A gradient-based approach to feedback control of quantum systems

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Abstract—The paper proposes a gradient-based feedback control approach to the stabilization of quantum systems. The spin model is used to define the eigenstates of the quantum system. Using Lindblad's differential equation an estimate of the state of the quantum system is obtained. Moreover, by applying Lyapunov's stability theory and LaSalle's invariance principle a gradient control law is derived which assures that the quantum system's state will track the desirable state within acceptable accuracy levels. The performance of the control loop is studied through simulation experiments for the case of a two-qubit quantum system.

Keywords-gradient-based feedback control, quantum systems, Schrödinger's equation, Lindblad's equation, Lyapunov stability, LaSalle's invariance principle.

I. INTRODUCTION

The main approaches to the control of quantum systems are: (i) open-loop control and (ii) measurement-based feedback control [1]. In open-loop control, the control signal is obtained using prior knowledge about the quantum system dynamics and assuming a model that describes its evolution in time. Some open-loop control schemes for quantum systems have been studied in [2-3]. Previous work on quantum open-loop control includes flatness-based control on a single qubit gate [4]. On the other hand measurement-based quantum feedback control provides more robustness to noise and model uncertainty [5]. In measurement-based quantum feedback control, the overall system dynamics are described by the estimation equation called stochastic master equation or Belavkin's equation [6]. An equivalent approach can be obtained using Lindblad's differential equation [1]. Several researchers have presented results on measurement-based feedback control of quantum systems using the stochastic master equation or the Lindblad differential equation, while theoretical analysis of the stability for the associated control loop has been also attempted in several cases [7-8]. In this paper, a gradient-based approach to the control of quantum systems will be examined. Previous results on control laws which are derived through the calculation of the gradient of an energy function of the quantum system can be found in [9-12]. Convergence properties of gradient algorithms have been associated to Lyapunov stability theory in [13]. The paper considers a quantum system confined in a cavity that is weakly coupled to a probe laser. The spin

model is used to define the eigenstates of the quantum system. The dynamics of the quantum model are described by Lindblad's differential equation and thus an estimate of the system's state can be obtained. Using Lyapunov's stability theory a gradient-based control law is derived. Furthermore, by applying LaSalle's invariance principle it can be assured that under the proposed gradient-based control the quantum system's state will track the desirable state within acceptable accuracy levels. The performance of the control loop is studied through simulation experiments for the case of a two-qubit quantum system.

The structure of the paper is as follows: In Section II the spin eigenstates are used to define a two-level quantum system. In Section III the Lindblad and Belavkin description of the quantum system dynamics are introduced as an analogous to Schrödinger's equation. In Section IV the feedback control approach to quantum system stabilization is explained. A gradient-based feedback control law is derived using Lyapunov stability analysis and LaSalle's invariance principle, both for the case that Schrödinger's equation and Lindblad's equation are used to describe the evolution of the quantum system in time. In Section V simulation tests are given on the performance of the proposed measurement-based feedback control scheme for the case of a two-qubit (four level) quantum system. Finally, in Section VI concluding remarks are stated.

II. THE SPIN AS A TWO-LEVEL QUANTUM SYSTEM

A. Description of a particle in spin coordinates

The basic equation of quantum mechanics is *Schrödinger's equation*, i.e.

$$i\frac{\partial \psi}{\partial t} = H\psi(x,t) \tag{1}$$

where $|\psi(x,t)|^2$ is the probability density function of finding the particle at position x at time instant t, and H is the system's Hamiltonian, i.e. the sum of its kinetic and potential energy, which is given by $H = p^2/2m + V$, with p being the momentum of the particle, m the mass and V an external potential. The solution of Eq. (1) is given by $\psi(x,t) = e^{-iHt}\psi(x,0)$ [14].

However, cartesian coordinates are not sufficient to describe the particle's behavior in a magnetic field and thus the

spin variable taking values in SU(2) has been introduced. In that case the solution ψ of Schrödinger's equation can be represented in the basis $|r, \epsilon >$ where r is the position vector and ϵ is the spin's value which belongs in $\{-\frac{1}{2}, \frac{1}{2}\}$ (fermion). Thus vector ψ which appears in Schrödinger's equation can be decomposed in the vector space $|r, \epsilon >$ according to $|\psi > = \sum_\epsilon \int d^3 r |r, \epsilon >, < r, \epsilon |\psi >$. The projection of $|\psi >$ in the coordinates system r, ϵ is denoted as $< r, \epsilon |\psi > = \psi_\epsilon(r)$. Equivalently one has $\psi_+(r) = < r, +|\psi >$ and $\psi_-(r) = < r, -|\psi >$. Thus one can write $\psi(r) = [\psi_+(r), \psi_-(r)]^T$.

B. Measurement operators in the spin state-space

It has been proven that the eigenvalues of the particle's magnetic moment are $\pm \frac{1}{2}$ or $\pm \hbar \frac{1}{2}$. The corresponding eigenvectors are denoted as $|+ >$ and $|- >$. Then the relation between eigenvectors and eigenvalues is given by $\sigma_z |+ > = +(\hbar/2)|+ >$, $\sigma_z |- > = +(\hbar/2)|- >$, which shows the two possible eigenvalues of the magnetic moment [14]. In general the particle's state, with reference to the spin eigenvectors, is described by

$$|\psi > = \alpha |+ > + \beta |- > \qquad (2)$$

with $|\alpha|^2 + |\beta|^2 = 1$ while matrix σ_z has the eigenvectors $|+ > = [1, 0]$ and $|- > = [0, 1]$ and is given by

$$\sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (3)$$

Similarly, if one assumes components of magnetic moment along axes x and z, one obtains the other two measurement (Pauli) operators

$$\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (4)$$

C. The spin eigenstates define a two-level quantum system

The spin eigenstates correspond to two different energy levels. A neutral particle is considered in a magnetic field of intensity B_z. The particle's magnetic moment M and the associated kinetic moment Γ are collinear and are related to each-other through the relation $M = \gamma \Gamma$. The potential energy of the particle is $W = -M_z B_z = -\gamma B_z \Gamma_z$. Variable $\omega_0 = -\gamma B_z$ is introduced, while parameter Γ_z is substituted by the spin's measurement operator S_z.

Thus the Hamiltonian H which describes the evolution of the spin of the particle due to field B_z becomes $H_0 = \omega_0 S_z$, and the following relations between eigenvectors and eigenvalues are introduced:

$$H|+ > = +\frac{\hbar \omega_0}{2}|+ >, \quad H|- > = +\frac{\hbar \omega_0}{2}|- > \qquad (5)$$

Therefore, one can distinguish 2 different energy levels (states of the quantum system) $E_+ = +\frac{\hbar \omega_0}{2}$, $E_- = -\frac{\hbar \omega_0}{2}$. By applying an external magnetic field the probability of

finding the particle's magnetic moment at one of the two eigenstates (spin up or down) can be changed. This can be observed for instance in the Nuclear Magnetic Resonance (NMR) model and is the objective of quantum control [14].

III. THE LINDBLAD AND BELAVKIN DESCRIPTION OF QUANTUM SYSTEMS

A. The Lindblad description of quantum systems

It will be shown that the Lindblad and the Belavkin equation can be used in place of Schrödinger's equation to describe the dynamics of a quantum system. These equation use as state variable the probability density matrix $\rho = |\psi > < \psi|$, associated to the probability of locating the particle at a certain eigenstate. The Lindblad and Belavkin equations are actually the quantum analogous of the Kushner-Stratonovich stochastic differential equation which denotes that the change of the probability of the state vector x to take a particular value depends on the difference (innovation) between the measurement $y(x)$ and the mean value of the estimation of the measurement $E[y(x)]$. It is also known that the Kushner-Stratonovich SDE can be written in the form of a Langevin SDE [11]

$$dx = \alpha(x)dt + b(x)dv \qquad (6)$$

which finally means that the Lindblad and Belavkin description of a quantum system are a generalization of Langevin's SDE for quantum systems [1]. For a quantum system with state vector x and eigenvalues $\lambda(x) \in R$, the Lindblad equation is written as [1], [15]

$$\hbar \dot\rho = -i[\hat{H}, \rho] + D[\hat{c}]\rho \qquad (7)$$

where ρ is the associated probability density matrix for state x, i.e. it defines the probability to locate the particle at a certain eigenstate of the quantum system and the probabilities of transition to other eigenstates. The variable $\hat{H}$ is the system's Hamiltonian, operator $[A, B]$ is a Lie bracket defined as $[A, B] = AB - BA$, the vector $\hat{c} = (\hat{c}_1, \cdots, \hat{c}_L)^T$ is also a vector of operators, variable D is defined as $D[\hat{c}] = \sum_{l=1}^{L} D[\hat{c}_l]$, and finally $\hbar$ is Planck's constant.

B. The Belavkin description of quantum systems

The Lindblad equation (also known as stochastic master equation), given in Eq. (7), is actually a differential equation which can be also written in the form of a stochastic differential equation that is known as *Belavkin equation*. The most general form of the Belavkin equation is:

$$\hbar d\rho_c = dt D[\hat{c}]\rho_c + H[-i\hat{H}dt + dz^+(t)\hat{c}]\rho_c \qquad (8)$$

Variable H is an operator which is defined as follows

$$H[\hat{r}]\rho = \hat{r}\rho + \rho\hat{r}^+ - Tr[\hat{r}\rho + \rho\hat{r}^+]\rho \qquad (9)$$

Variable $\hat{H}$ stands for the Hamiltonian of the quantum system. Variable $\hat{c}$ is an arbitrary operator obeying $\hat{c}^+\hat{c} = \hat{R}$, where $\hat{R}$ is an hermitian operator. The infinite dimensional complex variables vector dz is defined as $dz = (dz_1, \cdots, dz_L)^T$, and in analogy to the innovation dv of the Langevin equation (see Eq. (6)), variable dz expresses innovation for the quantum case. Variable dz^+ denotes the conjugate-transpose $(dz^*)^T$. The statistical characteristics of dz are $dzdz^+ = \hbar H_c dt$, $dzdz^T = \hbar Y dt$. In the above equations matrix Y is a symmetric complex-valued matrix. Variable H_c is defined as $m_1 = \{H_c = diag(n_1, \cdots, n_L) : \forall l, n_l \in [0, 1]\}$, where n_l can be interpreted as the possibility of monitoring the l-th output channel. There is also a requirement for matrix U to be positive semi-definite. As far as the *measured output* of the Belavkin equation is concerned one has an equation of complex currents $J^T dt = <\hat{c}H_c + \hat{c}^+Y>_c + dz^T$, where $<>$ stands for the mean value of the variable contained in it [1]. Thus, in the description of the quantum system according to Belavkin's formulation, the state equation and the output equation are given by Eq. (10).

$$\hbar d\rho_c = dt D[\hat{c}]\rho_c + H[-i\hat{H}dt + dz^+(t)\hat{c}]\rho_c$$
$$J^T dt = <\hat{c}^T H_c + \hat{c}^+ Y_c> dt + dz^T \quad (10)$$

where ρ_c is the probability density matrix (state variable) for remaining at one of the quantum system eigenstates, and J is the measured output (current).

C. Formulation of the control problem

The control loop consists of a cavity where the multiparticle quantum system is confined and of a laser probe which excites the quantum system. Measurements about the condition of the quantum system are collected through photodetectors and thus the projections of the probability density matrix ρ of the quantum system are turned into weak current. By processing this current measurement and the estimate of the quantum system's state which is provided by Lindblad's or Belavkin's equation, a control law is generated which modifies a magnetic field applied to the cavity. In that manner, the state of the quantum system is driven from the initial value $\rho(0)$ to the final desirable value $\rho_d(t)$ (see Fig. 1).

When Schrödinger's equation is used to describe the dynamics of the quantum system the objective is to move the quantum system from a state ψ, that is associated to a certain energy level, to a different eigenstate associated with the desirable energy level. When Lindblad's or Belavkin's equation is used to describe the dynamics of the quantum system, the control objective is to stabilize the probability density matrix $\rho(t)$ on some desirable quantum state $\rho_d(t) \in C^n$, by controlling the intensity of the magnetic field. The value of the control signal is determined by processing

the measured output which in turn depends on the projection of $\rho(t)$ defined by $Tr\{P\rho(t)\}$.
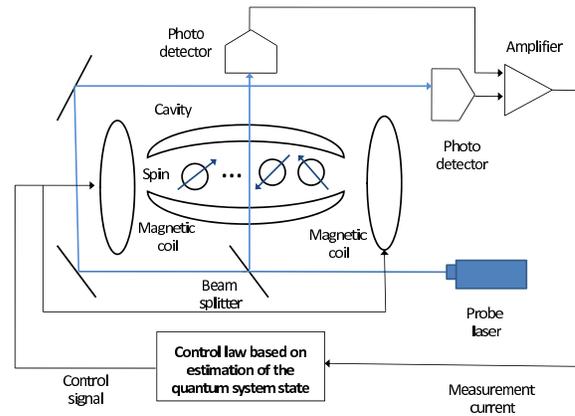
Figure 1. Feedback control loop for quantum systems

IV. A FEEDBACK CONTROL APPROACH FOR QUANTUM SYSTEM STABILIZATION

A. Control law calculation using Schrödinger's equation

It is assumed that the dynamics of the controlled quantum system is described by a Schrödinger equation of the form

$$i\hbar\dot{\psi}(t) = [H_0 + f(t)H_1]\psi(t) \quad \psi(t) \in C^n \quad (11)$$

where H_0 is the system's Hamiltonian, H_1 is the control Hamiltonian and $f(t)$ is the external control input. The following Lyapunov function is then introduced [9]

$$V(\psi) = (\psi^+ Z\psi - Z_d)^2 \quad (12)$$

where $+$ stands for the transposition and complex conjugation, Z is the quantum system's observable and is associated to the energy of the system. The term $\psi^+ Z\psi$ denotes the observed mean energy of the system at time instant t and Z_d is the desirable energy value. The first derivative of the Lyapunov function of Eq. (12) is

$$\dot{V}(\psi) = 2[\psi^+ Z\psi - Z_d][\dot{\psi}^+ Z\psi + \psi^+ Z\dot{\psi}] \quad (13)$$

while from Eq. (11) it also holds $\dot{\psi}(t) = -\frac{i}{\hbar}[H_0 + f(t)H_1]\psi(t)$, which results into

$$\dot{V}(\psi) = \frac{2i}{\hbar}(\psi^+ Z\psi - Z_d)\cdot$$
$$\cdot\psi^+\{H_0 Z - Z H_0 + f(H_1 Z - Z H_1)\}\psi \quad (14)$$

Choosing the control signal $f(t)$ to be proportional to the gradient with respect to f of the first derivative of the Lyapunov function with respect to time (velocity gradient) i.e. $f(t) = k\nabla_f\{\dot{V}(\psi)\}$, and for Z such that $\psi^+ H_0 Z\psi = \psi^+ Z H_0\psi$ (e.g. $Z = H_0$) one obtains

$$f(t) = \frac{2i}{\hbar}(\psi^+ Z\psi - Z_d)\psi^+(H_1 Z - ZH_1)\psi \qquad (15)$$

Substituting Eq. (15) into Eq. (14) provides $\dot{V}(\psi) = k\frac{2i}{\hbar}(\psi^+ Z\psi - Z_d)\psi^+[\frac{2i}{\hbar}(\psi^+ Z\psi - Z_d)\psi^+(H_1 Z - ZH_1)\psi(H_1 Z - ZH_1)]\psi$, and finally results in the following form of the first derivative of the Lyapunov function

$$\dot{V}(\psi) = -k\frac{4}{\hbar^2}(\psi^* Z\psi - Z_d)^2 \psi^{+2}(H_1 Z - ZH_1)^2 \psi^2 \leq 0 \qquad (16)$$

which is non-positive along the system trajectories. This implies stability for the quantum system and in such a case La Salle's principle shows convergence not to an equilibrium but to an area round this equilibrium, which is known as *invariant set*. La Salle's theorem is expressed as follows [16]:

Theorem 1: Assume the autonomous system $\dot{x} = f(x)$ where $f : D \rightarrow R^n$. Assume $C \subset D$ a compact set which is positively invariant with respect to $\dot{x} = f(x)$, i.e. if $x(0) \in C \Rightarrow x(t) \in C \; \forall \; t$. Assume that $V(x) : D \rightarrow R$ is a continuous and differentiable Lyapunov function such that $\dot{V}(x) \leq 0$ for $x \in C$, i.e. $V(x)$ is negative semi-definite in C. Denote by E the set of all points in C such that $\dot{V}(x) = 0$. Denote by M the largest invariant set in E and its boundary by L^+, i.e. for $x(t) \in E : lim_{t \rightarrow \infty} x(t) = L^+$, or in other words L^+ is the positive limit set of E. Then every solution $x(t) \in C$ will converge to M as $t \rightarrow \infty$.
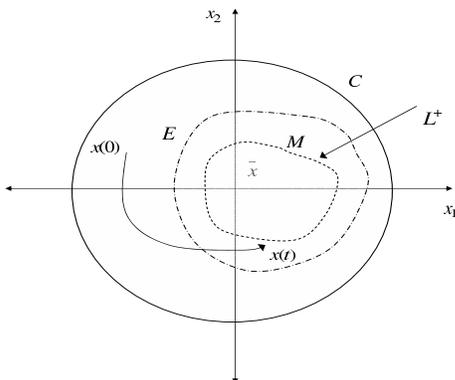
Figure 2. LaSalle's theorem: C: invariant set, $E \subset C$: invariant set which satisfies $\dot{V}(x) = 0$, $M \subset E$: invariant set, which satisfies $\dot{V}(x) = 0$, and which contains the limit points of $x(t) \in E$, L^+ the set of limit points of $x(t) \in E$

Consequently, from Eq. (16) and LaSalle's theorem, any solution of the system $\psi(t)$ remains in the invariant set $M = \{\psi : \; \dot{V}(\psi) = 0\}$.

B. Control law calculation using Lindblad's equation

Next, it will be shown how a gradient-based control law can be formulated using the description of the quantum system

according to Lindblad's equation. The following bilinear Hamiltonian system is considered (Lidbland equation)

$$\dot{\rho}(t) = -i[H_0 + f(t)H_1, \rho(t)] \qquad (17)$$

where H_0 is the interaction Hamiltonian of the quantum system, H_1 is the control Hamiltonian of the quantum system and $f(t)$ is the real-valued control field for the quantum system. The control problem consists of calculating the control function $f(t)$ such that the system's state (probability transition matrix $\rho(t)$) with initial conditions $\rho(0) = \rho_0$ converges to the desirable final state ρ_d for $t \rightarrow \infty$. It is considered that the initial state ρ_0 and the final state ρ_d have the same spectrum and this is a condition needed for reachability of the final state through unitary evolutions.

Because of the existence of the interaction Hamiltonian H_0 it is also considered that the desirable target state also evolves in time according to the Lindblad equation, i.e.

$$\dot{\rho}_d(t) = -i[H_0, \rho_d(t)] \qquad (18)$$

The target state is considered to be stationary if it holds $[H_0, \rho_d(t)] = 0$, therefore in such a case it also holds $\dot{\rho}_d(t) = 0$. When $[H_0, \rho_d(t)] \neq 0$ then one has $\dot{\rho}_d \neq 0$ and the control problem of the quantum system is a tracking problem. The requirement $\rho(t) \rightarrow \rho_d(t)$ for $t \rightarrow \infty$ implies a trajectory tracking problem, while the requirement $\rho(t) \rightarrow O(\rho_d)(t)$ for $t \rightarrow \infty$ is an orbit tracking problem.

It will be shown that the calculation of the control function $f(t)$ which assures that $\rho(t)$ converges to $\rho_d(t)$ can be performed with the use of the Lyapunov method. To this end, a suitable Lyapunov function $V(\rho, \rho_d)$ will be chosen and it will be shown that there exists a gradient-based control law $f(t)$ such that $\dot{V}(\rho, \rho_d) \leq 0$.

The dynamics of the state of the quantum system, as well as the dynamics of the target state are jointly described by $\dot{\rho}(t) = -i[H_0 + f(t)H_1, \rho(t)]$, $\dot{\rho}_d(t) = -i[H_0, \rho_d(t)]$. A potential Lyapunov function for the considered quantum system is taken to be

$$V = 1 - Tr(\rho_d \rho) \qquad (19)$$

It holds that $V > 0$ if $\rho \neq \rho_d$. The Lyapunov function given in Eq. (19) can be also considered as equivalent to the Lyapunov function $V(\psi, \psi_d) = 1 - | < \psi_d(t)|\psi(t) > |^2$, which results from the description of the quantum system with the use of Schrödinger's equation given in Eq. (1). The term $< \psi_d(t)|\psi(t) >$ expresses an internal product which takes value 1 if $\psi_d(t)$ and $\psi(t)$ are aligned. The first derivative of the Lyapunov function defined in Eq. (19) is

$$\dot{V} = -Tr(\dot{\rho}_d \rho) - Tr(\rho_d \dot{\rho}) \qquad (20)$$

One can continue on the calculation of the first derivative of the Lyapunov function $\dot{V} = -Tr(\dot{\rho}_d \rho) - Tr(\rho_d \dot{\rho}) \Rightarrow \dot{V} = -Tr([-iH_0, \rho_d]\rho) -$

$Tr(\rho_d[-iH_0, \rho]) \quad - \quad f(t)Tr(\rho_d[-iH_1, \rho]) \Rightarrow \dot{V} \quad = -Tr([-iH_0\rho_d + \rho_d iH_0]\rho) - Tr(\rho_d[-iH_0\rho + \rho iH_0]) - f(t)Tr(\rho_d[-iH_1, \rho]) \Rightarrow \dot{V} = -Tr(-iH_0\rho_d\rho + \rho_d iH_0\rho - \rho_d iH_0\rho + \rho_d\rho iH_0) - f(t)Tr(\rho_d[-iH_1, \rho])$

Using that $Tr(iH_0\rho_d\rho) = Tr(\rho_d\rho iH_0)$ one obtains

$$\dot{V} = -f(t)Tr(\rho_d[-iH_1, \rho]) \qquad (21)$$

The control signal $f(t)$ is taken to be the gradient with respect to f of the first derivative of the Lyapunov function i.e. $f(t) = -k\nabla_f\dot{V}(t)$, which gives

$$f(t) = kTr(\rho_d[-iH_1, \rho]) \quad k > 0 \qquad (22)$$

and which results in a negative semi-definite Lyapunov function $\dot{V} \leq 0$. Choosing the control signal $f(t)$ according to Eq. (22) assures that for the Lyapunov function of the quantum system given by Eq. (19) it holds

$$V > 0 \quad \forall \ (\rho, \rho_d) \neq 0$$
$$\dot{V} \leq 0 \qquad (23)$$

and since a negative semi-definite Lyapunov function is examined, LaSalle's theorem is again applicable [16].

According to LaSalle's theorem, explained in subsection IV-A, the state $(\rho(t), \rho_d(t))$ of the quantum system converges to the invariant set $M = \{(\rho, \rho_d)|\dot{V}(\rho(t), \rho_d(t)) = 0\}$. Attempts to define more precisely the convergence area for the trajectories of $\rho(t)$ when applying La Salle's theorem can be found in [8], [17].

V. SIMULATION TESTS

Simulation tests about the performance of the gradient-based quantum control loop are given for the case of a two-qubit (four-level) quantum system. Indicative results from two different simulation experiments are presented, each one associated to different initial conditions of the target trajectory and different desirable final state.

The Hamiltonian of the quantum system was considered to be ideal, i.e. $H_0 \in C^4$ was taken to be strongly regular and $H_1 \in C^4$ contained non-zero non-diagonal elements. The two-qubit quantum system has four eigenstates which are denoted as $\psi_1 = (1000), \psi_2 = (0100)$, $\psi_3 = (0010)$ and $\psi_4 = (0001)$. For the first case, the desirable values of elements $\rho_{ii}^d, \ i = 1, \cdots, 4$ corresponding to quantum states ψ_1 to ψ_4 are depicted in Fig. 3(a), while the convergence of the actual values $\rho_{ii}, \ i = 1, \cdots, 4$ towards the associated desirable values is shown in Fig. 3(b). Similarly, for the second case, the desirable values of elements $\rho_{ii}^d, \ i = 1, \cdots, 4$ are shown in Fig. Fig. 4(a), while the associated actual values are depicted in Fig. 4(b). It can be observed that the gradient-based control calculated according to Eq. (22) enabled convergence of ρ_{ii} to ρ_{ii}^d, within acceptable accuracy levels. Fig. 5 presents the evolution in time of the Lyapunov

function of the two simulated quantum control systems. It can be noticed that the Lyapunov function decreases, in accordance to the negative semi-definiteness proven in Eq. (23). Finally, in Fig. 6, the control signals for the two aforementioned simulation experiments are presented. The simulation tests verify the theoretically proven effectiveness of the proposed gradient-based quantum control scheme. The results can be extended to the case of control loops with multiple control inputs f_i and associated Hamiltonians $H_i, \ i = 1, \cdots, n$.

(a) (b)
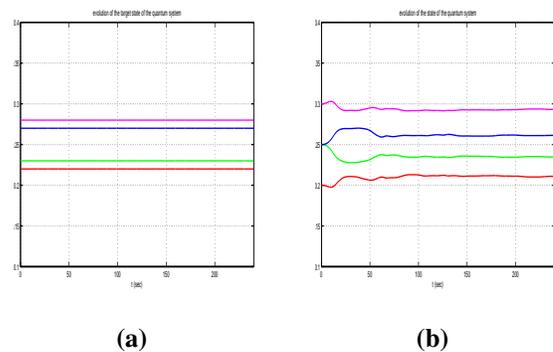
Figure 3. (a) Desirable quantum states in the first test case, (b) Actual quantum states in the first test case

(a) (b)
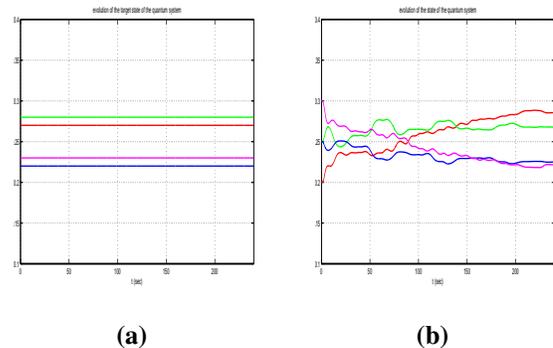
Figure 4. (a) Desirable quantum states in the second test case, (b) Actual quantum states in the second test case

VI. CONCLUSIONS

The paper has presented a gradient-based approach to feedback control of quantum systems. Different descriptions of the quantum system dynamics were formulated using Schrödinger's and Lindblad's differential equations, as well as Belavkin's stochastic differential equation. When Scrödinger's equation is used to describe the dynamics of the quantum system the objective is to move the quantum system form an eigenstate associated to a certain energy level to a different eigenstate associated to the desirable

(a)   **(b)**
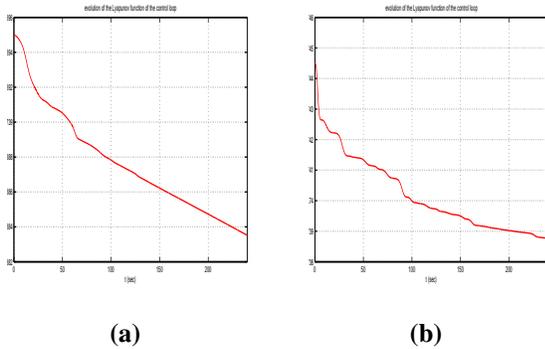
Figure 5.   (a) Lyapunov function of the first test case, (b) Lyapunov function of the second test case

(a)   **(b)**
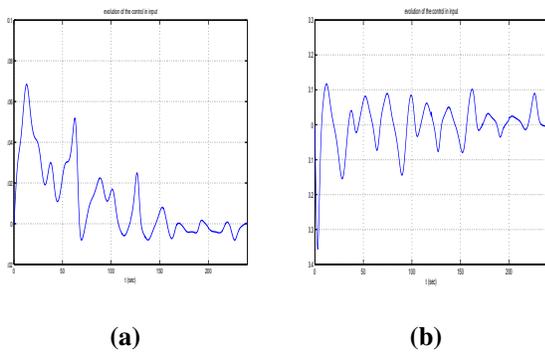
Figure 6.   (a) Control input of the first test case, (b) Control input of the second test case

energy level. When Lindblad's or Belavkin's equations are used to describe the dynamics of the quantum system the control objective is to stabilize the probability density matrix ρ on some desirable quantum state $\rho_d \in C^n$ by controlling the intensity of the magnetic field. The control input is calculated by processing the measured output, which in turn depends on the projection of the probability density matrix ρ, as well as on processing of the estimate of ρ provided by Lindblad's or Belavkin's equation. It was shown that using either the Schrödinger or the Lindblad description of the quantum system a gradient-based control law can be formulated which assures tracking of the desirable quantum state within acceptable accuracy levels. The convergence properties of the gradient-based control scheme were proven using Lyapunov stability theory and LaSalle's invariance principle. Finally, simulation experiments for the case of a two-qubit (four-level) quantum system verified the theoretically established efficiency of the proposed gradient-based control approach.

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