

# Combinatorial Structures of Quantum Entangled States

Hoshang Heydari  
 Physics Department  
 Stockholm University  
 Stockholm, Sweden  
 Email: hoshang@fysik.su.se

**Abstract**—We investigate the combinatorial structures of multipartite quantum systems based on algebraic varieties. In particular, we study the relations between resolution of conifold, toric varieties, separable states, and quantum entangled states. We show that the resolved or deformed conifold is equivalent with the space of a pure entangled two-qubit state. We also generalize this result into multi-qubit states. The results give new insight about multipartite systems and also a new way of representing quantum entangled multipartite systems and quantum operations with potential applications in quantum computing.

**Keywords**—Quantum entanglement, multipartite quantum systems, quantum information.

## I. INTRODUCTION

Pure quantum states are usually defined on complex Hilbert spaces which are very complicated to visualize. The simplest case, namely, the space of a single qubit state can be visualized with Bloch or Riemann sphere. Beyond that there have been little progresses to visualize quantum state. Recently, we have established a relation between quantum states and toric varieties. Based on such a construction or mapping it is possible to visualize the complex Hilbert space by lattice polytop.

In algebraic geometry [1], a conifold is a generalization of the notion of a manifold. But, a conifold can contain conical singularities, e.g., points whose neighborhood look like a cone with a certain base. The base is usually a five-dimensional manifold. However, the base of a complex conifold is a product of one dimensional complex projective space. Conifold are interesting space in string theory, e.g., in the process of compactification of Calabi-Yau manifolds. A Calabi-Yau manifold is a compact Kähler manifold with a vanishing first Chern class. A Calabi-Yau manifold can also be defined as a compact Ricci-flat Kähler manifold.

During recent decade toric varieties have been constructed in different contexts in mathematics [2], [3], [4]. A toric variety  $\mathbf{X}$  is a complex variety that contains an algebraic torus  $T = (\mathbf{C}^*)^n$  as a dense open set and with action of  $T$  on  $\mathbf{X}$  whose restriction to  $T \subset \mathbf{X}$  is the usual multiplication on  $T$ .

In this paper, we establish relations between toric varieties and space of entangled states of bipartite and multipartite quantum systems. In particular, we discuss resolving the

singularity and deformation of conifold and toric variety of the conifold. We show that by removing the singularity of conifold we get a space which is not anymore toric variety but it is the space of an entangled two-qubit state. We also investigate the combinatorial structure of multi-qubit systems based on deformation of each faces of cube (hypercube) which is equivalent to deformation of conifold. In particular, in section II we give a short introduction to conifold. In section III we review the construction of toric variety. In section IV and V we investigate conifold and resolution of toric singularity for two-qubits and three qubits states. Finally, in section VI we generalize our results to multi-qubits states. Through this paper we will use the following notation

$$|\Psi\rangle = \sum_{x_m=0}^1 \sum_{x_{m-1}=0}^1 \cdots \sum_{x_1=0}^1 \alpha_{x_m x_{m-1} \cdots x_1} |x_m x_{m-1} \cdots x_1\rangle, \quad (1)$$

with  $|x_m x_{m-1} \cdots x_1\rangle = |x_m\rangle \otimes |x_{m-1}\rangle \otimes \cdots \otimes |x_1\rangle \in \mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$  for a pure multi-qubit state, where  $\mathcal{H}_{\mathcal{Q}_j}$  is the Hilbert space of  $j$ th subsystem. Our reviewer also has pointed out that there are other ways of visualization of entangled states, especially, recently proposed visualization with the aid of classical random fields [5], [6].

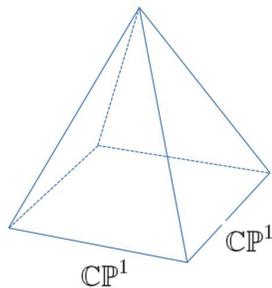
## II. CONIFOLD

In this section we will give a short review of conifold. Let  $\mathbf{C}$  be a complex algebraic field. Then, an affine  $n$ -space over  $\mathbf{C}$  denoted  $\mathbf{C}^n$  is the set of all  $n$ -tuples of elements of  $\mathbf{C}$ . An element  $P \in \mathbf{C}^n$  is called a point of  $\mathbf{C}^n$  and if  $P = (a_1, a_2, \dots, a_n)$  with  $a_j \in \mathbf{C}$ , then  $a_j$  is called the coordinates of  $P$ . A complex projective space  $\mathbf{P}_{\mathbf{C}}^n$  is defined to be the set of lines through the origin in  $\mathbf{C}^{n+1}$ , that is,

$$\mathbf{P}_{\mathbf{C}}^n = \frac{\mathbf{C}^{n+1} - \{0\}}{(x_0, \dots, x_n) \sim (y_0, \dots, y_n)}, \quad \lambda \in \mathbf{C} - 0, \quad y_i = \lambda x_i \quad (2)$$

for all  $0 \leq i \leq n$ . An example of real (complex) affine variety is conifold which is defined by

$$\mathcal{V}_{\mathbf{C}}(z) = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : \sum_{i=1}^4 z_i^2 = 0\}. \quad (3)$$


 Figure 1. Complex cone over  $\mathbf{CP}^1 \times \mathbf{CP}^1$ .

Conifold as a real affine variety is define by

$$\begin{aligned} \mathcal{V}_{\mathbf{R}}(f_1, f_2) &= \{(x_1, \dots, x_4, y_1, \dots, y_4) \in \mathbf{R}^8 : \sum_{i=1}^4 x_i^2 = \\ &= \sum_{j=1}^4 y_j^2, \sum_{i=1}^4 x_i y_i = 0\}. \end{aligned} \quad (4)$$

where  $f_1 = \sum_{i=1}^4 (x_i^2 - y_i^2)$  and  $f_2 = \sum_{i=1}^4 x_i y_i$ . This can be seen by defining  $z = x + iy$  and identifying imaginary and real part of equation  $\sum_{i=1}^4 z_i^2 = 0$ . As a real space, the conifold is cone in  $\mathbf{R}^8$  with top the origin and base space the compact manifold  $\mathbf{S}^2 \times \mathbf{S}^3$ . One can reformulate this relation in term of a theorem. The conifold  $\mathcal{V}_{\mathbf{C}}(\sum_{i=1}^4 z_i^2)$  is the complex cone over the Segre variety  $\mathbf{CP}^1 \times \mathbf{CP}^1 \rightarrow \mathbf{CP}^3$ . To see this let us make a complex linear change of coordinate

$$\begin{pmatrix} \alpha'_{00} & \alpha'_{01} \\ \alpha'_{10} & \alpha'_{11} \end{pmatrix} \rightarrow \begin{pmatrix} z_1 + iz_2 & -z_4 + iz_3 \\ z_4 + iz_3 & z_1 - iz_2 \end{pmatrix}. \quad (5)$$

Thus after this linear coordinate transformation we have

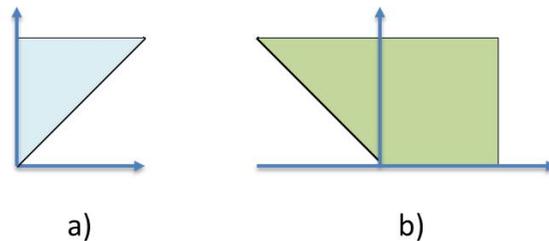
$$\mathcal{V}_{\mathbf{C}}(\alpha'_{00}\alpha'_{11} - \alpha'_{01}\alpha'_{10}) = \mathcal{V}_{\mathbf{C}}(\sum_{i=1}^4 z_i^2) \subset \mathbf{C}^4. \quad (6)$$

Thus we can think of conifold as a complex cone over  $\mathbf{CP}^1 \times \mathbf{CP}^1$  see Figure 1. We will comeback to this result in section IV where we establish a relation between these varieties, two-qubit state, resolution of singulary, and deformation theory.

### III. TORIC VARIETIES

The construction of toric varieties usually are based on two different branches of mathematics, namely, combinatorial geometry and algebraic geometry. Here, we will review the basic notations and structures of toric varieties [2], [3], [4].

A general toric variety is an irreducible variety  $\mathbf{X}$  that satisfies the following conditions. First of all  $(\mathbf{C}^*)^n$  is a Zariski open subset of  $\mathbf{X}$  and the action of  $(\mathbf{C}^*)^n$  on itself can extend to an action of  $(\mathbf{C}^*)^n$  on the variety  $\mathbf{X}$ . As an example we will show that the complex projective space  $\mathbf{P}^n$  is a toric variety. If  $z_0, z_1, \dots, z_n$  are homogeneous


 Figure 2. Example of a cone  $\sigma$  a) and its dual  $\sigma^\wedge$  b).

coordinate of  $\mathbf{P}^n$ . Then, the map  $(\mathbf{C}^*)^n \rightarrow \mathbf{P}^n$  is defined by  $(t_1, t_2, \dots, t_n) \mapsto (1, t_1, \dots, t_n)$  and we have

$$(t_1, t_2, \dots, t_n) \cdot (a_0, a_1, \dots, a_n) = (a_0, t_1 a_1, \dots, t_n a_n) \quad (7)$$

which proof our claim that  $\mathbf{P}^n$  is a toric variety. We can also define toric varieties with combinatorial information such as polytope and fan (which we will define next). But first we will give a short introduction to the basic of combinatorial geometry which is important in definition of toric varieties. Let  $S \subset \mathbf{R}^n$  be finite subset, then a convex polyhedral cone is defined by

$$\sigma = \text{Cone}(S) = \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0 \right\}. \quad (8)$$

In this case  $\sigma$  is generated by  $S$ . In a similar way we define a polytope by

$$P = \text{Conv}(S) = \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \geq 0, \sum_{v \in S} \lambda_v = 1 \right\}. \quad (9)$$

We also could say that  $P$  is convex hull of  $S$ . A convex polyhedral cone is called simplicial if it is generated by linearly independent set. Now, let  $\sigma \subset \mathbf{R}^n$  be a convex polyhedral cone and  $\langle u, v \rangle$  be a natural pairing between  $u \in \mathbf{R}^n$  and  $v \in \mathbf{R}^n$ . Then, the dual cone of the  $\sigma$  is define by

$$\sigma^\wedge = \{u \in \mathbf{R}^{n*} \mid \langle u, v \rangle \geq 0 \forall v \in \sigma\}, \quad (10)$$

where  $\mathbf{R}^{n*}$  is dual of  $\mathbf{R}^n$ . We also define the polar of  $\sigma$  as

$$\sigma^\circ = \{u \in \mathbf{R}^{n*} \mid \langle u, v \rangle \geq -1 \forall v \in \sigma\}. \quad (11)$$

As an example we consider the cone  $\sigma = \text{Cone}(e_1 + e_2, e_2) \subset \mathbf{R}^2$ . In this case the cone  $\sigma$  and its dual are illustrated in Figure 1. We call a convex polyhedral cone strongly convex if  $\sigma \cap (-\sigma) = \{0\}$ .

Next we will define rational polyhedral cones. A free Abelian group of finite rank is called a lattice, e.g.,  $N \simeq \mathbf{Z}^n$ . The dual of a lattice  $N$  is defined by  $M = \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$  which has rank  $n$ . We also define a vector space and its dual by  $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^n$  and  $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R} \simeq \mathbf{R}^{n*}$  respectively. Moreover, if  $\sigma = \text{Cone}(S)$  for some finite

set  $S \subset N$ , then  $\sigma \subset N_{\mathbf{R}}$  is a rational polyhedral cone. Furthermore, if  $\sigma \subset N_{\mathbf{R}}$  is a rational polyhedral cone, then  $S_{\sigma} = \sigma^{\wedge} \cap M$  is a semigroup under addition with  $0 \in S_{\sigma}$  as additive identity which is finitely generated by Gordan's lemma [2].

Here we will define a fan which is important in the construction of toric varieties. Let  $\Sigma \subset N_{\mathbf{R}}$  be a finite non-empty set of strongly convex rational polyhedral cones. Then  $\Sigma$  is called a fan if each face of a cone in  $\Sigma$  belongs to  $\Sigma$  and the intersection of any two cones in  $\Sigma$  is a face of each.

Now, we can obtain the coordinate ring of a variety by associating to the semigroup  $S$  a finitely generated commutative  $\mathbf{C}$ -algebra without nilpotent as follows. We associate to an arbitrary additive semigroup its semigroup algebra  $\mathbf{C}[S]$  which as a vector space has the set  $S$  as basis. The elements of  $\mathbf{C}[S]$  are linear combinations  $\sum_{u \in S} a_u \chi^u$  and the product in  $\mathbf{C}[S]$  is determined by the addition in  $S$  using  $\chi^u \chi^{u'} = \chi^{u+u'}$  which is called the exponential rule. Moreover, a set of semigroup generators  $\{u_i : i \in I\}$  for  $S$  gives algebra generators  $\{\chi^{u_i} : i \in I\}$  for  $\mathbf{C}[S]$ .

Now, let  $\sigma \subset N_{\mathbf{R}}$  be a strongly convex rational polyhedral cone and  $A_{\sigma} = \mathbf{C}[S_{\sigma}]$  be an algebra which is a normal domain. Then,

$$\mathbf{X}_{\sigma} = \text{Spec}(\mathbf{C}[S_{\sigma}]) = \text{Spec}(A_{\sigma}) \quad (12)$$

is called a affine toric variety. Next we need to define Laurent polynomials and monomial algebras. But first we observe that the dual cone  $\sigma^{\vee}$  of the zero cone  $\{0\} \subset N_{\mathbf{R}}$  is all of  $M_{\mathbf{R}}$  and the associated semigroup  $S_{\sigma}$  is the group  $M \simeq \mathbf{Z}^n$ . Moreover, let  $(e_1, e_2, \dots, e_n)$  be a basis of  $N$  and  $(e_1^*, e_2^*, \dots, e_n^*)$  be its dual basis for  $M$ . Then, the elements  $\pm e_1^*, \pm e_2^*, \dots, \pm e_n^*$  generate  $M$  as semigroup. The algebra of Laurent polynomials is defined by

$$\mathbf{C}[z, z^{-1}] = \mathbf{C}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}], \quad (13)$$

where  $z_i = \chi^{e_i^*}$ . The terms of the form  $\lambda \cdot z^{\beta} = \lambda z_1^{\beta_1} z_2^{\beta_2} \dots z_n^{\beta_n}$  for  $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbf{Z}$  and  $\lambda \in \mathbf{C}^*$  are called Laurent monomials. A ring  $R$  of Laurent polynomials is called a monomial algebra if it is a  $\mathbf{C}$ -algebra generated by Laurent monomials. Moreover, for a lattice cone  $\sigma$ , the ring  $R_{\sigma} = \{f \in \mathbf{C}[z, z^{-1}] : \text{supp}(f) \subset \sigma\}$  is a finitely generated monomial algebra, where the support of a Laurent polynomial  $f = \sum \lambda_i z^i$  is defined by  $\text{supp}(f) = \{i \in \mathbf{Z}^n : \lambda_i \neq 0\}$ . Now, for a lattice cone  $\sigma$  we can define an affine toric variety to be the maximal spectrum  $\mathbf{X}_{\sigma} = \text{Spec} R_{\sigma}$ . A toric variety  $\mathbf{X}_{\Sigma}$  associated to a fan  $\Sigma$  is the result of gluing affine varieties  $\mathbf{X}_{\sigma} = \text{Spec} R_{\sigma}$  for all  $\sigma \in \Sigma$  by identifying  $\mathbf{X}_{\sigma}$  with the corresponding Zariski open subset in  $\mathbf{X}_{\sigma'}$  if  $\sigma$  is a face of  $\sigma'$ . That is, first we take the disjoint union of all affine toric varieties  $\mathbf{X}_{\sigma}$  corresponding to the cones of  $\Sigma$ . Then by gluing all these affine toric varieties together we get  $\mathbf{X}_{\Sigma}$ . A affine toric variety  $\mathbf{X}_{\sigma}$  is non-singular if and only if the normal polytope has a unit volume.

#### IV. CONIFOLD AND RESOLUTION OF TORIC SINGULARITY FOR TWO-QUBITS

In this section we study the simplicial decomposition of affine toric variety. For two qubits this simplicial decomposition coincides with desingularizing a conifold [8]. We also show that resolved conifold is space of an entangled two-qubit state. For a pairs of qubits  $|\Psi\rangle = \sum_{x_2=0}^1 \sum_{x_1=0}^1 \alpha_{x_2 x_1} |x_2 x_1\rangle$  we can also construct following simplex. For this two qubit state the separable state is given by the Segre embedding of  $\mathbf{CP}^1 \times \mathbf{CP}^1 = \{((\alpha_0^1, \alpha_1^1), (\alpha_0^2, \alpha_1^2)) : (\alpha_0^1, \alpha_1^1) \neq 0, (\alpha_0^2, \alpha_1^2) \neq 0\}$ . Let  $z_1 = \alpha_1^1 (\alpha_0^1)^{-1}$  and  $z_2 = \alpha_1^2 (\alpha_0^2)^{-1}$ . Then we can cover  $\mathbf{CP}^1 \times \mathbf{CP}^1$  by four charts

$$\mathbf{X}_{\Delta_1} = \{(z_1, z_2)\}, \quad \mathbf{X}_{\Delta_2} = \{(z_1^{-1}, z_2)\}, \quad (14)$$

$$\mathbf{X}_{\Delta_3} = \{(z_1, z_2^{-1})\}, \quad \mathbf{X}_{\Delta_4} = \{(z_1^{-1}, z_2^{-1})\}, \quad (15)$$

The fan  $\Sigma$  for  $\mathbf{CP}^1 \times \mathbf{CP}^1$  has edges spanned by  $(1, 0), (0, 1), (-1, 0), (0, -1)$ . Next we observe that the space  $\mathbf{CP}^1 \times \mathbf{CP}^1$  and the conifold have the same toric variety. If we split the conifold into a fan which has two cones as shown in Figure 3. Then this process converts the conifold into a resolved conifold. The cones are three dimensional and the dual cones are two copies of  $\mathbf{C}^3$ . The procedure of replacing an isolated singularity by a holomorphic cycle is called a resolution of the singularity. We can also remove the singularity by deformation. The process of deformation modifies the complex structure manifolds or algebraic varieties. Based on our discussion of conifold we know that this space is defined by  $\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}$ . Now, if we rewrite this equation in the following form

$$\alpha_{00}\alpha_{11} - \Gamma\alpha_{01}\alpha_{10} + \Lambda\alpha_{10} = 0, \quad (16)$$

then the constant  $\Gamma$  and  $\Lambda$  can be absorbed in new definition of  $\alpha_{10}$  such as  $\alpha'_{10} = \Gamma\alpha_{10} - \Lambda$ . Next let  $T_n$  be the group of translations. Then an affine variety over complex field of dimension  $n$  can be transformed using the following action  $GL(n, \mathbf{C}) \times T_n$ . For a generic polynomial of degree two we have 15 possible parameters, but most of them can be removed with the action of  $GL(4, \mathbf{C}) \times T_4$ . However, we cannot remove the constant term with such transformation and we end up with the following variety

$$\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10} = \Omega. \quad (17)$$

which is called deformed conifold. This space is now non-singular, but it is not a toric variety since the deformation break one action of torus. Thus we also could proposed that the deformed conifold is the space of an entangled pure two-qubit state. Moreover, if we take the absolute value of this equation that is  $|\Omega|$ , then this value is proportional to concurrence which is a measure of entanglement for a pure two-qubit state, that is

$$|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}| = |\Omega| = C(\Psi)/2. \quad (18)$$

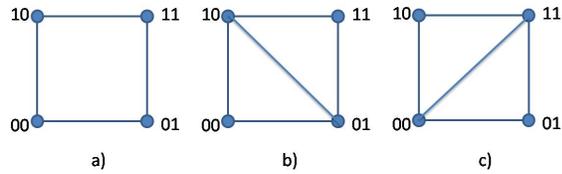


Figure 3. Two-qubit system. a) toric polytope of a two-qubit systems. b) and c) two ways of removing the singularity of conifold.

In general let  $X$  be an algebraic variety, then the space of all complex deformations of  $X$  is called the complex moduli space of  $X$ .

### V. THREE-QUBIT STATES

Next, we will discuss a three-qubit state  $|\Psi\rangle = \sum_{x_3, x_2, x_1=0}^1 \alpha_{x_3 x_2 x_1} |x_3 x_2 x_1\rangle$ . For this state the separable state is given by the Segre embedding of  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1 = \{((\alpha_0^1, \alpha_1^1), (\alpha_0^2, \alpha_1^2), (\alpha_0^3, \alpha_1^3)) : (\alpha_0^1, \alpha_1^1) \neq 0, (\alpha_0^2, \alpha_1^2) \neq 0, (\alpha_0^3, \alpha_1^3) \neq 0\}$ . Now, for example, let  $z_1 = \alpha_1^1/\alpha_0^1$ ,  $z_2 = \alpha_1^2/\alpha_0^2$ , and  $z_3 = \alpha_1^3/\alpha_0^3$ . Then we can cover  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$  by eight charts

$$\begin{aligned} \mathbf{X}_{\Delta_1} &= \{(z_1, z_2, z_3)\}, & \mathbf{X}_{\Delta_2} &= \{(z_1^{-1}, z_2, z_3)\}, \\ \mathbf{X}_{\Delta_3} &= \{(z_1, z_2^{-1}, z_3)\}, & \mathbf{X}_{\Delta_4} &= \{(z_1, z_2, z_3^{-1})\}, \\ \mathbf{X}_{\Delta_5} &= \{(z_1^{-1}, z_2^{-1}, z_3)\}, & \mathbf{X}_{\Delta_6} &= \{(z_1^{-1}, z_2, z_3^{-1})\}, \\ \mathbf{X}_{\Delta_7} &= \{(z_1, z_2^{-1}, z_3^{-1})\}, & \mathbf{X}_{\Delta_8} &= \{(z_1^{-1}, z_2^{-1}, z_3^{-1})\}, \end{aligned}$$

The fan  $\Sigma$  for  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$  has edges spanned by  $(\pm 1, \pm 1, \pm 1)$ . Now, let  $S = \mathbf{Z}^3$  and consider the polytope  $\Delta$  centered at the origin with vertices  $(\pm 1, \pm 1, \pm 1)$ . This gives the toric variety  $\mathbf{X}_\Delta = \text{SpecC}[S_\Delta]$ . To describe the fan of  $\mathbf{X}_\Delta$ , we observe that the polar  $\Delta^\circ$  is the octahedron with vertices  $\pm e_1, \pm e_2, \pm e_3$ . Thus the normal fan is formed from the faces of the octahedron which gives a fan  $\Sigma$  whose 3-dimensional cones are octants of  $\mathbf{R}^3$ . Thus this shows that the toric variety  $\mathbf{X}_\Sigma = \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ .

In this case we split the faces of 3-cube  $E_{2,3} = 2^{3-2} \frac{3(3-1)}{2} = 6$  into two cones see Figure 4. Then, this process converts the 3-cube into a nonsingular space which is not anymore toric variety. Following the same procedure, we can also remove all singularities of toric variety of three-qubits by deformation. Based on our discussion of conifold we can write six equations describing the faces of 3-cube. Here we will analyze one face of this 3-cube, namely

$$\alpha_{000}\alpha_{011} - \alpha_{001}\alpha_{010} = \alpha_0 \otimes (\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10})$$

Now, if we rewrite these equations e.g., in the following form

$$\alpha_0(\alpha_{00}\alpha_{11} - \Gamma\alpha_{01}\alpha_{10} + \Lambda\alpha_{10}) = 0, \quad (19)$$

then the constant  $\Gamma$  and  $\Lambda$  can be absorbed in new definition of  $\alpha_{10}$  such as  $\alpha'_{10} = \Gamma\alpha_{10} - \Lambda$ . At the end e.g., we have the following variety

$$\alpha_{000}\alpha_{011} - \alpha_{001}\alpha_{010} = \Omega \quad (20)$$

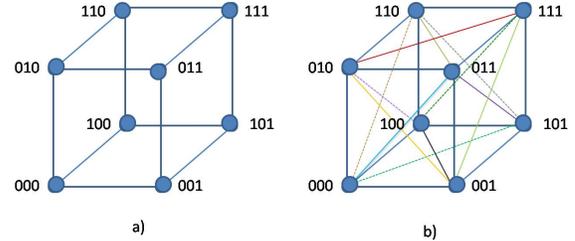


Figure 4. Three-qubit systems. a) toric polytope of a separable three-qubit systems. b) resolved space of entangled state, where each diagonal line is equivalent to the resolution of singularity of a conifold.

which is equivalent to the deformed conifold. If we do this procedure for all faces of the 3-cube, then the whole space becomes non-singular, but it is not a toric variety anymore. Thus we also could proposed that the deformed conifold is the space of an entangled pure three-qubit state. There are other relations between toric variety and measures of quantum entanglement that can be seen from the toric structures of multipartite systems. For example three-angle or 3-hyperdeterminant can be constructed from the toric variety.

### VI. MULTI-QUBIT STATES

Next, we will discuss a multi-qubit state  $|\Psi\rangle$  defined by equation (1). For this state the separable state is given by the Segre embedding of  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \dots \times \mathbf{CP}^1 = \{((\alpha_0^1, \alpha_1^1), (\alpha_0^2, \alpha_1^2), \dots, (\alpha_0^m, \alpha_1^m)) : (\alpha_0^1, \alpha_1^1) \neq 0, (\alpha_0^2, \alpha_1^2) \neq 0, \dots, (\alpha_0^m, \alpha_1^m) \neq 0\}$ . Now, for example, let  $z_1 = \alpha_1^1/\alpha_0^1, z_2 = \alpha_1^2/\alpha_0^2, \dots, z_m = \alpha_1^m/\alpha_0^m$ . Then we can cover  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \dots \times \mathbf{CP}^1$  by  $2^m$  charts

$$\begin{aligned} \mathbf{X}_{\Delta_1} &= \{(z_1, z_2, \dots, z_m)\}, \\ \mathbf{X}_{\Delta_2} &= \{(z_1^{-1}, z_2, \dots, z_m)\}, \\ &\vdots \\ \mathbf{X}_{\Delta_{2^{m-1}}} &= \{(z_1, z_2^{-1}, \dots, z_m^{-1})\}, \\ \mathbf{X}_{\Delta_{2^m}} &= \{(z_1^{-1}, z_2^{-1}, \dots, z_m^{-1})\} \end{aligned}$$

The fan  $\Sigma$  for  $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \dots \times \mathbf{CP}^1$  has edges spanned by  $(\pm 1, \pm 1, \dots, \pm 1)$ . Now, let  $S = \mathbf{Z}^m$  and consider the polytope  $\Delta$  centered at the origin with vertices  $(\pm 1, \pm 1, \dots, \pm 1)$ . This gives the toric variety  $\mathbf{X}_\Delta = \text{SpecC}[S_\Delta]$ . To describe the fan of  $\mathbf{X}_\Delta$ , we observe that the polar  $\Delta^\circ$  is the octahedron with vertices  $\pm e_1, \pm e_2, \dots, \pm e_m$ . Thus this shows that the toric variety  $\mathbf{X}_\Sigma = \mathbf{CP}^1 \times \mathbf{CP}^1 \times \dots \times \mathbf{CP}^1$ . In this case we split the faces of  $m$ -cube

$$E_{2,m} = 2^{m-2} \frac{m(m-1)}{2} \quad (21)$$

into two cones. Then this process converts the  $m$ -cube into a nonsingular space which is not anymore toric variety.

Following the same procedure, we can also remove all singularities of toric variety of a multi-qubit state by deformation. Based on our discussion of conifold we can write six equations describing the faces of  $m$ -cube. For example for one face (2-cube) of this  $m$ -cube, we have

$$\alpha_{00\dots 0}\alpha_{0\dots 011} - \alpha_{0\dots 01}\alpha_{0\dots 010} = \Omega \quad (22)$$

which is equivalent to the deformed conifold, since e.g., we could have  $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\dots 000\rangle + |00\dots 011\rangle) = \frac{1}{\sqrt{2}}|00\dots 0\rangle \otimes (|00\rangle + |11\rangle)$ . If we do this procedure for all faces of the  $m$ -cube, then the whole space becomes non-singular, but it is not a toric variety anymore. Thus we also could propose that this space is the space of an entangled pure multi-qubit state.

## VII. CONCLUSION

In this paper we have investigated the geometrical and combinatorial structures of entangled multipartite systems. We have shown that by removing singularity of conifold or by deforming the conifold we obtain the space of a pure entangled two-qubit state. We have also generalized our construction into multipartite entangled systems. The space of multipartite systems are difficult to visualize but the transformation from complex spaces to the combinatorial one makes this task much easier to realize. Hence our results give new insight about multipartite systems and also a new way of representing quantum entangled bipartite and multipartite systems with many possible applications in the field quantum computing. For example we could visualize action of holonomic quantum gates entangler on multi-qubit states based on these combinatorial structures. However, these issues need further investigations.

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