A LT-Based Distributed Codes Over Erasure Channel

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Abstract—For the future space network scenarios, where multirovers can return science data through one obiter simultaneously, we design a new distributed LT (NDLT) code between two sources, one relay and a single sink, which could enhance the efficiency and reliability of packets transferring from source to sink. This paper proposes a method to decompose weaken robust soliton distribution (WRSD) exploited by relay node into two deconvolved weaken robust soliton distributions (DWSDs), which are used in the sources and have lower average degree than traditional RSD-based deconvolved soliton distributions (DSDs). Low operation complexity at sources and only including a simple XOR operation at relay node could be provided by the proposed degree distribution. This paper provides analytical results about the coding complexity of proposed and traditional distributed LT (DLT) code. The simulation results show that the proposed NDLT has a higher decoding probability than that of traditional DLT, under the conditions of recovering a certain proportion of original information, such as some space mission scenarios with specific data completeness requirement.

Keywords- Distributed coding; Relay; LT code; WRSD

I. INTRODUCTION

An erasure channel in [1] is a communication channel model wherein errors are described as erasures events. In the packetized data transmission, such as scientific data return in space communication, packets may be deleted if the destination fails to recover the packets. In order to recover the packets lost in file delivery, a kind of forward erasure coding is proposed, i.e., fountain codes [1]. Luby transform (LT) code is the first realization of fountain code [2], where the input symbols are encoded according to RSD (robust soliton distribution). Raptor code in [3], as a kind of improved LT code, consists of pre-coding and LT-coding. Raptor codes have better decoding efficiency than LT codes by using pre-coding. The intermediate symbols are the symbols generated by pre-code from the input symbols. The output symbols are the symbols by LT-code from the intermediate symbols.

In space explorations, several scenarios are common, where probes on the explored planet are much more than the obiter or relay satellites. To provide the simultaneous bulk data return capacity for multi-sources through limited relay satellites and enhance the efficiency of the relay satellites, in this paper, we consider such a scenario, where two sources transmit information to a sink through a relay, as shown in Fig. 1.

In [4, 5], the method to decompose LT codes into DLT codes based on the deconvolution of RSD is proposed. The constructed DLT has large redundancy, coding complexity and large decode failure probability. Density evolution is used in [6] to find optimal codes over the network in which multiple sources transmit information to a sink through a relay. The work in [7] introduces Soliton-like rateless coding, which exploits the benefits of fountain coding and network coding over a Y-network. This method has larger operation complexity than that in [5, 6]. In space communication scenarios with multiple sources, single relay and single sink node, a new distributed LT code is designed for enhancing the efficiency and reliability of packets transferring from source to sink. In this network of Fig. 1, as a first step, the symbols at each of the two sources are encoded using the DWSD as the degree distribution. Then, the relay selectively XORs the bit streams it receives from each source and transmits the resulted NDLT code which approximatively follows WRSD to sink node. In this paper, we propose a method to deconvolve the WRSD for yielding DWSD, by which less coding packets is consumed to cover original packets with the maximum degree of \( D + 1 \). In this paper, the belief propagation (BP) decoding algorithm is adopted in the sink node. The proposed method could obtain the DWSD-based NDLT code by constructing a special function, which has lower coding complexity and lower decode failure probability compared with DLT under the condition of only recovering a certain proportion of original packets. We provide analytical results about coding complexity of proposed and traditional distributed LT code.

This paper is organized as follows. Section II gives a review on WRSD and proposes a specific method to deconvolve the WRSD into DWSD. Section III provides analytical results about the coding complexity of distributed LT code. The simulation results and discussion about overhead and decode failure probability are shown in Section IV. Section V presents a conclusion of this paper.

II. DECOMPOSING WRSD INTO DWSD

A. WRSD

WRSD is applicable to the LT-coding in Raptor codes [3]. In the decoding process, it could recover a certain proportion of original packets and the rest packets are recovered by pre-coding. WRSD has the lower average degree than RSD if they have the same number of original symbols. WRSD has lower decode failure probability than
RSD under the condition of recovering a certain proportion of original information.

**Definition 1:** For constants 0 < ε < 0, \( D = \left\lfloor 4(1+\varepsilon) / \varepsilon \right\rfloor \)
and \( S = (\varepsilon / 2) + (\varepsilon / 2)^2 \), the weaken robust soliton distribution (WRSD) \( R(\cdot) \) is given by

\[
R(i) = \begin{cases} 
\frac{S}{S+1}, & \text{for } i = 1 \\
\frac{1}{(i-1)(S+1)}, & \text{for } 2 \leq i \leq D \\
\frac{1}{D(S+1)}, & \text{for } i = D+1 \\
0, & \text{for } D + 2 \leq i \leq k
\end{cases}
\]  

(1)

WRSD has above advantages because of its degree distribution which has largest degree-D+1 and two obvious spiky values, as shown in Fig. 2. It is apparent to see that the first spiky value is in degree-2, which could enhance the decode success probability in BP decoding process. The other spiky value is in largest degree-D+1, which could consume low coding packets to cover original packets. Thus, this could reduce the decode failure probability.

**B. Deconvolution of the WRSD**

We consider that, as shown in Fig. 1, two sources \( s_1 \) and \( s_2 \) transfer packets to the same sink \( t \) through the relay \( r \). Each source has \( k / 2 \) input symbols to be transferred to the same sink. In this scenario, the source-to-relay link is lossless and all erasures occur on the relay -to -sink link. We define \( X_1 \) as a code symbol generated at \( s_1 \) with a degree \( d_1 \) and \( X_2 \) is a code symbol generated at \( s_2 \) with a degree \( d_2 \). Both \( d_1 \) and \( d_2 \) have the same degree distribution. Symbols from the two sources encoded in the relay are the same as [5]. We expect that the degree of \( X_1 \oplus X_2 \) is a random variable with the degree distribution of WRSD \( R(\cdot) \), which WRSD has the lower average degree decode failure probability than RSD under the condition of recovering a certain proportion of original information. We define this problem as follows.

\[
(p \ast p)(\cdot) = R(\cdot),
\]  

(2)

where \( d_1 \) and \( d_2 \) both obey \( p(\cdot) \) and \( d_1 + d_2 \) obey \( R(\cdot) \). We could obtain \( p(\cdot) \) by deconvolving WRSD directly.

Direct deconvolution of the WRSD \( R(\cdot) \) in (2), however, does not necessarily yield a valid probability distribution similar with [5]. To avoid direct deconvolution, we attempt to split the WRSD \( R(\cdot) \) into two distributions \( R1(\cdot) \) and \( R2(\cdot) \). \( R2(\cdot) \) captures the problematic part of the WRSD in [5](i.e., the degree-one symbols and the spike at \( i = D+1 \) and \( R1(\cdot) \) is a smooth distribution that is easier to deconvolve.

Then we define \( R1(\cdot) \) as follows:

\[
R1(i) = \begin{cases} 
0, & \text{for } i = 1 \\
\frac{1}{(i-1)(S+1)b_i}, & \text{for } 2 \leq i \leq D+1 \\
0, & \text{otherwise}
\end{cases}
\]

(3)

with the normalization factor \( b_i = \sum_{i=2}^{D+1} \frac{1}{(i-1)(S+1)} \).

Similarly, \( R2(\cdot) \) is given by

\[
R2(i) = \begin{cases} 
\frac{S}{(S+1)b_1}, & \text{for } i = 1 \\
\frac{1}{(D+1)(S+1)b_2}, & \text{for } i = D+1 \\
0, & \text{otherwise}
\end{cases}
\]

(4)

with the normalization factor \( b_2 = \frac{S}{(S+1)} + \frac{1}{(D+1)(S+1)} \).

Thus, \( b_1 + b_2 = 1 \), and the WRSD can be rewritten as

\[
R(i) = b_1 \cdot R1(i) + b_2 \cdot R2(i) = b_1 \cdot R1(i) + (1 - b_1) \cdot R2(i)
\]

(5)

So the WRSD is a mixture of the distributions \( R1(\cdot) \) and \( R2(\cdot) \) with mixing parameter \( b_1 \).

The approach taken in this paper is to deconvolve the distribution \( R1(\cdot) \) and use the result in the construction of the new DLT codes.

\[
(f \ast f)(i) = R1(i), \text{ for } 2 \leq i \leq \frac{D+1}{2} + 1
\]

(6)

Direct deconvolution of \( R1(\cdot) \) in (6) yields \( f(\cdot) \), whose independent variable value is from 1 to \( \frac{D+1}{2} \). Give a degree distribution \( p(i) = \lambda \cdot f(i) + (1 - \lambda) \cdot R2(i) \), for \( 1 \leq i \leq \frac{D+1}{2} \), \( f(\cdot) \) from direct deconvolution of (6) could make \( p(i) \).
obtain the largest degree $\frac{D+1}{2}$, which could not make the degree distribution in relay to sink approximate WRSD using the low operation complexity protocol in the relay. Without improving obviously the operations complexity at the relay, we hope the probability of choosing the largest degree $D+1$ in the relay is larger. We define $f(\cdot)$ recursively by

$$f(i) = \begin{cases} \sqrt{R(2)} , & \text{for } i = 1 \\ \frac{R(i+1) - \sum_{j=2}^{i-1} f(j) f(i+1-j)}{2 f(l)}, & \text{for } 2 \leq i \leq D+1 \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

Let $R(1) = \left( f^* f \right)(i)$.

We can get the relationship between $f(\cdot)$ and $\hat{f}(\cdot)$.

$$f(i) = \begin{cases} \sqrt{R(i)}, & \text{for } 1 \leq i \leq \frac{D+1}{2} \\ \frac{R(i+1) - \sum_{j=2}^{i-1} f(j) f(i+1-j)}{2 f(l)}, & \text{for } \frac{D+1}{2} \leq i \leq D+1 \end{cases} \quad (8)$$

Let $R(1) = \left( f^* f \right)(i)$.

We now define a new distribution constructed by $f(\cdot)$ and $R(2)(\cdot)$.

**Definition 2:** The deconvolved weak robust soliton distribution (DWSD) $p(\cdot)$ is given by

$$p(i) = \lambda \cdot f(i) + (1 - \lambda) \cdot R(2)(i) , \text{ for } 1 \leq i \leq D+1 , \quad (9)$$

where $\lambda = \sqrt{R}$.

According to definition of probability distribution, we must prove $p = \sum_{i=1}^{D+1} p(i) \approx 1$. In order to verify $p = \sum_{i=1}^{D+1} p(i) \approx 1$, we need to proof $f = \sum_{i=1}^{D+1} f(i) \approx 1$.

And in order to verify the result of new deconvolution $f(\cdot)$ is reasonable, we hope to get

$$\left( f^* f \right)(i) = \hat{R}(i) \approx R(i) , \text{ for } 1 \leq i \leq k .$$

**Proposition 1:**

$\hat{f} = \sum_{i=1}^{D+1} f(i) \approx 1$ and $\hat{R}(i) \approx R(i) , \text{ for } 1 \leq i \leq k$.

**Proof:**

Similar with [5], we have $\lim_{D \to \infty} \sum_{i=1}^{D+1} f(i) = 1$.

Let $f = \sum_{i=1}^{D+1} f(i)$ and $\hat{f} = \sum_{i=1}^{D+1} \hat{f}(i)$.

We have

$$\Delta \hat{f} = \hat{f} - f = \sum_{i=1}^{D+1} \hat{f}(i)$$

$$\leq \sum_{i=1}^{D+1} R(i) = \frac{1}{D}$$

with $D$ increasing, $\Delta \hat{f}$ tends to a arbitrarily small value $\varepsilon_i$.

Consequently,

$$\hat{f} = \sum_{i=1}^{D+1} f(i) \approx 1 \quad (10)$$

According to [5], we have

$$\sum_{i=1}^{D+1} R(1) = \sum_{i=1}^{D+1} \left( f^* f(i) \right) = \left( \sum_{i=1}^{D+1} f(i) \right)^2 \approx 1. \quad (12)$$

We can easily get $R(1) = R(1)$, for $2 \leq i \leq \frac{D+1}{2}$,

and add $R(1)$ from 2 to $\frac{D+1}{2}$ as follows.

$$\sum_{i=2}^{D+1} R(1) = \frac{D+1}{D} \cdot \frac{D+1}{D+3} \quad (13)$$

with $D$ increasing, $\sum_{i=2}^{D+1} R(1)$ tends to 0.

Obviously, $R(1) = R(1) = 0$, for $2D + 3 \leq i \leq k$.

And for $\frac{D+1}{2} \leq i \leq D+2$, we give a maximum limit which has the most number of items and each takes the biggest probability value as follows.

$$\Delta \hat{R}(i) = \hat{R}(i) - R(1)$$

$$\leq 2 f(1) f(D+1) + \cdots + 2 f(1) f(D+1)$$

$$\leq 2 \cdot \frac{D+1}{2} \cdot f(D+1)^2 \leq (D+1) \cdot R(1) \left( \frac{D+1}{2} + 2 \right)$$

$$= \frac{4(D+1)^2}{D(D+3)(D+5)}$$

with $D$ increasing, $\Delta \hat{R}(i)$ tends to a arbitrarily small value $\varepsilon_i$.

Consequently,

$$R(1) = R(1) , \text{ for } \frac{D+1}{2} \leq i \leq D+2$$

Similarly, for $D+2 \leq i \leq 2D+2$
\[ \Delta R_1(i) = R_1(i) \]
\[ \leq 2 f(1) f \left( \frac{D+1}{2} + 1 \right) \]
\[ \leq 2 \cdot \frac{D+1}{2} f \left( \frac{D+1}{2} + 1 \right) \leq (D+1) \cdot R_1 \left( \frac{D+1}{2} + 2 \right) \]
\[ \leq \frac{4(D+1)^2}{D(D+3)(D+5)} \]

When \( D \) increases, \( R_1(i) \) tends to a arbitrarily small value \( \varepsilon_1 \).

Consequently, \( \hat{R}_1(i) \approx R_1(i) \), for \( D+2 \leq i \leq 2D+2 \)

In summary,
\[ \hat{R}_1(i) \approx R_1(i), \text{ for } 1 \leq i \leq k \]

\[ (15) \]

We simulate \( p(i) \) in Fig. 2 and find that DWSD is the same as WRSD. It is noted that, the largest probability value of DWSD is about of 71% in degree-one, while the largest probability value of WRSD is in degree-two. However, DWSD also has a spiky value in degree-D+1, and is the same to WRSD. Thus, it has similar properties with WRSD.

C. DWSD Applied to a Two-Source Single-Sink Relay Network

Similar with [5], the two sources encode symbols following the DWSD, which can be used to encode information in the network of Fig. 1. We define a sequence of code symbols produced in this process as a NDLT-2 code, and the sequence of symbols transmitted by the relay as a NMLT-2 code. It is necessary to construct a randomized decision protocol in which the relay transmits a symbol whose degree approximately follows the WRSD to sink. This approach is applied in fountain network coding in [7-10].

\[ \mu(i) = \frac{i(i+1)}{2} \]

Considering the Fig. 3, the probability distribution produced by the decision protocol from [5] in the relay is approximate to WRSD. The distribution approximates WRSD in the Fig. 3. They have the same soliton waveforms coinciding with the theoretical analysis. WRSD has the lower average degree than RSD if they have the same number of original symbols. WRSD has lower decode failure probability than RSD in the condition of recovering a certain proportion of original information.

III. THE CODING COMPLEXITY OF DISTRIBUTED LT CODE

Definition 3: For constants \( c > 0 \) and \( \delta \in [0,1] \), the robust soliton distribution (RSD) \( \mu(\cdot) \) is given by
\[ \mu(i) = \frac{\rho(i) + \tau(i)}{\beta}, \text{ for } 1 \leq i \leq k \]

where \( \beta = \sum_{i=1}^{k} (\rho(i) + \tau(i)) \).

And, \( \rho(i) \) and \( \tau(i) \) are given by
\[ \rho(i) = \begin{cases} 1/k, & \text{for } i = 1 \\ 1/(i+\delta), & \text{for } i = 2 \ldots K \end{cases} \]
\[ \tau(i) = \begin{cases} \lfloor S/ik \rfloor, & \text{for } 1 \leq i \leq k \lfloor S/\delta \rfloor - 1 \\ \lfloor S/ik \rfloor, & \text{for } i = k/S \lfloor S/\delta \rfloor \\ 0, & \text{otherwise} \end{cases} \]

The RSD \( \mu(\cdot) \) could be split into two distributions \( \mu'(i) \) and \( \mu''(i) \):
The complexity of DLT codes is

\[ \rho(i) + \tau(i) + \frac{\ell}{S}, \quad \text{for } i = k \]

\[ \frac{\rho(i)}{\beta'}, \quad \text{otherwise} \]

where \( \beta' = \sum_{i=1}^{k} \rho(i) + \sum_{i=2}^{k} \tau(i) \):

\[ \mu'(i) = \begin{cases} 
0, & \text{for } i = 1 \\
\frac{\rho(i) + \tau(i)}{\beta'}, & \text{for } i = k \\
\frac{\tau(k/S)}{\beta''}, & \text{for } i = k/S \\
0, & \text{otherwise}
\end{cases} \]

(21)

where \( \beta'' = \rho(1) + \tau(1) + \tau(k/S) \).

A. Analysis of the Coding Complexity

In some specified scenarios with requirement of low complexity, such as deep space communication, overhead and decode failure probability and encoding complexity are important factors to measure the encoding algorithm. The algorithm of coding complexity is the same as [3].

We derive the expression of coding complexity as follows: assuming that the number of encoding symbols is \( n = (1 + \varepsilon) \cdot k \); and defining

\[ H(i) = \sum_{j=2}^{k} f(j) f(i + 1 - j), \quad G(i) = \sum_{j=2}^{k} H(i) \]

(22)

1) Encoding complexity of traditional distributed LT

Similar with methods in [3], the average degree of DSD in the source is

\[ \sum_{i=1}^{k} p(i) \]

(23)

The average degree of one and two degree distribution in the relay is

\[ 2 \cdot \frac{\beta}{\beta'} + 1 \cdot \left(1 - \frac{\beta}{\beta'}\right) \]

(24)

Thus, encoding complexity of distributed LT is

\[ 2n \sum_{i=1}^{k} p(i) + n \left(2 \cdot \frac{\beta}{\beta'} + 1 \cdot \left(1 - \frac{\beta}{\beta'}\right)\right) \]

(25)

\[ = 2n \left(\frac{\beta}{\beta'} f(1) + \left(1 - \frac{\beta}{\beta'}\right) \mu'(1) + \frac{k}{S} \left(\frac{\mu(k/S + 1) - H(k/S)}{2 f(1)}\right)\right) \]

\[ + \left(1 - \frac{\beta}{\beta'}\right) \frac{\ell}{S} \sum_{i=1}^{k} \frac{\mu(i + 1) - H(i)}{2 f(1)} + n \left(1 + \frac{\beta}{\beta'}\right) \]

2) Encoding complexity of new distributed LT

The average degree of DSD in the source is

\[ \sum_{i=1}^{k} ip(i) \]

(26)

The average degree of one and two degree distribution in the relay is

\[ 2h + 1 \cdot (1 - h) \]

(27)

Thus, encoding complexity of new distributed LT is

\[ = n \left(2 \sqrt{h'} R(1) + \left(1 - \sqrt{h'}\right) \frac{h'}{f(1)} - h' f(1)\right) \]

(28)

B. Results about the Coding Complexity

We assume that the number of encoding symbols is \( k = 500, 800, 1000 \); and DSD with constants \( c = 0.05 \), \( \delta = 0.5 \) and DWSD with \( e = 0.04 \).

<table>
<thead>
<tr>
<th>TABLE I. DWSD AND DSD</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
</tr>
<tr>
<td>DWSD</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
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<tr>
<td>64</td>
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<tr>
<td>76</td>
</tr>
<tr>
<td>84</td>
</tr>
<tr>
<td>105</td>
</tr>
<tr>
<td>al</td>
</tr>
</tbody>
</table>

Table I shows the probability value from degree one to five and max degree of DWSD and DSD. The average degree \( al \) is given in table I. The average degree of DWSD is far less than DSD. Coding complexity at source of NDLT codes is \( 3.1102 \times n \), and the complexity of DLT codes is \( 4.4985 \times n \).

<table>
<thead>
<tr>
<th>TABLE II. DEGREE DISTRIBUTION IN THE RELAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
</tr>
<tr>
<td>NDLT</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>a2</td>
</tr>
</tbody>
</table>

Table II shows the probability value from degree one to two in the relay of new and traditional DLT codes. The average degree \( a2 \) is given in the end. But the average degree of distribution in the relay from DWSD is slightly larger than DLT.
Combining complexity of coding at source with relay, the total encoding complexity of new distributed LT is
\[ 3.1102 \times 2n + 1.9707 \times n = 4.2497 \times n \]. \hspace{1cm} (29)
Meanwhile encoding complexity of traditional distributed LT is
\[ 4.4985 \times 2n \times 1.9464 \times n \times 7.0506 \times n \]. \hspace{1cm} (30)
It can be simply seen that, encoding complexity of new distributed LT is less than traditional distributed LT with the same overhead. In the practical application scenarios, such as the deep space communication, this communication requests low complexity which can save the consumption of energy. Thus, it can enhance efficiency of information transferring from detectors on the objective planet to the earth station.

IV. SIMULATION AND DISCUSSION

In the simulation, two sources encode symbols by DWSD or DSD, and the sink node decodes by belief propagation decoding algorithm. If it dose not recover all the original symbols or a certain proportion of original symbols, we will call this decode failure. Overhead is the ration between the number of extra symbols required for decoding to succeed and \( k \). The formula of calculating overhead is \((K-k)/k\). We assume that the number of original symbols is \( k=500, 800, 1000 \), DLT with constants \( c = 0.05, \delta = 0.5 \) and NDLT with \( \epsilon = 0.04 \).

Decode failure probability is shown in Fig. 4. with total coding overhead. It is indicated that distributed LT can recover all the original symbols, but its decode failure probability is larger than the NDLT codes under the condition of recovering a certain proportion of the original symbols. The NDLT codes only recover a certain proportion of the original symbols, but the rest proportion of the information could be recoverd by traditonal code in [3]. The results show if the new distributed LT code recover the quality of original information below 98%, decode failure probability is far less than traditional distributed LT. It ensures the efficiency and reliability of the information transmission in some degree. But when the NDLT recovers more than 99% of total original data packets, decode failure probability is more than traditional distributed LT. Thus, we should choose proper traditional code, which can be more effective to recover all the original symbols. Moreover, it also show that the NDLT code has lower decode failure probability than the traditional DLT code on the condition of recovering the same proportion of original information. Thus, the NDLT code could be applicable to several specified scenarios for recovery on a certain proportion of original packets.

V. CONCLUSION

In this paper, we proposed an efficient method of deconvolving degree distribution of decentralized LT code and designed a specified distributed LT codes about the network model with communication between two sources and single sink through the relay. The proposed NDLT has lower coding complexity than traditional DLT. The simulation results indicated that coding complexity of NDLT is approximate one half of DLT. The NDLT has lower decode failure probability than traditional DLT under the condition of recovering a certain proportion of original packets. The decode failure probability is far less than DLT when the NDLT recover below 98% original information. The proposed NDLT is applicable to several scenarios to recover a certain proportion of original packets, such as file delivery in space communication.
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