Reduced Complexity Algorithm for Quantized Equal Gain Transmission Codebook over Closed Loop MIMO Systems

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***Abstract*—In this paper, a novel index search algorithm is proposed for quantized equal gain transmission (QEGT) codebook based closed-loop multiple-input multiple-output systems. The codewords are grouped into a small number of categories by utilizing Grassmannian beamforming criterion. Then, the optimal centroid is selected within those of the groups, which maximizes the signal-to-noise ratio or capacity criterion. Hereby, the optimal index of QEGT codebook is determined within the group of the selected centroid. Monte-Carlo simulations are presented to show that the codebook index search complexity is halved, whilst maintaining almost the same throughput.**

Keywords-Codebook; Closed-loop MIMO; Limited feedback; Grouping; Grassmannian beamforming.

I. Introduction

For closed-loop multiple-input multiple-output (CL-MIMO) communication systems, the transmit beamforming (TxBF) technique alleviates the negative effect of fading channel by exploiting spatial diversity due to the increased number of MIMO fading channel paths. The TxBF requires feedback of channel state information (CSI) from the receiver to the transmitter. Such CSI feedback can potentially incur excessive overhead due to the multiplicity of channel coefficients, and thus a small number of feedback bits are sent via a feedback path for the transmitter to recreate the TxBF vector. These systems are known as limited feedback systems (see [1] and the references therein). To reduce the bandwidth requirement of the feedback systems, finite rate techniques have been proposed for the cases of TxBF [2]-[4]. In these limited feedback systems, the receiver chooses a precoding matrix from a finite set of precoding matrices, called codebook, on the basis maximizing the effective capacity or signal-to-noise (SNR) after combining, and sends the corresponding bits, which denotes index, to the transmitter. The codebook design strategies which have been suggested use either numerical optimization methods [4]-[8] or random vector quantization (RVQ) method [9]. Such random codebooks have been shown to be asymptotically optimal as the number of feedback bits and transmitted antennas increase [10], [11].

Among various codebooks for the TxBF, the QEGT codebook is the optimal precoding matrix for maximizing the capacity with a per-antenna equal power constraint. Also, it has modest transmit amplifier requirements than other TxBF techniques, since it does not require the antenna amplifiers to modify the amplitudes of the transmitted input signals [5]. Similar to other codebook based CL-MIMO systems, QEGT codebook needs a larger size codebook which gives better performance than a small size one, as the number of transmit antennas increases. In other words, the codebook size increases exponentially with the number of transmit antennas to maintain a given effective capacity or SNR loss with respect to the ideal non-quantized system [3], [4], [8]. The increased codebook size can cause a feedback delay due to the large amount calculation of excessive search (full index search) [4],[5], and also reduce the effectiveness of the precoding matrix at the transmitter [12]. It can be seen that the feedback delay may lead to negative effects on information capacity or symbol error rate [13]-[17]. Non-exhaustive methods for searching unstructured codebooks at the expense of increased memory requirements have been well researched in [18]. Hence, employing an efficient codebook index search algorithm becomes essential.

In this paper, an efficient codebook index search algorithm with Grassmannian beamforming criterion is proposed for finding optimal precoding matrix of QEGT codebook based CL-MIMO systems. Monte-Carlo simulations are presented to show that the normalized complexity is halved, whilst maintaining almost the same achievable throughput comparing to the full index search algorithm when the number of transmit antennas is more than three.

The remainder of the paper is organized as follows. Section II reviews CL-MIMO communication with TxBF. In Section III, we propose reduced complexity QEGT codebook index search algorithm which relies on a grouping strategy. In Section IV, the results of previous sections are demonstrated, and related discussions are given. Finally, concluding remarks are given in Section V.

II. System Overview

The CL-MIMO system relying on the TxBF and using N_t transmit as well as N_r receive antennas is considered. The M-dimensional complex transmit symbol vector at the

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channel instant m (for $m = 0, 1, \ldots$) is denoted by [1] $\mathbf{s}_m = [\mathbf{s}_{m,1} \cdots \mathbf{s}_{m,M}]^T \in \mathbb{C}^{M\times 1}$ with $\mathbf{s}_m \sim \mathcal{CN}(\mathbf{0}_{M\times 1}, \mathbf{I}_M)$ and $M \le \min\{N_t, N_r\}$. The vector $\mathbf{W}\mathbf{s}_m$ is sent through the channel where $\mathbf{W} \in \mathbb{C}^{N_t\times M}$ is the precoding matrix. Then, the received signal vector $\mathbf{y}_m$ at the N_r receive antennas can be written as

$$\mathbf{y}_m = \sqrt{\frac{\rho}{M}}\mathbf{H}_m\mathbf{W}\mathbf{s}_m + \mathbf{n}_m, \tag{1}$$

where $\mathbf{n}_m \in \mathbb{C}^{N_r\times 1}$ denotes the noise vector with $\mathbf{n}_m \sim \mathcal{CN}(\mathbf{0}_{N_r\times 1}, \mathbf{I}_{N_r})$ and ρ denotes the SNR. The matrix $\mathbf{H}_m \in \mathbb{C}^{N_r\times N_t}$ represents uncorrelated Rayleigh falt fading channel matrix with i.i.d. entries distributed according to $\mathcal{CN}(0, 1)$.

The evolution of $\mathbf{H}_m$ is modeled by a first-order Gauss-Markov process [19]

$$\mathbf{H}_m = \epsilon\mathbf{H}_{m-1} + \sqrt{1-\epsilon^2}\mathbf{G}_m, \tag{2}$$

where $\mathbf{G}_m \in \mathbb{C}^{N_r\times N_t}$ has i.i.d. entries with distribution $\sim \mathcal{CN}(0, 1)$. The noise process $\mathbf{n}_m$ in (1) is independent of $\mathbf{G}_m$ and $\mathbf{H}_0$. The time correlation coefficient $\epsilon\,(0 \le \epsilon \le 1)$ represents the correlation between elements $h_{m,i,j}$ and $h_{m-1,i,j}$ (where $h_{m,i,j}$ denotes the (i, j) entry of $\mathbf{H}_m$). We assume all the elements of $\mathbf{H}_m$ have the same ϵ. The evolution variable ϵ obeys Jakes' model [20] according to $\epsilon = J_0(2\pi f_D T)$, where $J_0(\cdot)$ is the zeroth order Bessel function, T denotes the channel instantiation interval, and $f_D = \frac{vf_c}{c}$ denotes the maximum Doppler frequency using terminal velocity v, carrier frequency f_c, and $c = 3\times 10^8 m/s$.

In a TxBF system, the key question is how to design $\mathbf{W}$ to maximize the system performance. For this reason, $\mathbf{W}$ should be chosen to maximize the receive SNR in order to minimize the average probability of error and maximize the capacity. In general, it can be determined by applying the singular vector decomposition (SVD) [21]. However, accurate quantization and feedback of this precoding matrix can require a large number of feedback bits quantized TxBF techniques provide a solution for this problem by quantizing the optimal precoder at the receiver. Specifically, the precoder is constrained to be one of N matrices, which is called a codebook. If the codebook of N matrices is known to both the transmitter and the receiver, $L = log_2 N$ bits of feedback are required for indicating the index of the appropriate precoding matrix [22]. Denote the precoding QEGT codebook $\mathcal{W} = \{\mathbf{W}_k\}_{k=1}^N$ and $\mathbf{W}_k \in \mathcal{U}(N_t, M)$. A procedure to generate the QEGT codebook for TxBF system is clearly given in six steps [5].

[1] a bold lower case letter $\mathbf{a}$ denotes the vector, a bold capital letter $\mathbf{A}$ denotes a matrix, $\mathbf{A} \in \mathbb{C}^{m\times n}$ denotes complex matrix $\mathbf{A}$ having m row and n column, $\mathbf{A}^H$ denotes the conjugation transposition of matrix $\mathbf{A}$, $\mathbf{I}_M$ denotes the $M\times M$ identity matrix, $\mathbf{0}_{m\times n}$ denotes the $m\times n$ zero matrix, $\mathcal{U}(m, n)$ denotes the set of $m\times n$ matrices with orthogonal columns, $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, $\log_2(\cdot)$ denotes the base two logarithm, and $\det(\cdot)$denotes the determinant of a matrix.

We assume that the receiver perfectly knows the current CSI by channel estimation algorithms. In order to maximize the system performance, two optimum codebook index selection criterions are presented: one is the SNR selection criterion and another is the capacity selection criterion. In the SNR selection criterion, the $\mathbf{W}$ should be selected to maximize the receive SNR, thus the selection criterion can be expressed as follows [4]

$$\mathbf{W} = \arg\max_{\mathbf{W}_k\in\mathcal{W}} \|\mathbf{H}_m\mathbf{W}_k\|_F. \tag{3}$$

Another precoding matrix selection criterion is the maximum capacity, which can be written as follows [23]

$$\mathbf{W} = \arg\max_{\mathbf{W}_k\in\mathcal{W}} \log_2\det\left(\mathbf{I}_M + \frac{\rho}{M}\mathbf{W}_k^H\mathbf{H}_m^H\mathbf{H}_m\mathbf{W}_k\right). \tag{4}$$

From (3), (4), the optimal precoding matrix $\mathbf{W}$ satisfy both SNR and capacity maximization criterions.

III. Proposed Index Search Algorithm

In order to reduce the complexity of index search algorithm for QEGT, we propose a new codeword grouping algorithm. In the second subsection, the optimal precoding matrix index selection algorithm was presented among the generated groups.

A. Grouping the codewords of QEGT

The prcoding matrices in the codebook are divided into a given number groups. Assuming that the QEGT codebook elements are uniformly quantized, the precoding matrices can be arranged into P groups, each group having Q precoding matrices $(N = P\times Q)$. The proposed grouping strategy uses Grassmannian beamforming criterion to generate P groups of codeword. Also, another grouping strategy using Lloyd's algorithm will be addressed briefly to compare performance for the proposed algorithm, which has already studied in [25]. The major difference between these two grouping strategies is that the key idea of proposed algorithm uses index elimination, while the Voronoi cell is used in [25]. Also, the proposed algorithm has characteristic that the precoding matrices in the QEGT codebook are distributed always evenly to each group, while the grouping strategy using Lloyd's algorithm is not always doing so. $i.e.$, if we have N = 16, P = 4, Q for the proposed algorithm is always 4, Q for Lloyd's algorithm has a value between 3 and 5. Thus, the calculation of proposed algorithm is always minimum comparing to the approach in [25]. This can be extended more generally, which will be shown in Section IV.

1) Grassmannian beamforming criterion: Using Grassmannian beamforming criterion, it can be maximized that the minimum distance between any pair of lines spanned by the codebook matrices on Grassmann manifold [4], [24]. And it provides an approach for finding optimal line packings. We outline the process as the following.

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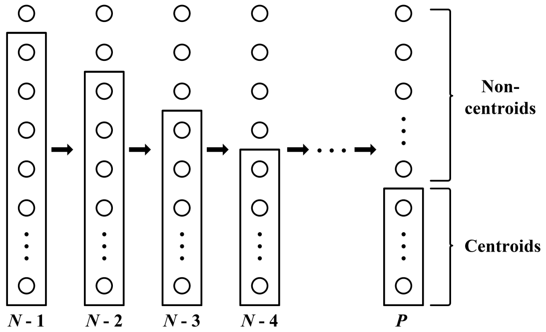

Figure 1. Centroid selection using Grassmannian beamforming criterion.

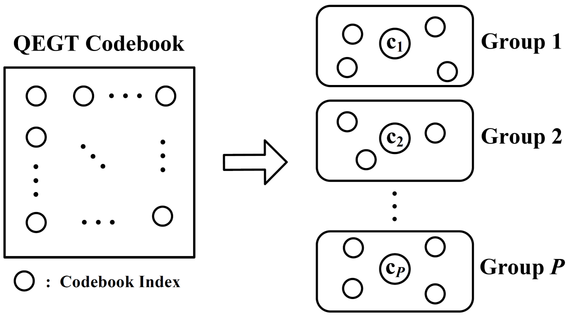

Figure 2. Grouping the precoding matrix using chordal distances.

- Step 1: Generate candidate reference codebooks $\{\mathcal{W}_i^c\}_{i=1}^N$ by deleting the i-th index matrix from $\mathcal{W}$. Thereby the number of candidate reference codebooks is $N = card\,(\mathcal{W})$, where $card(\cdot)$ denotes the cardinality of a set. Let $N^c = card\,(\mathcal{W}_i^c)$.
- Step 2: Select the optimal candidate codebook $\mathcal{W}_{opt}^c$ from the candidate codebooks which has maximized minimum distance between each pair of matrices. This can be expressed as follows

$$d\left(\mathbf{W}_k, \mathbf{W}_l\right) = \sqrt{1 - \left|\mathbf{W}_k^H \mathbf{W}_l\right|^2}, \tag{5}$$

$$\delta\left(\mathcal{W}_i^c\right) = \min_{(\mathbf{W}_k, \mathbf{W}_l) \in \mathcal{W}_i^c} d\left(\mathbf{W}_k, \mathbf{W}_l\right), \tag{6}$$

and

$$\mathcal{W}_{opt}^c = \underset{\mathcal{W}_i^c \in \{\mathcal{W}_i^c\}_{i=1}^N}{\arg\max}\ \delta\left(\mathcal{W}_i^c\right). \tag{7}$$

- Step 3: If N^c is not equal to P, let $\mathcal{W} = \mathcal{W}_{opt}^c$ which is previously selected. And repeat step 1 and step 2 until $N^c = P$, as shown in Fig. 1.
- Step 4: The deleted precoding matrices (in step 1) are assigned to its nearest centroid by using the chordal distance as in (5). This procedure generates new groups which only have one centroid in each of them, as shown in Fig. 2, where c_p, $p = 1,\ 2,\ \ldots,\ P$ are the centroid indices.

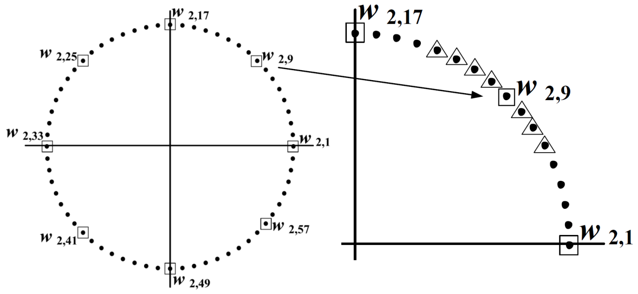

Figure 3. Example of QEGT codebook grouping when N_t=2, N=64 and M=1.

2) Lloyd's algorithm: To illustrate the grouping strategy with Lloyd's algorithm, we consider the case of N_t=2, N=64 and M=1. The 1st element of the TxBF vectors corresponding to the 1st transmit antenna can be forced to be real-valued owing to the rotation-invariant property of BF vectors [4]. Then, the 2nd element of the TxBF vectors corresponding to the 2nd transmit antenna can be plotted on the complex-valued plane, as seen in Fig. 3, which illustrates eight groups of TxBF vectors as the result of Lloyds clustering algorithm. In the notation of $\mathbf{w}_{i,j}$ seen in Fig. 3, i represents the i-th element of the TxBF vector corresponding to the i-th transmit antenna and j denotes the index of the TxBF vector in the QEGT codebook. Fig. 3 also magnifies a group of eight TxBF vectors at the right, where the cluster centroid is indicated by a square mark (□), while the others are indicated by triangles (△). In the same way, we can group the precoding matrices in the QEGT codebook, when the number of transmit antennas is higher than or equal to three. The more detailed grouping strategy using Lloyd's algorithm has been well documented in [25].

B. Group index selection criterion for QEGT codebook

Among the various centroids, the best of the centroids c_{opt} is selected, which maximized the receive SNR

$$c_{opt} = \underset{c_p \in C_{set}}{\arg\max} \left\|\mathbf{H}_m \mathbf{W}_{c_p}\right\|_F, \tag{8}$$

or channel capacity

$$c_{opt} = \underset{c_p \in C_{set}}{\arg\max} \log_2 \det\left(\mathbf{I}_M + \frac{\rho}{M} \mathbf{W}_{c_p}^H \mathbf{H}_m^H \mathbf{H}_m \mathbf{W}_{c_p}\right), \tag{9}$$

where C_{set} is the set of $\{c_1,\ c_2,\ \ldots,\ c_p\}$. And then, the optimal precoding matrix index m_{opt} is determined within the group of the selected best centroid, which maximized the receive SNR as follows

$$m_{opt} = \underset{m_q \in M_{set}}{\arg\max} \left\|\mathbf{H}_m \mathbf{W}_{m_q}\right\|_F, \tag{10}$$

or channel capacity

$$m_{opt} = \underset{m_q \in M_{set}}{\arg\max} \log_2 \det\left(\mathbf{I}_M + \frac{\rho}{M} \mathbf{W}_{m_q}^H \mathbf{H}_m^H \mathbf{H}_m \mathbf{W}_{m_q}\right), \tag{11}$$

where m_q, $(q = 1,\ 2,\ \ldots,\ Q)$ represents the precoding matrices of the chosen centroid group and M_{set} is the set of $\{m_1,\ m_2,\ \ldots,\ m_q\}$.

From the above process, proposed scheme needs only the search of $P+Q$ indices within QEGT codebook. It is much smaller than the number of full index search algorithm in (3), (4). For this reason, the entire search complexity is significantly reduced. Also, considering the highly temporally correlated fading channels, the group of the selected centroid as in (8), (9) almost constants. Therefore the entire search complexity is more reduced by researching only the Q indices.

The problem now becomes how to determine the number of groups, P and the number of elements Q in the group, when QEGT codebook size N is fixed. The proposed grouping rule can be applied that the number of groups, P should be the same or the nearest integer to the value of N/P when the precoding matrix in the QEGT codebook are distributed almost evenly to each group.

IV. Simulation Results

In this section, we perform Monte-Carlo simulations to investigate the achievable throughput performance of the proposed algorithm in MIMO channels. The CL-MIMO system is equipped with N_t transmit antennas, and N_r receive antennas, we use the notation $N_t \times N_r$ to denote N_t transmit and N_r receive antenna system. In order to show the performance in vehicular environments, simulation parameters employed in IEEE 802.11p/WAVE standard [26] are used. In IEEE 802.11p/WAVE, the TxBF system assumes mobile speed 50km/h, carrier frequency of 5.9GHz, bandwidth of 10MHz, and feedback interval of 5ms, thus, typical time correlation coefficient is ϵ=0.020. In our systems, only low mobile speed is supported (less than 60km/h), this is supported in 3GPP LTE and IEEE 802.16p standards. And adaptive modulation coding is assumed for orthogonal frequency division multiplexing system whose DFT/IDFT size is 512. The simulations were run with over 1.5 million iterations per SNR point. Codebook for the QEGT system was designed based on Grassmannian beamforming criterion [5]. The entire codebook size for 3 transmit antennas is 16, and the entire codebook size for 4 and 5 transmit antennas is 64. The maximum capacity selection criterion is used for the optimal precoding matrix selection. The feedback channel is assumed to be error free. Also, we assumed that the channel estimation and synchronization are perfect and there is no spatial correlation amongst transmit and receive antennas.

The achievable throughput performance of the proposed and the full index search ones which have 3×2, 4×3 and 5×4 systems are shown in Fig. 4. We assume P is 4 and 8 for 3 transmit antennas and 4 or 5 transmit antennas, respectively. Fig. 4 shows the performance of proposed algorithm, the proposed algorithm (curve label ‘proposed algorithm’) performs almost the same as those

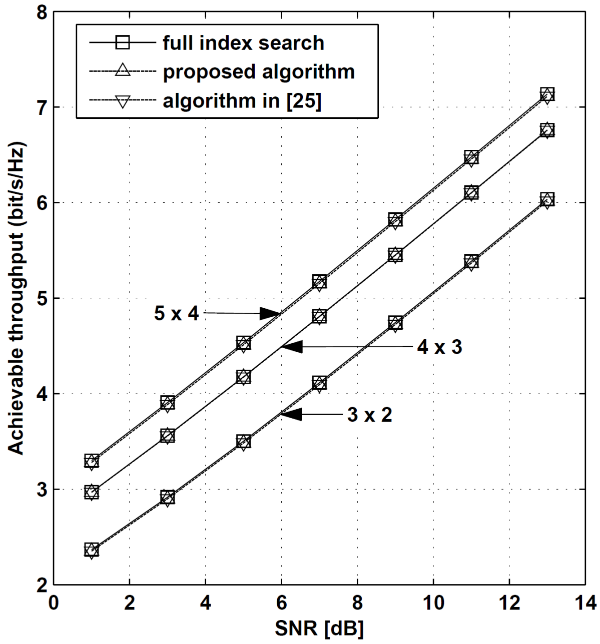

Figure 4. Throughput versus SNR when mobile speed is 50km/h.

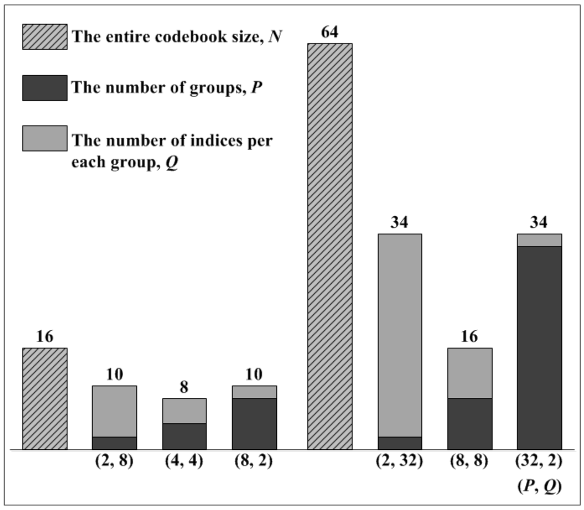

Figure 5. Comparison of the calculated indices for QEGT codebook both the full index search algorithm and the proposed algorithm.

of the approach in [25] (curve label ‘algorithm in [25]’). And those performance almost achieve the full index search algorithm (curve label ‘full index search’).

Fig. 5 shows how much the QEGT codebook index search complexity is reduced in comparison to the full index search algorithm. When the proposed codebook index search algorithm using group strategy based on Grassmannian beamforming criterion is applied, $i.e.$, we have $(N,\ P,\ Q) = (16,\ 4,\ 4)$ or $(64,\ 8,\ 8)$, we have almost the same performance comparing to the full index search ones, while maintaining the lowest codebook index search

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complexity. From the results of Fig. 5, the optimal grouping is that the value of P should be nearly the same at the value of Q. In other words, the value of Q may vary around the nearest integer of N/P. Thereby, the codebook index search complexity is halved, whilst maintaining almost the same throughput when the number of transmit antennas is more than three.

V. Conclusion

In this paper, we have investigated the codebook index search problem for CL-MIMO systems. A complexity reduced index search algorithm for QEGT codebook is proposed which uses grouping strategy based on Grassmannian beamforming criterion. From the simulation results, as the QEGT codebook size increases, the QEGT codebook index search complexity of the proposed algorithms were significantly decreased comparing with that of the full index search algorithm. Moreover, the achievable throughput performance of our proposed algorithm were almost the same as those of the existing full index approaches.

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