New Method for Representation of Multi-qubit Systems Using Fractals

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Abstract—**Visual representation is essential to share ideas, interpret previous achievements or formulate new algorithms quickly and intuitively, however most representations of multy-qubit systems either conceal the properties of individual qubits or fail to visualize entanglement. This study discusses a representation that overcomes these problems through the methodology of fractals. The proposed method visualizes individual qubits as constants of the fractal that corresponds to the whole system. The statistical self similarity allows the total number of qubits to be flexible, making it easy to study subsystems. Generalization of this method through labeled signed binary trees is also presented which makes it possible to create other representations with similar properties.**

Keywords - Quantum information; representation; visualization; fractals; binary trees

I. INTRODUCTION

Quantum informatics and communications already promises applications that outperform classical solutions, e.g. Shor's prime factorization [1], the unconditional security of quantum cryptography [2], or practical realization of quantum communication [3]. It is also likely that this discipline will become even more important during the up-coming years. However, quantum mechanics is well-known for its counterintuitive nature that is hard to visualize thus making it problematic to quickly share ideas, interpret previous achievements, or formulate new algorithms quickly and intuitively.

In order to be able to solve these issues, a visual representation could be useful. The Bloch-sphere sufficiently represents one qubit [4] [5], or more qubits that are separable, but entanglement – one of the most important phenomena in quantum informatics – eludes this type of visualization.

Another possible approach is to use objects that have enough degree of freedom to represent the whole system. However this method usually conceals the inner structure, and does not give us an idea of what happens if we measure the state of few qubits instead of the whole system (a method used in many algorithms and protocols). This approach does not handle well those cases where the addition of more qubits is decided or when dividing the system into smaller parts.

There are also methods to generalize the Bloch-sphere through the mathematical structure called Hopf-Fibrations [6], but the arising geometrical structures are vastly complex and hard to read, thus making the method useless as a visualization technique.

An ideal visualization scheme would preserve the mathematical structure of a multi-qubit system in a way that is easy to interpret by the naked eye using compact and two dimensional images. The ideal solution should also give at least some insight to the states of single qubits, would work for any finite number of qubits, as well as it should work show entanglement. This study aims to give an example of such a scheme based on fractals and a generalization rule to construct other schemes alike.

This paper is organized as follows: Section I introduces visualization of quantum states. Section II presents the new proposed approach using fractals. Section III generalizes this approach through labeled signed binary trees. Finally, Section IV concludes the paper.

II. FRACTAL-BASED REPRESENTATION

For the sake of clarity, the cases of single and multiple qubits should be presented separately.

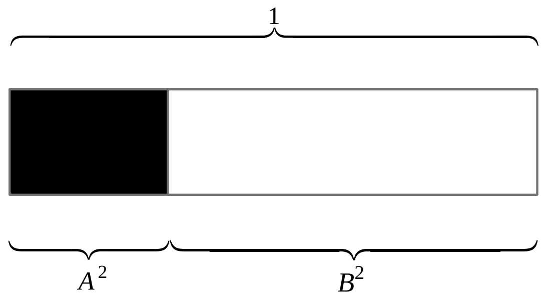

Figure 1. Representation of a single qubit (without phase). The respective lengths A^2 and B^2 of the black and white sides of the bar correspond to the probability of a measurement on the qubit yielding the bit value 0 or 1.

A. Single qubits

Let us write the probability amplitudes in an exponential form:

$$|\varphi\rangle = A \cdot \exp(i \cdot \alpha) \cdot |0\rangle + B \cdot \exp(i \cdot \beta) \cdot |1\rangle, \quad (1)$$

$$A^2 + B^2 = 1. \quad (2)$$

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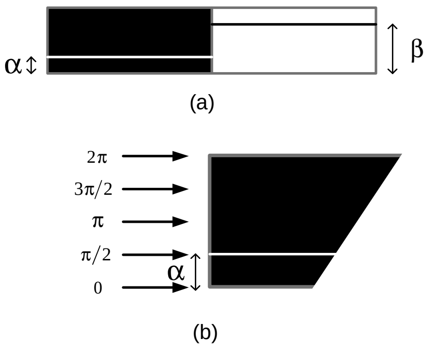

Figure 2. Representing the phase of a single qubit. (a) The vertical position of the lines on each part of the bar represents the phase. (b) An enlarged version of the bar's left side shows the phase. The bottom of the bar corresponds to 0 and the top to 2π.

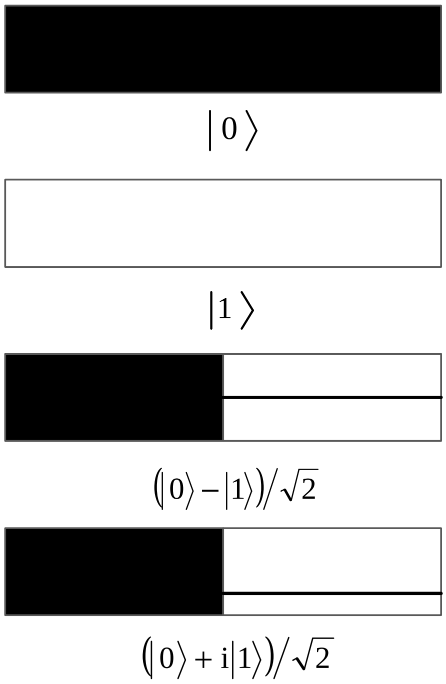

Figure 3. A few simple example of single qubits.

Let us draw a horizontal bar and then using a vertical gray line divide it into a black and a white side with respective lengths of A^2 and B^2 where the total length of the stripe is considered 1. This should give the probabilities of a measurement on the qubit producing the value 0 or 1. To avoid ambiguity, the black part of the bar corresponding to the measurement yielding 0 should always be placed first, and the white part corresponding to the measurement value 1 should placed second, thus representing them in ascending order.

A gray frame should also be added to the bar so that the white part can be easily seen in front of a white background.

To visualize the phase let's draw horizontal lines on the black and white parts of the bar, each with the opposite color (black on white and white on black), in a way, that the vertical position of the line represents α and β (the bottom of each stripe should correspond to 0 and the top to 2π).

B. Multiple qubits

To visualize multiple qubits statistically self-similar fractals should be constructed using the bars described in the previous section. It is well known that the complexity of a quantum system grows quickly as function of the number of qubits, but the complexity of a fractal can match this growth: every newly added qubit means a further iteration step in constructing the fractal representing our quantum system.

Therefore, at first, the qubits should be numbered in the order they will be measured at the end of our protocol. Multi qubit systems whose qubits are separable and whose are not should also be distinguished.

For separable qubits the representation of the whole system can be done, by simply copying the scaled down version of the bar corresponding to the subsequent qubit, under each parts of the previous bars.

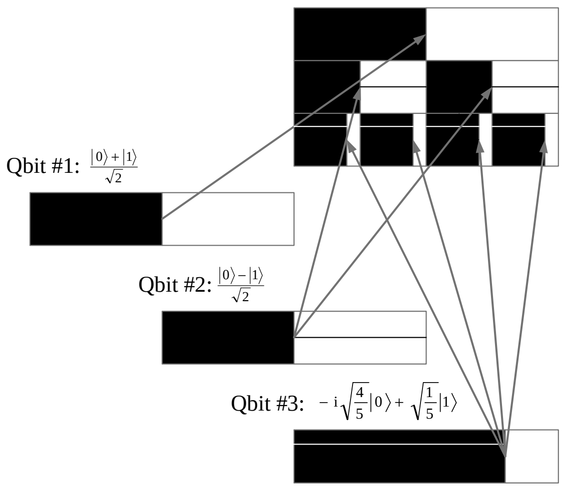

Figure 4. Fractal representation of a multi-qubit system and the separable qubits that serve as it's building blocks. Note that although the fractal displays stochastic attributes this is due to the varying building blocks that are not influenced by the construction process (which is deterministic).

In terms of the Lindenmayer system the representation of separable qubits can be formulated as follows:

Let the nth qubit be:

$$|\varphi_n\rangle = A_n \cdot \exp(i \cdot \alpha_n) \cdot |0\rangle + B_n \cdot \exp(i \cdot \beta_n) \cdot |1\rangle . \qquad (3)$$

Let us denote the bar representing this qubit by $C_n(w)$ where w stands for the total width of the bar. Let the constants of the fractals be the $C_n(w)$ bars as described above, except for the bottom part of the gray frame on each (black and white) sides. Let the variables be the bottom parts of the gray frame denoted by D_n^j with respective lengths of L_n^j where the index j runs from 1 to 2^n (odd indexes corresponding to the bottom parts of black and even indexes to the white sides of each bars).

Let the initiator be a straight horizontal gray line D_0^0, with the length of $L_0^0 = 1$, and the production rule be $\left(D_n^j \rightarrow C_{n+1}\left(L_n^j\right)\right)$ for all j.

Sticking to the convention that the black side is followed by the white on each bar, the widths on the resulting figure from left to right will give us the probabilities of measuring a bit string in ascending order.

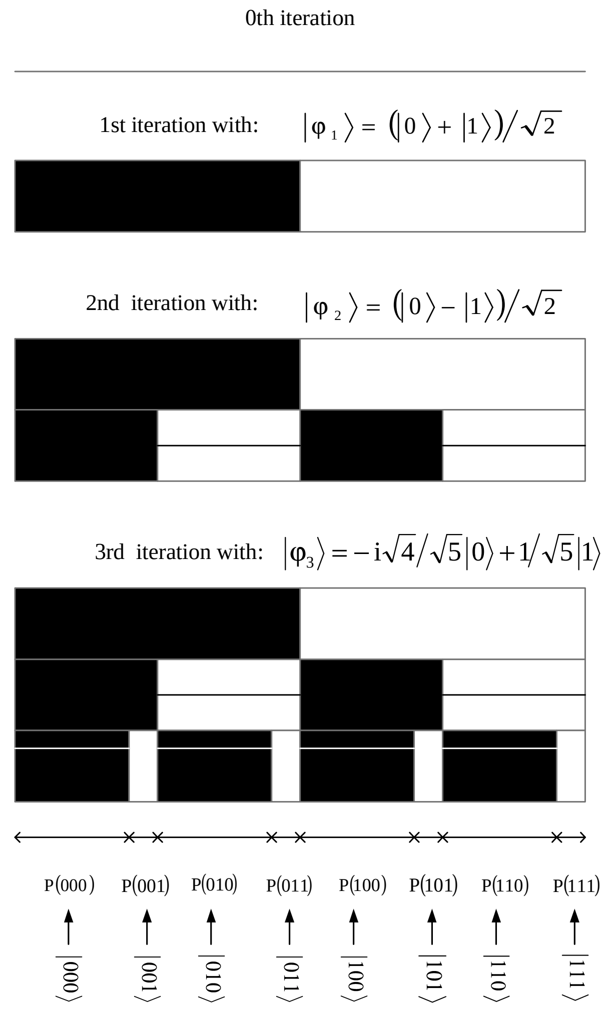

Figure 5. Iterational steps of the fractal representation. The width of the lowermost part of the bars represents the probability of a measurement yielding the value represented by the colors of the parts above it. (Read from top to bottom black means 0, white means 1.) To identify the individual qubits, compare with Figure 4.

However, there is another way of representing the phase. Sometimes, it is more useful to draw figures with all the phase information in the bars of the last iteration. This kind of representation will be equivalent to the description of the whole system using bracket formalism. If the state of the multi-qubit system is:

$$\begin{aligned} |\phi\rangle &= A_{00\ldots0} \cdot \exp(i \cdot \alpha_{00\ldots0}) \cdot |00\ldots0\rangle \\ &+ A_{00\ldots1} \cdot \exp(i \cdot \alpha_{00\ldots1}) \cdot |00\ldots1\rangle + \ldots \\ &+ A_{11\ldots1} \cdot \exp(i \cdot \alpha_{11\ldots1}) \cdot |11\ldots1\rangle, \end{aligned} \quad (4)$$

then the representation with this inherited phase can be drawn as follows. Let the black and white parts of the bar read from top to bottom indicate bit values in a certain i^{th} term of the sum (i being a binary number). Let the width of these parts (or to be more precise the width of the undermost part) be A_i^2, where the total width of all the bars is considered 1. Let the phase in this undermost part be represented by the height of a horizontal line of opposite color as described in part A.

$$\begin{aligned} |\phi\rangle &= A_{00\ldots0} \cdot \exp(i \cdot \alpha_{00\ldots0}) \cdot |00\ldots0\rangle \\ &+ A_{00\ldots1} \cdot \exp(i \cdot \alpha_{00\ldots1}) \cdot |00\ldots1\rangle + \ldots \\ &+ A_{11\ldots1} \cdot \exp(i \cdot \alpha_{11\ldots1}) \cdot |11\ldots1\rangle \end{aligned}$$

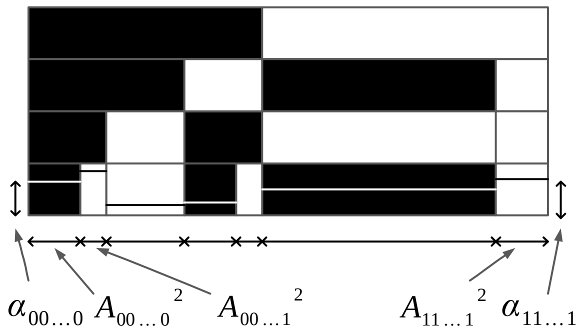

Figure 6. Representation using the collapsed phase. This is equivalent to the bracket description of the whole system. Note that some of the color combinations are missing from the graphs, meaning their width (and detection probability) is zero. One such combination is white, white, white, black (read from top to bottom), meaning the measurement will never yield 1110.

Converting the previous, recursively defined version of the representation to the non-inherited phases can be done by adding all the phases above each other and drawing a horizontal line on the lowermost bar in the height of the sum (where the bar is considered a torus due to 2π periodicity).

This kind of inherited phase can be useful because all the phase information is at the lowermost bars, and not separable states cannot be represented otherwise.

While the inherited phase version is equivalent to the bracket description in a sum form, the non-inherited phase form of separable qubits is equivalent to a tensor product form, and of course many transitional 'mixed' stages are possible between the two.

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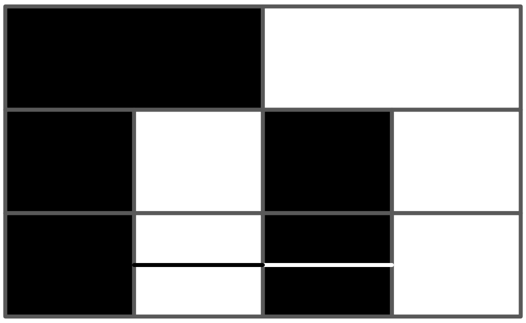
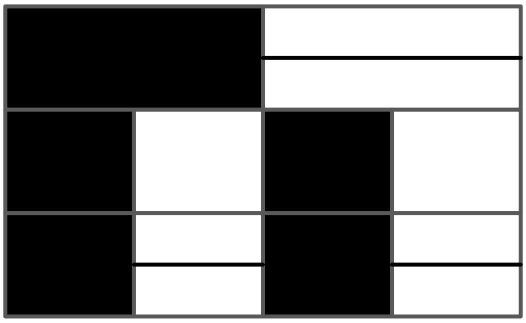
These mixed representations can be constructed with the same iterative process as the non-inherited phase variants, but they use the inherited phase representation as building blocks instead of the bars representing single qubits. (Of course, if a single qubit is separable from the whole system than its inherited phase, non-inherited phase and single bar representation are the same.)

$$\left|\phi\right\rangle = \\ = \frac{\left|000\right\rangle - \left|011\right\rangle - \left|100\right\rangle + \left|111\right\rangle}{2}$$



(a)

$$\left|\phi\right\rangle = \\ = \frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}} \otimes \frac{\left|00\right\rangle - \left|11\right\rangle}{\sqrt{2}}$$



(b)

Figure 7. Possible representations of the same multi-qubit system. (a) In the inherited phase version it is easy to read the phase of the total qubit triplets, making this useful in case when a phase-sensitive measurement or operation is performed on the whole system. (b) The partially inherited phase representation is useful when the first qubit is treated separately from the other two. Note that the inherited phase representation can only be used in case of separable qubits, thus the last two qubits that are entangled, cannot be represented that way, making this mixed representation a 'maximally non-inherited' one.

III. GENERALIZATION THROUGH BINARY TREES

The previously described representation is only one of many possibilities.

To construct another one, first we need to choose a building block that can represent a complex number. It is best to choose the blocks to have one more degree of freedom that can symbolize a binary value. Two blocks are needed to represent a single qubit, although the two can be handled as one object. For example in case of the representation described in Section II, the blocks symbolizing complex numbers are the sides of the bars, the additional degree of freedom is their color, and two of these (with opposite color) are handled as a single bar. Instead of these bars the two complex numbers could be represented using the Bloch-sphere although it is not the best choice since two dimensional representations are easier to interpret, and squared shapes are easier to pack compactly.

To represent multiple qubits a rule is needed that makes it possible to connect two of the building blocks representing complex numbers to each previous block. The resulting structure will be equivalent to a signed, labeled binary tree, where the labels are complex numbers. The signs will correspond to the bit values, although if the blocks representing complex numbers have an additional degree of freedom, it does not have to be visualized through the connection rule.

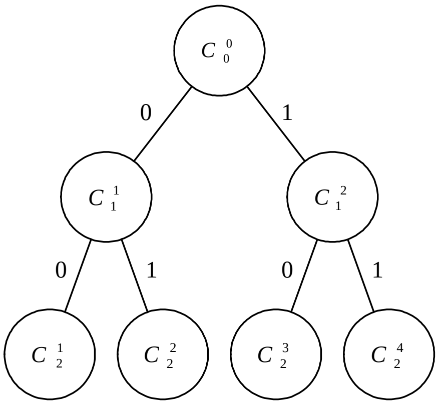


Figure 8. Other representations can be constructed, which are equivalent to a signed labeled binary tree whose labels are complex numbers.

The labels can be calculated either in a fashion when they inherit some of the properties of their ancestors, or in a way when they do not. The later can only be done for separable qubits but can be used to illustrate individual states of single qubits. In this case the n^{th} level of the tree will correspond to the n^{th} qubit, and the label C_n^j will correspond to the probability amplitude of qubit value 0 if j is odd, and 1 if even (where j runs from 1 to 2^n). In the inheriting case the label C_n^j will correspond to the complex number that can be calculated as the product of the probability amplitude described in the non-inheriting case and every other probability amplitude corresponding to the ancestors of that

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particular node. This method is useful for representing the system as a whole, and compresses most of the information in the labels of the leaf nodes.

Mixed representations are also possible, as described in Section II. Note that in the first representation the width (and thus the detection probability) is always inherited while the phase can be inherited, not inherited or mixed, making different attributes of the same complex number behave differently.

It is also worth noting, that the complex number associated with the root should have an absolute value of 1. Although the root can represent the global phase, it is not necessary to describe a multi-qubit system, and as such it was not used in the first representation.

IV. CONCLUSION

As discussed, fractals self-similarity and complexity make them ideal candidates for representing multi-qubit systems. In the present paper the properties of an ideal visualization were described, giving an example of such a representation in detail. This representation was generalized through labeled signed binary trees. Even though the two approaches might seem different, these binary trees can be constructed recursively using the methodology of fractals.

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REFERENCES

[1] P. W. Shor, "Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer", SIAM J. Comput. 26 (5): 1484–1509 (1997)

[2] C. H. Bennett and G. Brassard, “Quantum cryptography: Public key distribution and coin tossing”, Proc. IEEE International Conference on Comput., Systems and Signal Process. 1984, pp. 175–179

[3] L. Bacsardi “Satellite Communication Over Quantum Channel”, Acta Astronautica 2007: 61(1–6):151–159

[4] M. Nielsen and I. Chuang, “Quantum Computation and Quantum Information” Cambridge, England: Cambridge University Press, 2000

[5] S. Imre and B. Ferenc, “Quantum Computing and Communications: An Engineering Approach”, Wiley, 2005

[6] D. Chruscinski, “Geometric Aspects of Quantum Mechanics and Quantum Entanglement”, Journal of Physics: Conference Series 30 (2006) 9–16