, 2, \ldots, m.
\end{aligned} \tag{39}$$

Let $\lambda^1 = \left(\lambda^1_{kjw}\right)$, δ^1_{kjw}, and ν^1 be the Lagrange multipliers of constraints $(i), (ii)$, and equality constraints given in the constraint (iii) of (39), respectively. Here, for any $j = 1, \ldots, N_1, w = 1, \ldots, P_1, k \in \mathcal{I}_1$, λ^1_{kjw} is a real number, δ^1_{kjw} is an $m \times 1$ real vector, and ν^1 is a $K_1 \times 1$ real vector. Then, the Lagrangian dual problem of the SOCP (39) can be written as

$$\begin{aligned}
&\min_{\lambda_1 \geq 0,\; \delta^1_{kjw},\; \nu^1} \; \max_{x \geq 0,\; t^1_{kjw}} \Big\{ x^{\mathrm{T}} G y + g^{\mathrm{T}} x + h^{\mathrm{T}} y \\
&+ \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \Big[\lambda^1_{kjw} \big(-x^{\mathrm{T}} \mu^1_{kj} - \kappa_{\alpha^1_k} \left\| t^1_{kjw} \right\|_2 + b^1_k \big) \\
&+ (\delta^1_{kjw})^{\mathrm{T}} \Big(t^1_{kjw} - \left(\Sigma^1_{kw}\right)^{\frac{1}{2}} x \Big) \Big] + (\nu^1)^{\mathrm{T}} (d^1 - C^1 x) \Big\}.
\end{aligned}$$

By reformulating the objective function of the above optimization problem as the sum of two functions such that one depends on x and other depends on t^1_{kjw}, we have

$$\begin{aligned}
&\min_{\lambda_1 \geq 0, \delta^1_{kjw}, \nu^1} \max_{x \geq 0} \Big\langle \; x^{\mathrm{T}} \Big[G y - (C^1)^{\mathrm{T}} \nu^1 + g \\
&- \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \left(\lambda^1_{kjw} \mu^1_{kj} + \left(\Sigma^1_{kw}\right)^{\frac{1}{2}} \delta^1_{kjw} \right) \Big] \\
&+ \max_{t^1_{kjw}} \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \; \Big[(\delta^1_{kjw})^{\mathrm{T}} t^1_{kjw} - \kappa_{\alpha^1_k} \lambda^1_{kjw} \left\| t^1_{kjw} \right\|_2 \Big] \\
&+ h^{\mathrm{T}} y + (\nu^1)^{\mathrm{T}} d^1 + \sum_{k \in \mathcal{I}_1} \sum_{j=1}^{N_1} \sum_{w=1}^{P_1} \lambda^1_{kjw} b^1_k \Big\rangle.
\end{aligned}$$

The first term of the objective function is a function of x

$$x^{\mathrm{T}}\Big[Gy-(C^1)^{\mathrm{T}}\nu^1+g-\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\big(\lambda^1_{kjw}\mu^1_{kj}+\big(\Sigma^1_{kw}\big)^{\frac{1}{2}}\delta^1_{kjw}\big)\Big]. \tag{40}$$

The above term is unbounded on the domaine $x\geq 0$, unless the following condition holds

$$Gy-\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\big(\lambda^1_{kjw}\mu^1_{kj}+\big(\Sigma^1_{kw}\big)^{\frac{1}{2}}\delta^1_{kjw}\big)-(C^1)^{\mathrm{T}}\nu^1+g\leq 0.$$

When the above condition holds, it is clear that the maximum value of (40) is zero and it holds at $x=0$. The second term of the objective function is a function of t^1_{kjw}

$$\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\Big[(\delta^1_{kjw})^{\mathrm{T}}t^1_{kjw}-\kappa_{\alpha^1_k}\lambda^1_{kjw}\left\|t^1_{kjw}\right\|_2\Big]. \tag{41}$$

The above term is unbounded on the domaine $t^1_{kjw}\in\mathbb{R}^m$, unless the following condition holds

$$||\delta^1_{kjw}||\leq\kappa_{\alpha^1_k}\lambda^1_{kjw},$$
$$\forall\ k\in\mathcal{I}_1, j=1,2\ldots,N_1, w=1,2\ldots,P_1.$$

When the above condition holds, it is clear that the maximum value of (41) is zero and it holds at $t^1_{kjw}=0$. Then, the Lagrangian dual problem of the SOCP (39) can be written as

$$\min_{\lambda_1\geq 0,\delta^1_{kjw},\nu^1}\big(h^{\mathrm{T}}y+(\nu^1)^{\mathrm{T}}d^1+\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda^1_{kjw}b^1_k\big)$$
$$\text{s.t. } (i)\quad Gy-\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\big[\lambda^1_{kjw}\mu^1_{kj}+\big(\Sigma^1_{kw}\big)^{\frac{1}{2}}\delta^1_{kjw}\big]-(C^1)^{\mathrm{T}}\nu^1+g\leq 0,$$
$$(ii)\quad ||\delta^1_{kjw}||\leq\kappa_{\alpha^1_k}\lambda^1_{kjw},$$
$$\forall\ k\in\mathcal{I}_1, j=1,2\ldots,N_1, w=1,2\ldots,P_1.$$

Under Assumption 1, the Lagrangian dual problem of (39) has zero duality gap [24], which implies that the above optimization problem is equivalent to the problem $\max_{x\in S^1_{\alpha^1}}u(x,y)$. Hence, the problem $\min_{y\in S^2_{\alpha^2}}\max_{x\in S^1_{\alpha^1}}u(x,y)$ is equivalent to the following SOCP

$$\min_{y,\ \nu^1,\ \delta^1_{kjw},\ \lambda^1_{kjw}\geq 0} h^{\mathrm{T}}y+(\nu^1)^{\mathrm{T}}d^1+\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda^1_{kjw}b^1_k$$
$$\text{s.t.}$$
$$(i)\quad Gy-\sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\big[\lambda^1_{kjw}\mu^1_{kj}+\big(\Sigma^1_{kw}\big)^{\frac{1}{2}}\delta^1_{kjw}\big]-(C^1)^{\mathrm{T}}\nu^1+g\leq 0,$$
$$(ii)\quad ||\delta^1_{kjw}||\leq\kappa_{\alpha^1_k}\lambda^1_{kjw},$$
$$\forall\ k\in\mathcal{I}_1,\ j=1,2\ldots,N_1,\ w=1,2\ldots,P_1,$$
$$(iii)\quad -(\mu^2_{lj})^{\mathrm{T}}y+\kappa_{\alpha^2_l}||(\Sigma^2_{lw})^{\frac{1}{2}}y||\leq -b^2_l,$$
$$\forall\ j=1,2,\ldots,N_2,\ w=1,2,\ldots,P_2,\ l\in\mathcal{I}_2,$$
$$(iv)\quad C^2y=d^2,\ y_s\geq 0,\ \forall\ s=1,2,\ldots,n, \tag{42}$$

where the constraints (iii) and (iv) are due to the fact that $y\in S^2_{\alpha^2}$ and the representation of $S^2_{\alpha^2}$ in (36). Similarly, problem $\max_{x\in S^1_{\alpha^1}}\min_{y\in S^2_{\alpha^2}}u(x,y)$ is equivalent to the following SOCP

$$\max_{x,\ \nu^2,\ \delta^2_{ljw},\ \lambda^2_{ljw}\geq 0} g^{\mathrm{T}}x+(\nu^2)^{\mathrm{T}}d^2-\sum_{l\in\mathcal{I}_2}\sum_{j=1}^{N_2}\sum_{w=1}^{P_2}\lambda^2_{ljw}b^2_l$$
$$\text{s.t.}$$
$$(i)\quad G^{\mathrm{T}}x-\sum_{l\in\mathcal{I}_2}\sum_{j=1}^{N_2}\sum_{w=1}^{P_2}\big[-\lambda^2_{ljw}\mu^2_{lj}+\big(\Sigma^2_{lw}\big)^{\frac{1}{2}}\delta^2_{ljw}\big]-(C^2)^{\mathrm{T}}\nu^2+h\geq 0,$$
$$(ii)\quad ||\delta^2_{ljw}||\leq\kappa_{\alpha^2_l}\lambda^2_{ljw},\ \lambda^2_{ljw}\geq 0,$$
$$\forall\ \ l\in\mathcal{I}_2,\ j=1,2,\ldots,N_2,\ w=1,2,\ldots,P_2,$$
$$(iii)\quad (\mu^1_{kj})^{\mathrm{T}}x+\kappa_{\alpha^1_k}||(\Sigma^1_{kw})^{\frac{1}{2}}x||\leq b^1_k,$$
$$\forall\ j=1,2,\ldots,N_2,\ w=1,2,\ldots,P_2,\ k\in\mathcal{I}_1,$$
$$(iv)\quad C^1x=d^1,\ x_r\geq 0,\ \forall\ r=1,2,\ldots,m. \tag{43}$$

It follows from the duality theory of SOCPs that (42) and (43) form a primal-dual pair of SOCPs [24]. Next, we show that the equivalence between the optimal solutions of (42)-(43) and an SPE of the game Z_α.

Theorem 2. *Consider the zero-sum game Z_α, where the feasible strategy sets of player 1 and player 2 are given by* (35) *and* (36)*, respectively. Let Assumption 1 holds. Then, for a given $\alpha\in(0,1)^p\times(0,1)^q$, (x^*,y^*) is an SPE of the game Z_α if and only if there exists $(\nu^{1*},\delta^{1*}_{kjw},\lambda^{1*}_{kjw}\geq 0)$ and $(\nu^{2*},\delta^{2*}_{ljw},\lambda^{2*}_{ljw}\geq 0)$ such that $(y^*,\nu^{1*},\delta^{1*}_{kjw},\lambda^{1*}_{kjw})$ and $(x^*,\nu^{2*},\delta^{2*}_{ljw},\lambda^{2*}_{ljw})$ are optimal solutions of* (42) *and* (43)*, respectively.*

Proof. Let (x^*,y^*) be an SPE of the game Z_α. Then, x^* and y^* are the solutions of (37) and (38), respectively. Therefore, there exists $(\nu^{1*},\delta^{1*}_{kjw},\lambda^{1*}_{kjw}\geq 0)$ and $(\nu^{2*},\delta^{2*}_{ljw},\lambda^{2*}_{ljw}\geq 0)$ such that $(y^*,\nu^{1*},\delta^{1*}_{kjw},\lambda^{1*}_{kjw})$ and $(x^*,\nu^{2*},\delta^{2*}_{ljw},\lambda^{2*}_{ljw})$ are optimal solutions of (42) and (43) respectively. On the other hand, let $(y^*,\nu^{1*},\delta^{1*}_{kjw},\lambda^{1*}_{kjw})$ and $(x^*,\nu^{2*},\delta^{2*}_{ljw},\lambda^{2*}_{ljw})$

be optimal solutions of (42) and (43), respectively. Under Assumption 1, (42) and (43) are strictly feasible. Therefore, strong duality holds for primal-dual pair (42)-(43). Then, we have

$$\begin{aligned} &g^{\mathrm{T}}x^* + (\nu^{2*})^{\mathrm{T}}d^2 - \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{N_2}\sum_{w=1}^{P_2}\lambda_{ljw}^{2*}b_l^2 \\ &= h^{\mathrm{T}}y^* + (\nu^{1*})^{\mathrm{T}}d^1 + \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda_{kjw}^{1*}b_k^1. \end{aligned} \tag{44}$$

Consider the constraint (i) of (42) at optimal solution $(y^*, \nu^{1*}, \delta_{kjw}^{1*}, \lambda_{kjw}^{1*})$ and multiply it by x^{T}, for any $x \in S_{\alpha^1}^1$, we have

$$\begin{aligned} &x^{\mathrm{T}}Gy^* + g^{\mathrm{T}}x \le x^{\mathrm{T}}(C^1)^{\mathrm{T}}\nu^{1*} \\ &+ \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\left[x^{\mathrm{T}}\mu_{kj}^1\lambda_{kjw}^{1*} + x^{\mathrm{T}}(\Sigma_{kw}^1)^{\frac{1}{2}}\delta_{kjw}^{1*}\right]. \end{aligned} \tag{45}$$

By using the Cauchy-Schwartz inequality, for any $k \in \mathcal{I}_1,\ j = 1,2\dots,N_1,\ w = 1,2\dots,P_1$, we have

$$x^{\mathrm{T}}(\Sigma_{kw}^1)^{\frac{1}{2}}\delta_{kjw}^{1*} \le \|(\Sigma_{kw}^1)^{\frac{1}{2}}x\|_2\|\delta_{kjw}^{1*}\|_2.$$

Using the constraint (ii) of (43), the above constraint implies that

$$(x^*)^{\mathrm{T}}(\Sigma_{kw}^1)^{\frac{1}{2}}\delta_{kjw}^{1*} \le \|(\Sigma_{kw}^1)^{\frac{1}{2}}x\|_2\kappa_{\alpha_k^1}\lambda_{kjw}^{1*}.$$

Since $x \in S_{\alpha^1}^1$, we have

$$C^1x = d^1.$$

Then, the constraint (45) implies that

$$\begin{aligned} &x^{\mathrm{T}}Gy^* + g^{\mathrm{T}}x \le (\nu^{1*})^{\mathrm{T}}d^1 \\ &+ \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\left[x\mathrm{T}\mu_{kj}^1\lambda_{kjw}^{1*} + (\Sigma_{kw}^1)^{\frac{1}{2}}x\|_2\kappa_{\alpha_k^1}\lambda_{kjw}^{1*}\right], \end{aligned}$$

which in turn implies by using the constraint (iii) of (43) that

$$x^{\mathrm{T}}Gy^* + g^{\mathrm{T}}x \le (\nu^{1*})^{\mathrm{T}}d^1 + \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda_{kjw}^{1*}b_k^1.$$

Then, for any $x \in S_{\alpha^1}^1$, we have

$$\begin{aligned} &x^{\mathrm{T}}Gy^* + g^{\mathrm{T}}x + h^{\mathrm{T}}y^* \le h^{\mathrm{T}}y^* + (\nu^{1*})^{\mathrm{T}}d^1 \\ &+ \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda_{kjw}^{1*}b_k^1. \end{aligned} \tag{46}$$

Similarly, for any $y \in S_{\alpha^2}^2$, we have

$$\begin{aligned} &(x^*)^{\mathrm{T}}Gy + g^{\mathrm{T}}x^* + h^{\mathrm{T}}y \ge g^{\mathrm{T}}x^* \\ &+ (\nu^{2*})^{\mathrm{T}}d^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{N_2}\sum_{w=1}^{P_2}\lambda_{ljw}^{2*}b_l^2. \end{aligned} \tag{47}$$

Take $x = x^*$ and $y = y^*$ in (46) and (47), then from (44), we get

$$\begin{aligned} u(x^*, y^*) &= h^{\mathrm{T}}y^* + (\nu^{1*})^{\mathrm{T}}d^1 + \sum_{k\in\mathcal{I}_1}\sum_{j=1}^{N_1}\sum_{w=1}^{P_1}\lambda_{kjw}^{1*}b_k^1 \\ &= g^{\mathrm{T}}x^* + (\nu^{2*})^{\mathrm{T}}d^2 + \sum_{l\in\mathcal{I}_2}\sum_{j=1}^{N_2}\sum_{w=1}^{P_2}\lambda_{ljw}^{2*}b_l^2. \end{aligned} \tag{48}$$

It follows from (46), (47), and (48) that

$$u(x, y^*) \le u(x^*, y^*) \le u(x^*, y),\ \forall\, x \in S_{\alpha^1}^1,\ y \in S_{\alpha^2}^2,$$

which in turn implies that (x^*, y^*) is an SPE of the game Z_α. □

VI. NUMERICAL RESULTS

A. Competition in Financial Market

In this section, we consider a competition of two firms in financial market. They invest in the same set of portfolios. Let $P = \{1, 2, \dots, N_P\}$ be the set of portfolios. Let $\mathcal{A}_j$ be the set of assets in the portfolio j. Assume that the sets $\mathcal{A}_j$ and $\mathcal{A}_k$ are disjoint, for any $j \neq k$. Let $x_k = (x_{kj})_{j\in\mathcal{A}_k}$ be the investment vector of firm 1 in portfolio k and $y_k = (y_{kj})_{j\in\mathcal{A}_k}$ be the investment vector of firm 2 in portfolio k. Let $x = (x_k)_{k\in P}$ and $y = (y_k)_{k\in P}$ be the investment vector of firm 1 (resp. firm 2). The set of investments X of firm 1 is defined as follows

$$X = \left\{x \,\Big|\, \sum_{j\in\mathcal{A}_k} x_{kj} = W_k^1,\ \forall j \in \mathcal{A}_k,\ k \in P\right\},$$

and the set of investments Y of firm 2 is defined as follows

$$Y = \left\{y \,\Big|\, \sum_{j\in\mathcal{A}_k} y_{kj} = W_k^2,\ \forall j \in \mathcal{A}_k,\ k \in P\right\},$$

where W_k^i is the total investment of firm i in portfolio k, for any $i = 1, 2$ and $k \in P$. Let $L_k^i = (L_{kj}^i)_{j\in\mathcal{A}_k}$ be a random loss vector of firm i from portfolio k. Then, for a given investment vector x_k and y_k, the total loss of firm 1 (resp. firm 2) caused by portfolio k is defined as $(L_k^1)^{\mathrm{T}}x_k$ (resp. $(L_k^1)^{\mathrm{T}}y_k$). Each firm wants to make sure that their random loss is below a maximal allowable loss level with high probability. This condition is modeled by the following inequality

$$\mathbb{P}\left\{(L_k^1)^{\mathrm{T}}x_k \le b_k^1\right\} \ge \alpha_k^1, \tag{49}$$

and

$$\mathbb{P}\left\{(L_l^2)^{\mathrm{T}}y_l \le b_l^2\right\} \ge \alpha_l^2, \tag{50}$$

where b_k^i are deterministic vectors and α_k^i are confidence levels, $i = 1, 2,\ k \in P$. We assume that the true distribution of random loss vectors is unknown, but only known to belong to some uncertainty set $\mathcal{D}_k^i$ defined in Section IV. Then, the feasible strategy sets of two firms are given by

$$\inf_{F_k^1\in\mathcal{D}_k^1} \mathbb{P}\left\{(L_k^1)^{\mathrm{T}}x_k \le b_k^1\right\} \ge \alpha_k^1,\ \forall\, k \in P,$$

and

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\left\{(L_l^2)^{\mathrm{T}} y_l \leq b_l^2\right\} \geq \alpha_l^2,\ \forall\, l \in P.$$

We assume that the total profit of both firm is zero, i.e., for each profile of strategies $(x, y) \in X \times Y$, if firm 1 gains a profit $u(x, y)$, then firm 2 gains a profit $-u(x, y)$. Firm 1 wants to maximize u w.r.t x, for $y \in S^2_{\alpha^2}$ and firm 2 wants to minimize u w.r.t y, for $x \in S^1_{\alpha^1}$. We assume that u has the form (1), i.e., $u(x, y) = x^{\mathrm{T}} G y + g^{\mathrm{T}} x + h^{\mathrm{T}} y$.

In order to find an SPE of (8), we solve the two SOCP problems (42) and (43) using **coneqp** solver in CVXOPT. We compare the uncertainty sets defined in Section (IV) with the true model, in which we assume that the true distribution of random loss vectors is known and follows Gaussian distribution. In this case, it is well known that the constraints (49) and (50) are equivalent to SOC constraints [25]. An SPE in true model can be computed by solving an SOCP reformulation [8].

B. Case Study

All the numerical results below are performed using Python 3.8.8 on an Intel Core i5-1135G7, Processor 2.4 GHz (8M Cache, up to 4.2 GHz), RAM 16G, 512G SSD. We consider two firms investing in a portfolio consists of four assets, i.e., $P = \{1\}$ and $\mathcal{A}_1 = \{1, 2, 3, 4\}$. We generate randomly the vectors g and h in (1) in $[-3, 3]^4$ by the command *"numpy.random.uniform(-3,3,size=(4,1))"*. The matrix G in (1) is randomly generated by the command *"numpy.random.uniform(-3,3,size=(4,4))"*. We take the confidence levels of two firms as $\alpha^1 = \alpha^2 = 0.9$, the total investment of two firms in the portfolio W_1^1 and W_1^2 are randomly generated on $[20, 80]$ by the command *"numpy.random.uniform(20,80)"*. The maximal allowable loss levels of two firms b_1^1 and b_1^2 are randomly generated on $[100, 500]$ by the command *"numpy.random.uniform(100,500)"*. The probability distribution of the loss of two firms L_1^1 and L_1^2 are assumed to be Normal distributions with mean vector μ_1^1 (resp. μ_1^2) and covariance matrix Σ_1^1 (resp. Σ_1^2). The mean vectors are randomly generated on $[8, 12]^4$ using the command *"numpy.random.uniform(8,12, size=(4,1))"*. The covariance matrix are defined as follows

$$\Sigma_1^i = \frac{AA^{\mathrm{T}}}{4} + \mathbf{I}_4,\ \forall\, i = 1, 2,$$

where A is a 4×4 random matrix whose all entries belong to $[0, 1]$ generated by the command *"A=numpy.random.random(size=(4,4))"* and $\mathbf{I}_4$ denotes 4×4 identity matrix. For any $i = 1, 2$, we define sample mean vector μ^i_{sample} and Σ^i_{sample} by generating randomly a sample of 100 observations $\xi_1^i, \ldots, \xi_{100}^i$, which follow Normal distribution with mean vector μ_1^i and covariance matrix Σ_1^i. To do that, we generate a standard Gaussian vector by the command "*x=numpy.random.normal(0,1,4)*". We generate a Gaussian vector with mean vector μ_1^i and Σ_1^i by taking $\xi_j^i = Bx + \mu_1^i$, where B is the Cholesky factorization of Σ_1^i. To get the Cholesky factorization of a matrix, we use the command "*numpy.linalg.cholesky*". The sample mean vector μ^i_{sample} and the covariance matrix Σ^i_{sample} are defined as follows

$$\mu^i_{sample} = \frac{1}{100} \sum_{j=1}^{100} \xi_j^i,$$

$$\Sigma^i_{sample} = \frac{1}{99} \sum_{j=1}^{100} (\xi_j^i - \mu^i_{sample})(\xi_j^i - \mu^i_{sample})^{\mathrm{T}}.$$

Now, we define other parameters for each model. For the uncertainty set (12), we take $\gamma_1^i = 1.1$, for any $i = 1, 2$. For the uncertainty set (16), we take $\gamma_{11}^i = \gamma_{12}^i = 1$, for any $i = 1, 2$. We take the uncertainty set (21) similarly as the uncertainty set (9) by choosing $M = 1$. For the uncertainty set (25), we take the radius vector $\epsilon^i_{\mu,1} = (0.1, 0.1, 0.1, 0.1)^4$ and the radius matrix $\epsilon^i_{\Sigma,1} = 0.1 \times \mathbf{I}_4$, for any $i = 1, 2$, where $\mathbf{I}_4$ is 4×4 identity matrix. For the uncertainty set (29), we take $\theta_1^i = 0.05$, for any $i = 1, 2$.

For the above instance, we compute an SPE of the true model, where the true distribution of random loss vectors L_1^1 and L_1^2 follow Gaussian distributions with mean vector μ_1^1 (resp. μ_1^2) and covariance matrix Σ_1^1 (resp. Σ_1^2). We obtain an SPE (x^*, y^*) given by

$$x^* = (18.91, 19.45, 19.45, 20.22)^{\mathrm{T}},$$
$$y^* = (19.01, 20.15, 20.45, 18.71)^{\mathrm{T}}.$$

The profit of firm 1 for this instance is $u(x^*, y^*) = -275.52$. Now, we calculate an SPE of the models defined in Section (IV). For the uncertainty sets (9), (12), (16), (21) and (25), we take $\mu_1^i = \mu^i_{sample}$ and $\Sigma_1^i = \Sigma^i_{sample}$, for any $i = 1, 2$. For the uncertainty set (29), we assume that the nominal distribution ν_1^i follows a Gaussian distribution with mean vector μ^i_{sample} and covariance matrix Σ^i_{sample}. We compare the optimal profit value of firm 1 in above models with the optimal profit value of firm 1 in the true model. The results are given in Table III. We can see that for this instance, the models defined by $\phi-$divergence give better solution than the models defined by moments since the optimal profit value in $\phi-$divergence uncertainty sets approximates well the optimal profit value in true model. We also present the time analysis

TABLE III
LIST OF OPTIMAL PROFIT VALUES $u(x^*, y^*)$

True model	Known Mean Known Covariance	Known Mean Unknown Covariance	Unknown Mean Unknown Covariance	Polytopic
-257.52	-221.11	-222.5	-224.8	-221.11
Componentwise Bounds	Kullback Leibler	Variation Distance	Modified χ^2 - distance	Hellinger Distance
-223.3	-255.1	-256.23	-255.8	-253.9

for a large numbers of assets size model by considering the number of assets between 100 and 1000. For each case of number of assets, we randomly generate 10 instances of the known mean known covariance model, where the parameters are defined similarly as above and we calculate the average running time (in seconds) to solve the two optimization problems (42) and (43). The results are given in Figure 1.

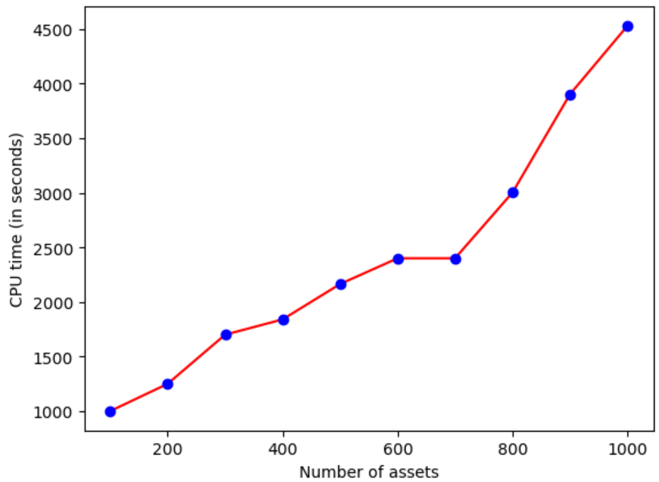

Fig. 1. CPU time (in seconds) to solve (42) and (43) in known mean known covariance cases with different number of assets.

It is clear from Figure 1 that our optimization problems can be solved efficiently in high dimension up to 1000 assets.

VII. Conclusion and Future Work

We study a more general two player zero-sum game than the model considered in [1] under various moments based and statistical based uncertainty sets. We propose a reformulation of the chance constraints using distributionally chance-constrained optimization framework and show that there exists a mixed strategy SPE of the game. Under Slater's condition, the SPE of the game can be obtained from the optimal solutions of a primal-dual pair of SOCPs. We present a competition of two firms in financial market as an application to figure out out theoretical results. The numerical experiments are performed using randomly generated data on the game up to 1000 assets and it is clear from our time analysis that the SOCPs problems can be computed efficiently. For our future works, we will study tractable reformulation of the zero-sum game problem with different payoff structure in a different game model and apply the game problem in a different application to the competition in financial market considered in this paper.

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Appendices

Appendix A: Proof of Lemma 7 - Case Hellinger distance

For $i = 1, 2$ and $k \in \mathcal{I}_i$, it suffices to calculate the value of $\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{f_k^i(\lambda,\beta)\right\}$ with Hellinger distance divergence. We consider two cases as follows

- **Case 1:** $\frac{\beta}{\lambda} < 1 \Leftrightarrow \beta < \lambda$. We have

$$\phi^*\left(\frac{\beta}{\lambda}\right) = \frac{\beta}{\lambda-\beta},$$
$$\phi^*\left(\frac{\beta-1}{\lambda}\right) = \frac{\beta-1}{\lambda+1-\beta}.$$

Therefore,

$$\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{f_k^i(\lambda,\beta)\right\} =$$
$$\sup_{\lambda>0,\beta\in\mathbb{R}} \mathbb{P}_{\nu_k^i}(M_k^i)\frac{\lambda^2}{(\lambda-\beta)(\lambda-\beta+1)} + \frac{\beta^2}{\beta-\lambda} - \lambda\theta_k^i.$$

Since $\lambda > 0$ and $\beta < \lambda$, let $\gamma = \lambda - \beta$, we deduce that

$$\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{f_k^i(\lambda,\beta)\right\}$$
$$= \sup_{\lambda>0,\gamma>0} \left\{\lambda^2\left(\frac{\mathbb{P}_{\nu_k^i}(M_k^i)}{\gamma(\gamma+1)} - \frac{1}{\gamma}\right) + \lambda(2-\theta_k^i) - \gamma\right\}.$$

Let $Q(\lambda,\gamma) = \lambda^2\left(\frac{\mathbb{P}_{\nu_k^i}(M_k^i)}{\gamma(\gamma+1)} - \frac{1}{\gamma}\right) + \lambda(2-\theta_k^i) - \gamma$. Note that $0 \le \mathbb{P}_{\nu_k^i}(M_k^i) \le 1$ and $\gamma > 0$. Therefore, $Q(\lambda,\gamma)$ is a second-order polynomial of λ and the coefficient of λ^2 is negative. It is well known that the maximum value of a second order function $f(x) = ax^2 + bx + c$ with $a < 0$ is $c - \frac{b^2}{4a}$ and it holds at $x = \frac{-b}{2a}$. Hence, the maximum value of $Q(\lambda,\gamma)$ holds at $\lambda^* = \frac{\gamma(\gamma+1)(2-\theta_k^i)}{2(1+\gamma-\mathbb{P}_{\nu_k^i}(M_k^i))}$. Since $\theta_k^i < 2$, it is clear that $\lambda^* > 0$. Then, the optimal value of $\sup_{\lambda>0,\beta\in\mathbb{R}}\left\{f_k^i(\lambda,\beta)\right\}$ holds when $\lambda = \lambda^*$ and we have

$$\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{f_k^i(\lambda,\beta)\right\}$$
$$= \sup_{\gamma>0}\left\{-\gamma + \frac{(2-\theta_k^i)^2\gamma(\gamma+1)}{4(\gamma+1-\mathbb{P}_{\nu_k^i}(M_k^i))}\right\}. \quad (51)$$

Let $u = \gamma + 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$, then $u > 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$. Rewriting (51) as a function of u, we have:

$$\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{f_k^i(\lambda,\beta)\right\} = \sup_{u>1-\mathbb{P}_{\nu_k^i}(M_k^i)} F(u),$$
$$= \sup_{u>1-\mathbb{P}_{\nu_k^i}(M_k^i)} \left\{au + \frac{b}{u} + c\right\},$$

where $a = \left(\frac{(2-\theta_k^i)^2}{4} - 1\right)$,
$b = \frac{(2-\theta_k^i)^2\mathbb{P}_{\nu_k^i}(M_k^i)(\mathbb{P}_{\nu_k^i}(M_k^i)-1)}{4}$,
$c = 1 - \mathbb{P}_{\nu_k^i}(M_k^i) + \frac{(2-\theta_k^i)^2(2\mathbb{P}_{\nu_k^i}(M_k^i)-1)}{4}$. Note that $a < 0$ and $b \le 0$. We have: $F'(u) = a - \frac{b}{u^2}$. Hence, it can be shown that F is decreasing on $(u^*, +\infty)$, increasing on $(-u^*, u^*)$ and decreasing on $(-\infty, -u^*)$, where $u^* = \sqrt{\frac{b}{a}}$. Or,

$$u^* = \sqrt{\frac{(2-\theta_k^i)^2}{4-(2-\theta_k^i)^2}\mathbb{P}_{\nu_k^i}(M_k^i)(1-\mathbb{P}_{\nu_k^i}(M_k^i))}. \quad (52)$$

We have $F(u^*) = -2\sqrt{ab} + c$. We consider 2 cases as follows

1: $u^* \leq 1 - \mathbb{P}_{\nu_k^i}(M_k^i)$. Since F is decreasing on $(u^*, +\infty)$, it is also decreasing on $(1 - \mathbb{P}_{\nu_k^i}(M_k^i), +\infty)$. Hence, $\sup_{u>1-\mathbb{P}_{\nu_k^i}(M_k^i)} F(u) = 0$, where the optimal value holds when $u \to 1 - \mathbb{P}_{\nu_k^i}(M_k^i) \Leftrightarrow \gamma \to 0$, which violates (30).

2: $u^* > 1 - \mathbb{P}_{\nu_k^i}(M_k^i) > 0$. Then, the optimal value of $\sup_{u>1-\mathbb{P}_{\nu_k^i}(M_k^i)} F(u)$ holds when $u = u^*$. Therefore,

$$\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{ f_k^i(\lambda,\beta) \right\} = F(u^*) = -2\sqrt{ab} + c.$$

Then, (30) is equivalent to

$$\begin{aligned} &-2\sqrt{\frac{(2-\theta_k^i)^2}{4}\left(1 - \frac{(2-\theta_k^i)^2}{4}\right)\mathbb{P}_{\nu_k^i}(M_k^i)(1-\mathbb{P}_{\nu_k^i}(M_k^i))} \\ &\geq \left(1 - \frac{(2-\theta_k^i)^2}{2}\right)\mathbb{P}_{\nu_k^i}(M_k^i) + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i). \end{aligned} \tag{53}$$

By taking square on both side of (53), we obtain a second order inequality of $\mathbb{P}_\nu(K)$ as follows

$$\mathbb{P}_{\nu_k^i}(M_k^i)^2 + B\mathbb{P}_{\nu_k^i}(M_k^i) + C \geq 0,$$

where B, C are defined in Table II. By solving the equality $x^2 + Bx + C = 0$, we have two solutions $x_{\text{min}} < x_{\text{max}}$ where $x_{\text{min}} = \frac{-B-\sqrt{\Delta}}{2}$, $x_{\text{max}} = \frac{-B+\sqrt{\Delta}}{2}$. It is clear that (53) is equivalent to either $\mathbb{P}_{\nu_k^i}(M_k^i) \geq x_{\text{max}}$ or $\mathbb{P}_{\nu_k^i}(M_k^i) \leq x_{\text{min}}$. Since $\theta_k^i < 2 - \sqrt{2}$, we deduce that $1 - \frac{(2-\theta_k^i)^2}{2} < 0$. Therefore, we have

$$\begin{aligned} &\left(1 - \frac{(2-\theta_k^i)^2}{2}\right)x_{\text{min}} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i) \\ &> \left(1 - \frac{(2-\theta_k^i)^2}{2}\right)x_{\text{max}} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i). \end{aligned} \tag{54}$$

On the other hand, we have

$$\begin{aligned} &-2\sqrt{\frac{(2-\theta_k^i)^2}{4}\left(1 - \frac{(2-\theta_k^i)^2}{4}\right)x(1-x)} \\ &= \pm\left[\left(1 - \frac{(2-\theta_k^i)^2}{2}\right)x + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i)\right], \end{aligned}$$

where $x = x_{\text{min}}$ or $x = x_{\text{max}}$. Note that $-2\sqrt{\frac{(2-\theta_k^i)^2}{4}\left(1 - \frac{(2-\theta_k^i)^2}{4}\right)x(1-x)} < 0$. Using (54), we deduce that

$$\begin{aligned} &-2\sqrt{\frac{(2-\theta_k^i)^2}{4}\left(1 - \frac{(2-\theta_k^i)^2}{4}\right)x_{\text{max}}(1-x_{\text{max}})} \\ &= \left[\left(1 - \frac{(2-\theta_k^i)^2}{2}\right)x_{\text{max}} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i)\right], \end{aligned}$$

and

$$\begin{aligned} &-2\sqrt{\frac{(2-\theta_k^i)^2}{4}\left(1 - \frac{(2-\theta_k^i)^2}{4}\right)x_{\text{min}}(1-x_{\text{min}})} \\ &= -\left[\left(1 - \frac{(2-\theta_k^i)^2}{2}\right)x_{\text{min}} + \frac{(2-\theta_k^i)^2}{4} - (1-\alpha_k^i)\right]. \end{aligned}$$

or x_{max} satisfies (53) while x_{min} does not satisfy (53). Then, (53) is equivalent to $\mathbb{P}_{\nu_k^i}(M_k^i) \geq x_{\text{max}}$.

- **Case 2:** $1 \leq \frac{\beta}{\lambda} \Leftrightarrow \lambda \leq \beta$. We have

$$\phi^*\left(\frac{\beta}{\lambda}\right) = +\infty,$$

which implies that $\sup_{\lambda>0,\beta\in\mathbb{R}} \left\{ f_k^i(\lambda,\beta) \right\} = -\infty$, which violates (30).

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