A Linear Matrix Inequality Based Strategy For Maximum Amplitude Analysis In Discrete Time Systems

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***Abstract*—A typical problem in engineering consists of quantifying the maximum amplitude of the signals of discrete-time linear time-invariant systems. This paper suggests a strategy based on the solution of a convex optimization problem, in particular, a linear matrix inequality feasibility problem. The proposed strategy is motivated by the fact that the existing methods are generally conservative.**

Keywords: Discrete-Time; Maximum Amplitude; LMI.

I. Introduction

Discrete-time linear time-invariant systems represent a wide class of control systems of interest [1]–[3]. For instance, they can model a worktable motion control system or a space station orientation control system where digital control is implemented through the use of digital-to-analog and analog-to-digital converters [4]. A typical problem in these systems consists of quantifying the maximum amplitude of their signals. Indeed, this problem is met in order to establish that physical quantities do not exceed their allowable limits, a necessary step for ensuring safety.

This paper suggests a strategy for dealing with this problem. This strategy exploits polynomials and the Gram matrix method [5] [6], and requires the solution of a Linear Matrix Inequality (LMI) feasibility problem, which belongs to the class of convex optimization.

The paper is organized as follows. Section II discusses some related works. Section III introduces the problem formulation. Section IV presents a motivating example. Section V described the proposed strategy. Section VI reports the conclusions.

II. Related Works

The problem of quantifying the maximum amplitude of the signals of linear time-invariant systems has been studied in the literature. In particular, LMI methods have been sought because they can be tested through convex optimization and because they may be exploited in the design of feedback controllers. A pioneering work is [7], which considers the canonical problem of quantifying the peak of the impulse response, and shows how upper bounds can be established via LMIs based on set invariance of quadratic Lyapunov functions. Another pioneering work is [8], which describes how this methodology can be used in the design of feedback controllers ensuring desired upper bounds on the peak of the impulse response. The reader is also referred to recent works such as [9] [10] where this methodology is exploited in optimal control and in model predictive control.

However, the methodology exploited in these related works is generally conservative and may provide upper bounds that are rather far from the sought quantities as it will be shown in Section IV. The idea proposed in this paper aims at providing an LMI-based strategy where conservatism is eliminated. This paper extends our previous work [11] where only continuous-time systems are considered.

III. Problem Formulation

The notation is as follows. The set of integer numbers is denoted by $\mathbb{Z}$. The set of real numbers is denoted by $\mathbb{R}$. The transpose of a matrix A is denoted by A'. The infinity norm of a matrix A is denoted by $\|A\|_\infty$, i.e., $\|A\|_\infty = \max_{i,j} |a_{i,j}|$ where $a_{i,j}$ is the entry of A on the (i,j)-th position.

We consider the discrete-time linear time-invariant system

$$\left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{1}$$

where $t \in \mathbb{Z}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are given matrices. For the system (1), we consider the problem in canonical form described hereafter. Let us start by introducing the following definition.

Definition 1: The impulse response of the system (1) with respect to the i-th input channel is the function $Y_i(t)$, defined as the solution $y(t)$ for initial condition $x(0) = 0$ and input $u(t) = \delta(t)E_i$, where E_i is the i-th column of the $m \times m$ identity matrix, and $\delta(t)$ is the impulse defined by

$$\delta(t) = \left\{ \begin{array}{ll} 1 & \text{if } t = 0 \\ 0 & \text{else.} \end{array} \right. \tag{2}$$

The problem addressed in this paper is as follows.

Problem 1: Given $c \in (0,\infty)$, establish whether c is an upper bound of the maximum amplitude of the impulse response of the system (1) with respect to all input channels, i.e.,

$$\|Y_i(t)\|_\infty < c \quad \forall t \geq 0 \quad \forall i = 1,\ldots,m. \tag{3}$$

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IV. Motivating Example

In order to motivate the strategy proposed in this paper, let us consider the simple discrete-time linear time-invariant system described by

$$\begin{cases} x(t+1) &= \begin{pmatrix} 0 & 1 \\ -0.3 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t). \end{cases}$$

This is a second-order system (i.e., $x(t) \in \mathbb{R}^2$), with scalar input and scalar output (i.e., $u(t), y(t) \in \mathbb{R}$). The impulse response of this system, as defined in Definition 1, is shown in Figure 1.

Let us use the LMI methods [7] [8] based on set invariance of quadratic Lyapunov functions for establishing upper bounds of the impulse response of this system. These methods are formulated as LMI feasibility problems, and provide the upper bound 1.307 shown in Figure 1. It is interesting to observe from the figure that, in spite of the simplicity of the system under consideration, this upper bound is conservative.

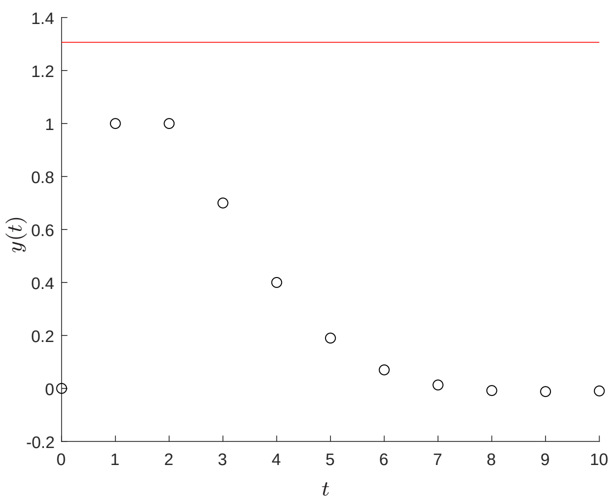

Fig. 1. Black circles: impulse response. Red line: upper bound provided by [7], [8].

V. Proposed Strategy

In this section, we describe the strategy proposed for dealing with Problem 1. The first idea is to describe the trajectories of the system (1) that correspond to its impulse responses through the sublevel set of a polynomial. Specifically, we denote this sublevel set as

$$\mathcal{V} = \{x \in \mathbb{R}^n : \ v(x) \leq 1\} \tag{4}$$

where $v(x)$ is a polynomial. The polynomial $v(x)$ should be determined under two main conditions. The first condition is that the trajectories of the system (1) that correspond to its impulse responses are included in the sublevel set $\mathcal{V}$. The second condition is that the sublevel set $\mathcal{V}$ is included in the set of states for which the maximum amplitude among the entries of the corresponding output is smaller than the desired upper bound c. By denoting with $\mathcal{W}$ the latter set, we have

$$\mathcal{W} = \{x \in \mathbb{R}^n : \ \|Cx\|_\infty < c\}. \tag{5}$$

In order to impose the first condition, a possibility is to require that $v(x)$ does not increase with the time starting from a point on the trajectories of the system (1) that correspond to its impulse responses. This can be expressed as follows. Let us consider the i-th input channel of the system (1), and let B_i be the i-th column of B. Let σ be a nonnegative integer. Then,

$$\begin{cases} c &> \left\|CA^k B_i\right\|_\infty \quad \forall k = 0, \ldots, \sigma - 1 \\ 1 &\geq v(A^\sigma B_i) \\ v(x) &\geq v(Ax). \end{cases} \tag{6}$$

In order to impose the second condition, let us observe that the inclusion of the sublevel set $\mathcal{V}$ in the set $\mathcal{W}$ can be expressed by the condition

$$v(x) > 1 \quad \forall x : \ \|Cx\|_\infty \geq c. \tag{7}$$

This condition could be imposed through the introduction of polynomial multipliers, which are typically exploited in order to convert local properties (such as, local positivity) into global properties (such as, global positivity).

Lastly, in order to search for a polynomial $v(x)$ that satisfies the conditions previously mentioned, one could exploit the Gram matrix method [5] [6]. Indeed, let us consider a polynomial $p(x)$ of degree not greater than $2d$, where d is a nonnegative integer. Then, $p(x)$ can be expressed through the Gram matrix method as

$$p(x) = b(x)' \left(P + L(\alpha)\right) b(x) \tag{8}$$

where $b(x)$ is a vector of monomials in x of degree not greater than d, P is a symmetric matrix, $L(\alpha)$ is a linear parameterization of the linear set

$$\mathcal{L} = \left\{\tilde{L} = \tilde{L}' : \ b(x)'\tilde{L}b(x) = 0\right\}, \tag{9}$$

and α is a free vector. The existence of α satisfying the LMI

$$P + L(\alpha) \geq 0 \tag{10}$$

is equivalent to the property that the polynomial $p(x)$ is a sum of squares of polynomials. Since this property implies that $p(x)$ is nonnegative, one could exploit the Gram matrix method to turn the search for a polynomial $v(x)$ into an LMI feasibility problem.

VI. Conclusions

This paper has suggested an LMI-based strategy for quantifying the maximum amplitude of the signals of discrete-time linear time-invariant systems. Future work will investigate the realization of this strategy and its use in the design of feedback controllers.

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