Metacognitive Support of Mathematical Abstraction Processes: Why and How - A Basic Reasoning

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***Abstract*—A significant and distinctive feature of human beings is the ability of performing abstraction operations, e.g., when forming categories of objects or even consciouls‌y creating abstract objects as it is typical in mathematics. Although the possible range of corresponding abilities is certainly predetermined by individual genetic factors, a high-level abstraction performance will typically be achieved gradually by an intensive long-run practice in solving abstraction prone problems. On the other hand, mathematical abstraction is often considered to be a serious obstacle in mathematics education. As metacognition has turned out to be helpful in overcoming some other obstacles in the students' process of knowledge acquisition, one may ask whether metacognition may also serve as a remedy against the abstraction obstacle and, in particular, whether it may help to accelerate the acquisition of abstraction abilities. The paper provides some inital reasoning w.r.t. this possibility, proposes some basic principles of abstraction that could be taught on a metacognitive level, and presents a concept of a corresponding teaching experiment. Hopefully, it will provide more effective teaching as well as a better understanding of cognitive processes underlying mathematical abstraction.**

***Index Terms*—Abstraction; Mathematical Abstraction; Mathematics Education; Mathematical Reading; Metacognition.**

I. Introduction

The present article is a substantially enlarged and extended version of the authors recent paper "Metacognitive Support of Mathematical Abstraction Processes" [1] presented at the 2016 IARIA Cognitive conference in Rome.

Basic mathematics courses belong to the greatest challenges for first year university students from many disciplines. The author's long run experience in conducting such courses at the University of Paderborn indicates that one main reason for that is the lack of appropriate study and working techniques. For many students, the major obstacle in understanding mathematics is the lacking aptitude to understand its language. As a remedy, a system of in-teaching *metacognitive* support instruments named "CAT" was introduced [2]. CAT provides instructions that help to improve the studying and working routines. But more than that, its core focus is on enhancing the ability to read and understand mathematical texts properly [3][4]. Accompanying empirical studies [5][6] showed that improvements could be achieved by using CAT's metacognitive instruction tools.

Even with this, one often sees refusal or even fear of the perceived *abstractness* of mathematics. Moreover, many of the beginning students are quite unfamiliar with any kind of abstractness. Hence, coping with mathematics becomes particularly hard for them. This raises the question how to facilitate the "access to abstraction" for them.

It is impossible to rise this question without referring to the aspect of time, because good abstraction abilities are typically achieved "by doing", i.e., by solving problems that require – or at least promote – a certain level of abstraction. Even mathematicians develop their abstraction skills within a lengthy process of education and mathematical work. However, in basic courses for non-mathematicians, there is not enough time to re-run along this path. As an alternative, the present paper proposes to support some basic aspects of abstraction on a *metacognitive* level, by explicitly "teaching abstraction principles", with the objective to accelerate the process of acquiring abstraction skills.

In order to derive such rules, several aspects of abstraction are discussed. A generally adopted hypothesis is that abstraction operations are organized hierarchically. Piaget [7] has described that, and how, this hierarchy is run through in children's development of mathematical thinking. The hierarchical nature of abstraction was also emphasized by Dubinsky [8] [9] and Arnon et al. [10]. In contrast to the forementioned ones, the approach pursued here aims to additionally support the construction of several layers of abstraction by *explicit metacognitive instruction*. Although this work is still in an early stage it can be hoped that it shall yield not only better teaching instruments but a better understanding of the underlying cognitive processes as well. In particular, the hope for better teaching instruments is supported by the empirical findings from [5] [6] regarding CAT's instructions, which are essentially metacognitive.

As compared to [1], the novel contribution of the present paper covers both a broader and deeper embedding of the paper's subject in the corresponding research history – including new insights from quite recent work –, and a more thorough and lucid exposition of relevant concepts, including new examples for metacognitive support.

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The paper is organized as follows: In Section II, the need of abstraction in economics education is highlighted. The nature of abstraction and its "economics" is discussed in Sections III, and IV. Section V deals with perceived abstraction aversity. The role of metacognition is discussed in Section VI. The following section deals with operational aspects of abstraction. Section VIII gives an outlook of a forthcoming teaching project and possible applications of the results.

II. Is Abstraction Educationally Needed?

It is often believed that abstraction is a matter of "pure mathematics" rather than of its applications. However, practically this is not true. Here are some fundamental reasons for that: In many applications, at least a basic understanding of the (abstract) language of mathematics is required. Further, abstraction is a core feature of building mathematical models for more or less "real" problems. Modeling requires a "translation" of representations of "real" objects, their attributes, and relations into the language of mathematics. And last but not least, precise reasoning often relies on the abstract logical structure rather than on the domain specific content of arguments. By these and other reasons, the ability to cope with abstraction is closely related to a high professional performance in several domains, as can be seen, e.g., from Greuel et al. [11].

Moreover, in economics, there is a particular demand of "abstraction" at least along four different lines. First, fundamental economic phenomena are explained with the help of abstract mathematical concepts. Look, e.g., at a preference relation as described here:

$$\underline{x} \preceq \underline{y} \quad :\Longleftrightarrow \quad 2x_1 + 3x_2 \leq 2y_1 + 3y_2. \tag{1}$$

The students must be able to read, understand, and handle symbolic expressions like this. Note that the context of considering such relations typically involves some set theory. From the economic point of view, set theory provides the appropriate language to describe, e.g., sets of possible economic decisions. Thus, the students should master this language to a certain minimal extent. However, the author's teaching experience says that even quite basic concepts of set theory are perceived as being rather abstract by many students.

Second, modern economics is interested in qualitative results that are valid under quite general assumptions. Accordingly, these results rely on abstract qualitative properties of the underlying models. For example, one can say a lot about the behavior of an enterprise that produces one good for a polypolistic market, given only that its internal cost function is *neoclassic*, regardless of its concrete form [3]. Hence the qualitative nature of the problems is in the foreground, while purely numerical calculations are of less significance.

Third, economics is concerned with complex systems and their equilibria. The study of such systems is often decomposed into studies of subsystems, the internal parameters of which are determined via optimization, given some exogenous parameters. But it has to be understood that these exogeneous parameters become endogenous when entering a higher level of consideration. That is, frequently there is no point in fixing numerical values, rather one has to develop kind of functional thinking.

And fourth, by employing modern and sophisticated results of mathematics, economics adopt the abstraction level of mathematics itself. This confirms that Devlin's [12] statement "The main benefit of learning and doing mathematics is not the specific content; rather it's the fact that it develops the ability to reason precisely and analytically about formally defined abstract structures" holds true for modern economics, as well as for other sciences.

III. What is Abstraction?

So far, "abstraction" was used in quite general way. For the purposes of this paper, some specific aspects of interest shall be described and put in the general context.

A. General Aspects

Everybody knows somehow and from somewhere "what is abstraction", as this word became present in a lot of domains. A common feature of many conceptions of "abstraction" refers to the latin word *abstrahere* in the philosophical sense of omitting unessential details of an object in the process of inductive thinking, resulting in a new – or simpler – entity, as it was described first by Aristotle. Until today, the term "abstraction" has conquered its place in various scientific disciplines: philosophy, psychology, linguistics, cognitive science, mathematics, computer science, education, and others – and beyond that as well in common understanding. Accordingly, the amount of varying interpretations as well as the number of publications on this subject is extremely large and indicates that "abstraction" is a rather rich and complex notion. It is neither possible nor the purpose of this paper to give a full account to all essential aspects of this notion. Rather it shall concentrate on some aspects that may be essential both from the cognitive point of view and for teaching mathematics.

B. Abstraction as Mental Processes

Henceforth, the term "abstraction" shall be used in the narrower sense to denote *individual mental processes*. Typically, these processes result in *abstract objects* or, more precisely, in new – and simpler – mental representations of previously present mental objects or their relations, respectively, or even in the creation of new mental objects. Clearly, there is a duality of abstraction processes and their results. Viewing abstraction as a process has a long history, as "... there is evidence in Aristotle's psychological and biological writings which suggests that abstraction is a component of the inductive process by which we reach universal concepts" (Smith [13]). An intrinsic feature of abstraction is that it can lead to a re-structured organization of mental knowledge structures (Hershkowitz et al. [14]). In a wider sense, "abstraction" will also be understood as individual mental processes of understanding and exploiting

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already existing abstract objects and concepts. It is obvious that these processes are of particular significance for learning mathematics and, thus, should be promoted.

Some Comments

The author is aware of the deficiencies of these explanations – after all, the given notion of abstraction refers to "mental processes" and "mental representations", or more specifically, to the change of the latter by means of the first. That is, these terms might need an explanation, too. Again, they are used frequently with varying senses. Given that both – processes, and representations – are present within our living organism, in the end they must be accomplished on a neuro-bio-physiological level, exist in distributed electric potentials, specific chemical substances or particular molecular structures and their changes. That is, the possibly most subtle explanation of cognitive functions may start from neuro-bio-physiological details.

However, the possibility to investigate these details with sufficiently high resolution has but developed within the last few years; knowledge about these basic mechanisms keeps on growing rapidly. On the other hand, reasoning about cognitive activities has a very long tradition, without having hands on that level. The key to this seeming puzzle is that research and reasoning on the subject "human thinking" used more or less abstract models of this subject, and more or less specific languages to describe these models as well.

With respect to research on cognitive systems these models can be understood, as C. Eliasmith [15] poses it, by different kinds of *metaphors*. He systematizes four mainstream metaphoric approaches known as *symbolicism*, *connectionism*, *dynamicism*, and the *Bayesian approach*. While symbolicism, roughly spoken, supposes that cognitive systems work similar like computers and process something like symbolic rules, as prominently stated by Fodor [16], connectionism sees brain functions better represented by abstract "neural" networks (Rumelhart and McClelland [17]). The dynamicism approach tries to tie brain functions closer to the continous flow of different input signals that have to be responded to within very short periods of time; it is related to systems and control theory. The Bayesian approach is prominently represented by Anderson [18] and the recent work of Tenenbaum, Griffiths, and others [19][20]. Eliasmith [15] points out that all of these metaphors are appropriate to explain *some* aspects of cognition, but perhaps neither of them is apt to explain *all*.

To sum up: Dependent on the respective purpose, it can suffice to use a metaphoric level of description. Abstraction (as performed by the mind) can perhaps be understood by abstract models of the mind's operation. For the purposes aimed at here, this exposition shall rely on verbal terms as used in mathematics education, education sciences, and in part in cognitive psychology. The view towards abstraction as a process of changing or creating mental representations in a particular manner seems to be widely accepted, even if not always a definition of abstraction is provided (see, e.g., Gentner [21]).

The role of abstraction in cognition

Although "... human cognition is certainly embodied, its embodiment is not what gives human cognition its advantage over that of other species. Its advantage depends on its ability to achieve abstraction in content and control" (Anderson et al. [22]). Indeed, much speaks for the view that abstraction is a *fundamental working principle of the brain*, that drives already early stages of mental development. Schulz et al. [23] come to the suggestion "... that children's ability to learn robust, abstract principles does not depend on extensive prior experience but can occur rapidly, on-line, and in tandem with inferences about specific relations. ... Researchers have proposed that such rapid learning is possible because children's inferences are constrained by more abstract theories ...". Skorstad et al. [24] stressed that "Current work in concept-formation suggests that abstraction does indeed take place during concept learning..." H. Ballard pointedly summarizes "brain computation as hierarchical abstraction" [25]. Shepard [26] argues "Possibly, behind the diverse behaviors of humans and animals, as behind the various motions of planets and stars, we may discern the operation of universal laws. "

In this line, the author conjectures that it is universal laws, too, that allow for abstraction and even drive it. Abstraction makes many cognitive operations feasible – by reducing the demand of ressources, energy consumption, and operational complexity. Hence the hypothesis that any particular abstractions occur as the result of optimization processes driven by universal laws.

Localization

It seems obvious that abstraction plays a prominent role in those brain domains that are responsible for conscious thinking and human language processing, but it is also quite reasonable to assume that abstraction mechanisms already work in more basic layers of the brain's functional architecture, in particular, when processing sensomotoric informations. Here, one of the most basic operations is visual pattern recognition, possibly followed by identifying simultaneously occuring similar patterns. The occurence of patterns – or patterns of patterns – is processed further by higher cognitive layers, associating these patterns with objects or events. The same can be said with respect to the parallel processing of other kinds of sensomotoric input. A particular task of even higher layers is to define or understand, respectively, categories of perceived objects, like "animal", "cat" vs. "dog", etc. This task is highly abstractive as it requires to detect essential common features and to neglect non-essential features of the objects; note that whether some features are "essential" or not depends on the underlying cognitive goal.

A further abstraction step is performed by creating category labels, and yet another by handling category labels instead of

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a variety of objects itself. From there, a much higher level of abstraction is achieved by including structural relations between categories or labels, respectively.

Some more words about the connection between abstraction and understanding, creating, or handling categories, as this connection is particularly strong: “The earliest theories of category representation (e.g., rule based and prototype) tended to view category knowledge as consisting of abstract information that summarizes the central tendency across examples. Rule-based theories and models hypothesize that category representations consist of one or more rules or definitions that can be be learned from experience ...” (Levering & Kurtz [27]). It appears to be obvious that *rules* themselves are abstract in its nature, even if they are physically represented. Gentner has pointed out that the human smartness is connected with the ability to create *relational* categories [21]. Relational properties have to be extracted from structured representations of objects, their attributes, and relations. The way this goes in comparison and analogy is described in terms of structure-mapping by Gentner et al. [28][29]. It seems to be natural that a similar description might be given w.r.t. abstraction as a particular form of structure-mapping.

Summarizing, it appears that abstraction processes are organized within a complex architecture that mirrors the functional brain architecture itself.

C. Mathematical Abstraction

Thinking about mathematical abstraction, too, goes back to Aristotle who, in “... the context of mathematics, ... uses the term ‘abstraction’ (aphairesis) to refer to the act of ignoring or disregarding matter and change from perceptible objects in order to isolate their specifically mathematical characteristics as distinct objects of thought” (Smith [13]). When talking about mathematical abstraction a slightly broader view shall be used; “mathematical abstraction” will refer to abstraction processes connected with “understanding mathematics” or “doing mathematics”, respectively. This means that the objects of cognition themselves are representations of mathematical objects or relations.

Formally, mathematical abstraction is often understood as a (non-injective) mapping a, say, with $a(x)$ beeing an *abstraction* of x and, vice versa, each x with $f(x) = y$ being an *instantiation* of y. An early source of viewing it that way is Rinkens [30]; more recently, this view is taken up in abstract diagrammatic reasoning (Stapleton [31]).

Here, we have to be more specific w.r.t. the teaching objectives. It will be distinguished between *receptive*, *applicative* and *creative* abstraction. *Receptive* abstraction refers to individual brain activities that provide “understanding” of abstract concepts that have been defined beforehand by other individuals. To the opposite, *creative* abstraction is concerned with the construction of new mental representations without external inspiration. *Applied* abstraction means to employ abstract objects and relations, regardless whether these have been created by other individuals or not. Accordingly, enhancing receptive abstraction is the primary concern of teaching, where active and creative abstraction play an important role in problem solving, which comes into the focus in the advanced stages of teaching.

Although being complex, there are some particular aspects of abstraction that can be isolated. The following activities will be considered as basic aspects of abstraction:

- *encapsulation:*
  i.e., to see a number of objects as a whole entitiy, e.g., to see
  $$e^{\frac{4x^2}{23x+17}} \quad \text{as} \quad e^{\boxed{\text{something}}} \tag{2}$$
- *symbolization:*
  i.e., introducing abstract referents (indices) for patterns like expressions, relations, statements etc.; e.g.,
  $$e^{\frac{4x^2}{23x+17}} = e^{\boxed{\text{a}}}, \tag{3}$$
- *analogization:*
  i.e., identifying common features in different objects or domains and creating a new object out of them, e.g., identifying the common property of squares, rectangles, rhombus, etc., as being a quadrangle:
  COMMON(square, rectangle, rhombus, ...) = quadrangle
- *class formation:*
  i.e., encapsulation of a number of analogized objects, e.g., forming the class (or set) of quadrangles.

The following activities work upon a certain stock of pre-established abstract objects:

- *structural synthesis:*
  e.g., grouping separate objects x and y to a pair (x, y) being considered as a new object
- *object embedding:*
  i.e., seeing a particular object as an element of an appropriate category (set) in order to use category properties rather than individual properties, e.g., as here:
  $$e^{\frac{4x^2}{23x+17}} = e^{\varphi(x)} \tag{4}$$
  In the example, the left hand superscript expression is interpreted as evaluation of some differentiable function φ; hence, results for the whole class can be applied (e.g., the chain rule of differentiation).
- *switching embedding levels:*
  i.e., embedding/outbedding in nested structures; e.g., the changes of focus between a set and its elements.

Further abstraction operations work on structures on collections of objects rather than on objects itself:

- *structural alignment*
  in the sense of a simultaneous encapsulation of objects

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and their connecting structures, typically followed by a symbolization.

Example: The expression

$$\begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 8 \\ 9 & 4 \end{pmatrix} \qquad (5)$$

(with the dot denoting usual matrix multiplication) could be seen as

$$\boxed{\textit{something}} \circ \boxed{\textit{something'}}$$

or even more briefly as

$$a \circ b,$$

say, with a, b denoting some objects of some category and $\circ$ denoting some operation, all of them still to be specified;

- *structure-object interchange:*

  that is, rendering *structures*, i.e., relations between different objects, to encapsulated *objects* of consideration
- *recursion:*

  i.e., establishing recursive structures within problems or within problem solving strategies; e.g., when trying to simplify the expression

$$A \cap (B \cup (A \cap (B \cup (A \cap (B \cup (A \cap B)))))). \qquad (6)$$

This enumeration is by no means complete, but may suffice for the purpose of this paper.

IV. The Economics of Abstraction

As already mentioned, a significant feature of creative abstraction is to omit "unessential" details of the object under consideration. However, what is "unessential" can vary heavily with the underlying cognitive task. This can be observed in a variety of domains and is particularly true in mathematics. For example, the set of the real numbers, equipped with the usual addition and multiplication, represents different abstract objects at the same time, e.g., a vector space, a ring, a field, etc. Which property is "essential" clearly depends on the problem under consideration. Typically, the choice of the appropriate abstraction will ease the solution of a problem – the problem can be solved with less mental effort, within less time, with deeper insight in its nature, etc. Sometimes, it is even impossible to solve a given problem without appropriate abstraction. So far, this phenomenon is clearly a social experience of the mathematical community, but on the other hand, it can be re-experienced by each individual that deals with mathematical problems.

Hence the author's hypothesis: *A latent aversion against abstraction can be reduced by the individual experience of "economic benefits" when using abstraction.*

V. Abstraction Aversity

As mentioned, mathematics education is perceived as being difficult by many students. Often it is claimed that this is due to the abstractness of mathematics. This does reflect an attitude of restraint up to rejection towards abstractness. This may seem a little bit puzzling as without the hierarchically abstractive organization of brain functions there would be no human life. However, many of the brain abstractions remain unconscious or, at least, are not subject to conscious attention. That is, the conscious mind seems to reject activities that are quite basic for its very existence. However, there is evidence of such phenomena. E.g., Medin and Ross write "We have repeatedly demonstrated the limitations of people as abstract, deductive reasoners and noted with chagrin the difficulty of producing transfer of training or generalized problem-solving skills" [32].

While this statement refers to abstraction-related *actions*, there is also rejection of abstraction before any action takes place. Here is an example from the author's teaching practice. From time to time, the author uses to check whether the students captured a given matter – by directly asking them in the lecture. Often, the students have questions. In many cases, the author can answer explaining the subject

- either by a concrete numerical example
- or by using symbolic notation (variables).

The students can choose between these possibilities by a "ballot", i.e., by by raising their hands in favor of one of these two possibilities. In an auditory of about 500-600 people, the number of hands cannot be counted exactly; however, it is possible to obtain reasonably good estimates. In almost all cases

- about 80% of the votes prefer the numerical example
- only 20% prefer a more abstract explanation.

This observation matches a statement of Österholm, saying that the presence of symbols renders mathematical texts more difficult [33].

VI. What can Metacognition do?

When arguing that metacognition might help to accelerate the acquisition of abstraction abilities, there should be some justification. After all, the possible use of metacognition in education is not undisputed. Also, the relation of metacognition to abstraction should be considered. A pessimistic view is offered by Hajek: "... the formation of a new abstraction seems *never* to be the outcome of a conscious process, nor something at which the mind can deliberately aim, but always a discovery of something which *already* guides its operation" [34]. However, the author's experience with the metacognitive support system CAT allows for some optimism, as shall be seen below.

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A. Some educational experience

As alredy mentioned, coping with mathematics belongs undoubtedly to the major challenges for many first-year university students. The author's long run experience in teaching basic mathematics courses for large numbers of future economists at the University of Paderborn indicated that one severe reason for the difficulties with mathematics is the lack of adequate techniques – for studying in general, and for coping with mathematics in particular.

As a remedy, the author started in 2010 to teach not only mathematical subjects itself, but as well appropriate strategies that allow to succcessfully deal with these subjects. These strategies, summarized by the logo "CAT", are rule based and of metacognitive type. The rules, referred to as *Checklists*, *Ampel* (german for traffic lights) and *Toolbox*, do not only address issues of a proper organization of the study process, but rather the organization of concept understanding, self assessment, and problem solving. As it is impossible to account to all details here, the reader is referred to [2][3][4] for a detailed description.

A distinctive feature of CAT is that teaching and exercising working methods became integrated part of the regular mathematics course. Already long before 2010, there had been several attempts to support the students – e.g., by offering them optional tutorials dealing with such methods. But these offers failed to be effective as most of the students did not take advantage of them. By introducing CAT in the regular course, all students can optionally benefit from this offer; they can learn what these methods aim at, how they work, judge them, and decide whether to employ them in their own work.

"Reading Mathematics"

CAT's major concern is to enable the students to read and understand mathematical texts and expressions properly – from the level of single signs, symbols, or words up to the level of valid mathematical concepts. The way to go is described by the Checklist "Reading" along five steps. This checklist is to be used together with the student's *vocabulary* – a written (and, hopefully, mental) list that keeps track of all new definitions and symbols. The five steps of the checklist are described in Table I.

TABLE I. The Checklist 'Reading'

S1: *Spell*
(Make sure to know the precise meaning/role of each single symbol, sign, or word.)
S2: *Read out*
(Try to read out the full phrase as a spoken sentence.)
S3: *Animate*
(Provide examples, and non-examples, respectively.)
S4: *Visualize*
(Provide a graphical visualization, if appropriate.)
S5: *Talk*
(Give an explaining talk to others, including questions and discussion.)

Basically, the first two steps of the Checklist "Reading" provide nothing but the lexical fundamentals of a concept. Consider, e.g., the phrase

$$M := \{\, n \in \mathbb{N} \,|\, \exists m \in \mathbb{N} : n = 2m \,\}. \tag{7}$$

In reasonably good course notes, all necessary ingredients are provided before this phrase occurs; here, we assume that the symbols " := ", "{...|...}", "∈", "ℕ", "∃", and ":" have been introduced beforehand. Clearly, the concept of sets has to be understood before, too. Note that a student who arrives at the above phrase when reading the course notes can easily perform the steps S1 and S2 of the Checklist "Reading", as all mentioned pre-defined ingredients can be imported from the vocabulary. After doing so, the student might arrive at a sentence like this:

(V1) *M is defined as the set of all natural numbers n with the property that there exists a natural number m such that n equals $2m$.*

One might think that such a – reasonably fluent – verbalization should reveal its meaning easily, hence the student should generate at a brief description like

(V2) *M is defined as the set of all even natural numbers.*

However, this is by far not true, as became evident from written exams, see, e.g., Dietz & Rohde [35]. In an appropriate task, about 50% of the students were able to provide a reasonable "read out" analogous to (V1), but only 10% could give a meaningful explanation like (V2) using their own words. That is, a fluent statement like (V1) does by far not guarantee a deeper concept understanding. Hence, (V1) has still to be followed by the steps S3, S4 and S5 of the Checklist "Reading" in order to arrive at a valid mental concept. Note that the mentioned results of the 2012 exams clearly indicate that many students, although they had been taught these steps, did not yet perform them in a satisfactory manner.

To henceforth support the students better in going through these steps, the instrument of a *Concept Base* was introduced – a kind of form sheet that augments the vocabulary entries of a concept – key word, definition, denomination, "read out", syntax – by compliant extensions like examples, non-examples (from S3), visualizations (from S4), useful statements, and applications (from the ongoing progress in the course). [4] gives an detailed account on how to perform these steps, in particular, how to obtain examples and visualizations.

Empirical findings

It should be stressed again that the mentioned rules are metacognitive in nature. Initially, these rules had been explained and exemplified within the lectures solely, based on the author's belief in their immediate persuasiveness. However, it

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turned out that this was not enough in order to change long-term working habits. Therefore, after 2010 the implementation and improvement of CAT was accompanied by several empirical studies.

An initial qualitative study [5] gave rise to the persuasion that metacognitive support, particularly for "reading mathematics", can be effective. From 2012 on, a quantitative pre-study and a three-stage main study were conducted. Several results are reported in [36] and [6]. The pre-study from 2012 showed clearly that, at that time, many students did not apply the CAT methods. Several reasons for that could be found:

- Essentially, the students missed support by the tutors.
- They did often not understand how the methods apply.
- Many students widely underestimated their own need and possible benefit of these methods.

As a consequence, all components of the teaching process – including tutorials, homework exercises, mentoring – have successively been aligned in order to "live" CAT. The investigations of the subsequent stages of the main study show that these measures led to a considerable increase in the acceptance of CAT. From [6] and from some yet unpublished data of the main study one can see the following: Recently, the Checklist Reading, Vocabulary, Toolbox, and Concept Base are ranked to be helpful by large majorities of the students (between 63% and 84%) and – except for the Concept Base – these tools are regularly used by the majority of students. Moreover, most of the students do use concept bases at least sometimes. Interestingly, many students esteem the helpfulness of these instruments even higher (Table II).

TABLE II. Rating of the helpfulness of CAT's instruments

Instrument of CAT	Percentage of students rating it "very helpful"
Checklist Reading	19.9
Vocabulary	38.5
Concept Base	16.4
Toolbox	32.2

As to the effects of CAT on the study success: At the described level of exploitation, CAT turns out to be definitively helpful for students with medium academic performance. In this group of students, the use frequency of CAT is positively correlated with the success rate of the final exam and negatively correlated with the grades (1='very good', 5 = 'insufficient'). Besides that, there are verbal comments of students like "... put more effort on rendering concept bases, as these are really important for own learning" (authors translation from german "mehr Aufwand auf die Erstellung von Konzeptbasen legen, da diese wirklich wichtig zum eigenen Lernen sind").

Admittedly, the utilization of CAT by the students did not yet reach the desired degree; in particular, the author believes that a more intense use of concept bases could promote deeper understanding a lot. The data from [6] suggest that many students do overestimate the time effort needed for utilizing concept bases. This problem might be overcome in the future by providing better information for the students. Further, both utilization and efficacy of the Ampel need to be substantially improved. But summarizing the results obtained so far, the metacognitive support has shown its potential to provide effective help for many students.

As to abstraction

In the same spirit, it appears that at least some aspects of conscious abstraction are amenable to meta-instructions. Goldstone & Sakamoto state that "If abstracting deep principles that cut across different domains is frequently valuable (see Anderson, Reder, & Simon, 1996 and Barnett & Ceci, 2002 for defenses of this assumption), then it is likewise valuable to find ways to promote this abstraction" [37].

B. "Value added" by metacognition

The metacognitive rules provided by CAT, although exemplified in the context of a mathematics course, are by far not bound to mathematics. Rather it appears that the provided working techniques could easily apply in other domains as well, with minor and obvious modifications. (This was at least one of the basic intentions when initiating CAT.) There is episodic evidence that some students applied Concept Bases in other courses, too. However, so far the long run effects of introducing CAT have not yet been investigated.

C. Metacognition as abstraction

It should be noted that metacognition itself is highly abstractive in its nature, because the rules of working and thinking become objects of interest rather than the proper subjects of the work and thinking. Of course, one may argue that metacognition is part of the brain's task management, which seems to be undeniable. However, in the author's opinion it seems to be more natural to consider task priorization, task ressource allocation, and time management as belonging to task management, whereas rules are abstract in nature.

VII. Operational Aspects of Abstraction

For the purposes of the project, we have to confine ourselves to selected aspects of abstraction. The selection takes into account:

- the needs of abstraction within the course
- the degree of operationability
- the degree of observability.

Recall that we want to support problem understanding and solving processes with the help of *metacognitive* abstraction rules. These can be understood as rules that guide and structure the *working process* rather than providing particular abstraction results. From this point of view, the focus will be on such aspects of abstraction that appear to be in reach of such metacognitive rules. Examples of such aspects are

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- encapsulation/analogization/symbolization
- structuring
- recursion techniques and
- qualitative reasoning.

To illustrate the idea of abstraction meta-rules suppose that the student's problem under consideration is given by some text, formula or so, henceforth called the *document*. The first of the forementioned abstraction aspects is closely related to the visual input. Hence, the following meta-rules are suggested:

(R1) *Provide a clear* visual *organization of the document.*

(R2) *Identify large substructures.*

(If appropriate, *put them into containers* or *symbolize* them, respectively.)

(R3) *Identify similar patterns.*

(If appropriate, *symbolize* them.)

(R4) *Identify repetition indicators w.r.t. tasks or structures, respectively.*

(Try to use *one* solution for all repeated tasks and *one* principle to work with repeated structures.)

For example, consider this task for students:

> **Task 1:** *Determine the operating minimum, given the following cost functions:* 1) $K_1(x) := 4x^2+15\,x+42, x \geq 0$, 2) $K_2(x) := 242x^2+72x+117, x \geq 0$, ... 5) $K_5(x) := 25x^2+5x+242, x \geq 0$.

Obviously, there are at least three different levels of abstraction on which this task could be fulfilled. We call the least one *level*

(A0) Without any experience in abstraction-aided working, the students would tend to solve each of the problems 1 to 5 individually, using only numerical computations. This would imply to perform the corresponding ansatzes and solving techniques altogether five times, and probably some of the students would try to facilitate the computation somehow "on the way".

We claim that by respecting the above rules progress to a higher abstraction level could be promoted. Indeed, a better visual organization of the task according to rule (R1) might already change the document as follows:

> **Task 1:** *Determine the operating minimum, given the following cost functions:*
> 1. $K_1(x) := 4\;x^2 + 15\;x + 42,\;\; x \geq 0$
> 2. $K_2(x) := 242\;x^2 + 72\;x + 117,\;\; x \geq 0$
>
> ... ...
>
> 5. $K_5(x) := 25\;x^2 + 5\;x + 242,\;\; x \geq 0.$

From here, looking both at the five *repetitions* as proposed by rule (R4) and at *similar patterns* as proposed by rule (R3), the students might more easily see the uniform structure

$$K_{\blacksquare}(x) := \blacksquare\; x^2 + \blacksquare\; x + \blacksquare,\;\; x \geq 0, \tag{8}$$

where the gray boxes symbolize containers with different contents. According to (R4), we recommend to find a unified solution from here. Thus, it is appropriate to follow (R3) and to symbolize the contents of the boxes as

$$K_{\blacksquare}(x) := a\; x^2 + b\; x + c\; x \geq 0. \tag{9}$$

Thus, the next *abstraction level* is attained:

(A1) The problem is solved in a *symbolic* rather than numeric way.

Using the symbolic approach, Task 1 can be solved *at once*, yielding a result in terms of the parameters a, b and c. Then, the desired five numerical results can easily be obtained by just plugging in the appropriate numbers.

Note that working on level (A1) rather than on level (A0) is quite obviously advantageous; it pays in time savings, less error sensitivity, qualitative insights, and also aesthetics. All these advantages can be experienced by the students themselves and they might also stimulate them to try such an approach again, when solving other problems. Analogous meta-rules can be formulated for structuring and recursion techniques, although there we shall need and exploit additional syntactical guidelines.

Let us look at another example. Suppose we are in a context where matrix multiplication was just introduced, defined in terms of abstract formulae and illustrated by numerical examples. No further rules of matrix multiplication have been given yet. Now the students are given the following task:

> **Task 2:** *Decide whether these two terms describe one and the same matrix:*
>
> $$\left(\begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 8 \\ 9 & 4 \end{pmatrix} \right) \cdot \begin{pmatrix} 6 & -1 \\ 0 & 11 \end{pmatrix}$$
>
> *and*
>
> $$\begin{pmatrix} 1 & 7 \\ 3 & 5 \end{pmatrix} \cdot \left(\begin{pmatrix} 2 & 8 \\ 9 & 4 \end{pmatrix} \cdot \begin{pmatrix} 6 & -1 \\ 0 & 11 \end{pmatrix} \right).$$

It seems to be quite natural to answer this question directly by simply calculating both double matrix products and obtaining two equal results, namely

$$\begin{pmatrix} 390 & 331 \\ 306 & 433 \end{pmatrix}.$$

The situation changes if the same task is to be performed again – either several times with different numbers or just once, but with "uneasy" numbers – like in this (artificial) example:

> **Task 2':** *Decide whether these two terms describe one and the same matrix:*
>
> $$\left(\begin{pmatrix} 4263 & 7377 \\ 3521 & 5769 \end{pmatrix} \cdot \begin{pmatrix} 74650 & 70294 \\ 19032 & 54280 \end{pmatrix} \right) \cdot \begin{pmatrix} 6279 & 5017 \\ 3592 & 2156 \end{pmatrix}$$
>
> *and*
>
> ...

Probably, some students might try again to apply numerical calculations, using some calculators. Nevertheless, it becomes plausible that the more effort an immediate calculation requires, the bigger the incentive to look for a simpler, more elegant, and typically more abstract solution. Some students might see, along the lines described before, as structure like that:

Task 2': *Decide whether these two terms describe one and the same matrix:*

$$(A \circ B) \circ C$$

and

$$A \circ (B \circ C).$$

So, the idea that associativity does also hold in matrix multiplication, which was just introduced, might occur – at least more easily than before.

At this point it should be stressed again that the initial progress in dealing with both forementioned tasks is heavily supported by our brain's ability to identify structural properties of the sensomotoric input, here more specifically: of the *visual* input. Processing any text-based mathematical task starts with processing its text. At a very raw level, even before reading the symbols and making sense of them, one can view such a text just like a picture. By doing so, one can detect structures like clusters, straight lines or axes, or even nesting of patterns. In our examples, the ideas governing abstraction are closely connected with such properties.

Turning back to Task 2', it remains to prove that this idea is really true – with A, B, C being matrices and "$\circ$" representing matrix multiplication " $\cdot$ ". It turns out that the attempt to do so reveals the possibility to obtain further neat abstractions on different levels.

To begin with, one has to understand that the following identity has to be proved:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C) \tag{10}$$

or briefly

$$D^L = D^R \tag{11}$$

with $D^L := (A \cdot B) \cdot C$ and $D^R := A \cdot (B \cdot C)$. Note that this idea involves a – more or less conscious – combined encapsulation/symbolization operation.

The superficially "easiest" version to prove (10) is bound to matrix dimension (2,2) by writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} f & g \\ h & j \end{pmatrix}, C = \begin{pmatrix} k & l \\ m & n \end{pmatrix},$$

say, and performing the necessary calculations symbolically. So one finds that

$$D^L_{11} = (af + bh)k + (ag + bj)m = D^R_{11};$$

analogously one shows that it holds

$$D^L_{ij} = D^R_{ij}$$

for *all* relevant pairs (i, j) of indices. Thus (11) is established. Again the solution is found by passing to abstraction level (A1). Its advantage is being valid for *all* (2,2) - matrices, irrespective of concrete numeric values.

Nevertheless, one may argue that this solution is still too laborious. Suppose one agrees upon this point of view. Then, it is time to look for another approach that might be more effective – i.e., more economic. Again, it turns out that this objective is attained via abstraction: The solution is promoted by CAT's Toolbox concept mentioned above. The first tools to be placed in the Toolbox when trying to prove anything are the necessary *definitions*. As assumed, the product of two matrices A and B of dimensions (L, M) and (M, N), respectively, was defined symbolically, namely to be the (L, N) matrix H with entries

$$H_{ij} = \sum_{m=1}^{M} a_{im} b_{mj} \tag{12}$$

$(i = 1, ..., L; j = 1, ..., N)$; one uses the notation $H =: A \cdot B$.

Now, in order to prove (10) by referring to this definition, one has to show that

$$\sum_{n=1}^{N} \sum_{m=1}^{M} a_{lm} b_{mn} c_{np} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{lm} b_{mn} c_{np} \tag{13}$$

holds for any matrices A, B and C of dimensions (L, M), (M, N) and (N, P), with $L, M, N, P \in \mathbb{N}$, respectively, and for all $l \in \{1, ..., L\}, P \in \{1, ..., P\}$, respectively. Note that it is quite straightforward to prove (13) by just exploiting the rules for addition and multiplication of real numbers. But, according to the author's experience, many students don't understand that (13) proves (10) right away. The reason for that lies in the fact that connecting (10), (13), and (12) involves a nested structural embedding; in particular, one and the same symbol – H – has to be identified subsequently with $A \cdot B$, $(AB) \cdot C$, $B \cdot C$, and $A(B \cdot C)$. That is, a targeted training of such operations might be quite desirable.

To summarize: Starting from abstraction level (A1) in connection with the desire for a higher working economy and guided by metacognitive instructions like Toolbox and abstraction rules, a higher degree of sophistication is attained. This higher degree of sophistication can be understood as a price that has to be paid in order to answer an initially simple question – but this price pays multiply, not only as proving (13) is that straightforward, but even more as, in the end, this solution of the initial task becomes quite easy and, moreover, provides a deeper conceptual insight into the matrix calculus.

Qualitative reasoning

Now what about *qualitative reasoning*? This refers to abstraction level

(A2) This level of abstraction is achieved when referring to more general – and thus more abstract – classes of objects when solving a given problem.

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Here, there are two main directions of interest: First, the mathematical generalization; the second direction might be called "economic generalization".

Mathematical generalization: Here, the more general classes of objects are defined *within* mathematics.

Consider, e.g., the following task:

Task 3: *Determine*

$$\min_D f \quad \left(= \min_{x \in D} f(x)\right) \tag{14}$$

for a given differentiable function f on some interval D. As long as f is described by a computable expression as, e.g., in Task 1 or in (9), it can be assumed that most of the students would try to run a standard approach by solving $f'(x) = 0$ for x and reasoning about the solution(s) that have been found (if so), possibly by additionally inspecting the boundary points of D.

However, a more abstract point of view is taken if they try to figure out whether f obeys some useful *qualitative* properties. E.g., if it can easily be seen that f is *increasing*, there is no point in trying to solve $f'(x) = 0$; instead it is known in advance that if $\min f$ exists it is attained in the left boundary of D. Similarly, if one knows that f is *convex* than one also knows that each local minimum of f is automatically global – saving energy, e.g., when reasoning about the number and types of solutions of $f'(x) = 0$.

Note that this kind of viewing the problem requires the readiness to invest some more abstract thoughts before starting to really tackle the problem; however, in many cases this investment will pay. In addition, the initial investment will not be that expensive, because the students do have some quite easy tests for mononocity and convexity at hand.

Economic generalization:

In modern economics one considers classes of functions named "cost functions", "production functions", "utility functions", etc., often in combination with qualitative attributes like being "neoclassic". Intrinsically, these classes and attributes, respectively, can be defined in terms of mathematical properties. At the same time, they obey a strong economic significance. To be able to reason in terms of these notions – and thus on the second abstraction level – is quite valuable for ongoing economists.

Returning to Task 1, the students might enter level (A2) by observing that *each K is a neoclassic cost function*. Thus, the operating minimum – as the minimum average variable costs - is nothing but the limit of the average variable costs as $x \downarrow 0$. Now it is quite easy to obtain the same results as above.

Clearly, to step here from level (A1) is quite complex and requires a solid theoretical background. It is clear that to work on this level cannot – and shall not – be trained before this solid theoretical background was laid out. But provided this was done, a corresponding meta-rule could be

(R5) *Try to work in economic categories rather than with numeric examples.*

To follow this rule, the students need a very clear overview over the mathematical tools at their disposal. This overview is supported by the toolbox concept as described in [3].

VIII. The Project

The forementioned meta-rules can only brought to life by an intense training that shows how to use them and how they can help to re-structure ones own work in order to gain more progress within the same time. We intend to test and to improve corresponding training measures within a voluntary project group. These measures should

- positively change the students' attitude towards abstraction
- increase the aceptance of (at least passive) abstraction
- enhance the ability of active abstraction
- enrich the regular teaching process.

The project group shall be constituted by random choice from a set of voluntary applicants, hence there shall be an untreated control group as well. The only incentive for participating shall be the perspective of being able to cope better with mathematics, but no examen credits shall be promised.

As to the program: Before and after the series of proper training units we shall perform guideline based interviews as well as observed and videotaped working sessions. Through appropriately designed tasks, it shall be observed whether the students become more apt to understand and use abstract approaches than before. The training sessions shall focus on the different aspects of abstraction, as mentioned above. Tasks 1 – 3 might serve as a possible examples: First, before the training starts, the students are asked to solve a task of this kind by their own. Their approaches and solutions are observed and video documented. After that, we introduce the meta-rules and explain how they work in these and other examples. It will be important to address the benefits of using abstract approaches as well. The students will be given a series of example tasks, with the help of which they can exercise their ability in pracising the meta-rules. At the end of the training sessions, the students shall be given another set of tasks, and again their approaches and solutions are documented. Ideally, there shall be a tendency to work on a (slightly) higher abstraction level as at the beginning of the training. It can be expected that this effect will be the more significant the more the students can gain positive own experience.

IX. Conclusion

In large and heterogeneous basic mathematics courses students need support to manage mathematical abstraction. The paper described some particular aspects of mathematical abstraction that, so the author's hypothesis, can be trained

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with the help of metacognitive rules. As a justification of this hypothesis it is shown that metacognitive rules already have proved to be helpful in another context, i.e., in supporting "reading mathematics" and problem solving. In addition, some examples of metacognitive rules that address abstraction are provided. Further a framework for an appropriate field study in order to investigate the possible effects of a metacognitive-rule based training of mathematical abstraction was presented.

Although performing such a field study as well as adjusting the training instruments is subject to future work, the present discussion might be inspiring for those that intend to deal with mathematical abstraction as a human cognitive ability, in particular by supporting abstraction in mathematics education.

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