On The Topological Entropy of Continuous-Time Polytopic Systems

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***Abstract*—This paper investigates the topological entropy of continuous-time polytopic systems. The topological entropy is a measure that quantifies the instability in dynamical linear systems and has important applications in autonomous systems. Polytopic systems are dynamical linear systems whose coefficients are functions of an uncertain vector constrained into a polytope. A novel approach is proposed for establishing upper bounds of the largest topological entropy of continuous-time polytopic systems based on the Routh-Hurwitz stability criterion. The upper bounds are established through Linear Matrix Inequality (LMI) feasibility tests, which amount to solving convex optimization problems. A numerical example illustrates the proposed approach.**

***Keywords*–Topological entropy; Polytopic systems; LMI.**

I. INTRODUCTION

The topological entropy is a measure that quantifies the instability in dynamical linear systems. This measure is defined as the sum of the real part of the unstable eigenvalues in the continuous-time case, and as the product of the magnitude of the unstable eigenvalues in the discrete-time case [1]. The topological entropy has important applications in autonomous systems where it is required to ensure stability with communication constraints [2]. For instance, this measure can be used to establish the existence of stabilizing state feedback controllers in the presence of constraints on the signal-to-noise ratio [3]. See also [4] [5] for other uses of this measure.

Unfortunately, the mathematical model of a control system is often affected by uncertainty, e.g., representing physical quantities that cannot be measured exactly or that are subject to changes. As a consequence, one has to consider a family of admissible models depending on the uncertainty. Clearly, the instability measures become functions of the uncertainty, and the target is to determine the largest instability measures over the admissible uncertainties.

In the literature, the topological entropy of continuous-time uncertain systems has been investigated in [6] [7] through convex optimization. However, these methods exploit Lyapunov functions [8] and determinants, and cannot be easily used for control design because the presence of an unknown controller would lead to the formulation of nonconvex optimization problems.

In order to deal with this drawback, a novel approach is proposed in this paper for investigating the topological entropy of continuous-time uncertain systems. Specifically, polytopic systems are considered, i.e., dynamical linear systems whose coefficients are functions of an uncertain vector constrained into a polytope. It is shown that upper bounds of the largest topological entropy can be established based on the Routh-Hurwitz stability criterion through LMI feasibility tests, which amount to solving convex optimization problems. A numerical example illustrates the proposed approach.

The paper is organized as follows. Section II introduces the preliminaries. Section III describes the proposed results. Section IV presents an illustrative examples. Lastly, Section V concludes the paper with some final remarks.

II. PRELIMINARIES

Notation: $\mathbb{R}, \mathbb{C}$: sets of real and complex numbers; $\Re(M), \Im(M)$: real and imaginary parts of M; I: identity matrix (of size specified by the context); M': transpose; $M > 0$, $M \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix; $\lambda_i(M)$: i-th eigenvalue of M; $\mathrm{spec}(M)$: set of eigenvalues of M; $\|M\|_2$: 2-norm of v; M^2: entry-wise square; Hurwitz matrix: a matrix whose eigenvalues have negative real parts.

Let us consider the continuous-time uncertain system

$$\dot{x}(t) = A(p)x(t) \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $p \in \mathbb{R}^q$ is an uncertain vector constrained by

$$p \in \mathcal{S} \tag{2}$$

where $\mathcal{S}$ is the simplex

$$\mathcal{S} = \left\{ p \in \mathbb{R}^q : \; p_i \geq 0, \; \sum_{i=1}^{q} p_i = 1 \right\}, \tag{3}$$

and $A(p) \in \mathbb{R}^{n \times n}$ is a matrix polynomial.

Let $B \in \mathbb{R}^{n \times n}$. The topological entropy of B is defined as

$$\mu(B) = \sum_{i=1}^{n} \max \left\{ 0, \Re(\lambda_i(B)) \right\}, \tag{4}$$

i.e., as the sum of the real part of the unstable eigenvalues of B.

Problem 1. The problem that we consider in this paper consists of determining the largest topological entropy of (1)–(3), i.e.,

$$\mu^* = \sup_{p \in \mathcal{S}} \mu(A(p)). \tag{5}$$

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III. Proposed Approach

The first step of the proposed approach is to introduce a matrix whose eigenvalues are all the possible sums of the eigenvalues of a given matrix. Specifically, let $B \in \mathbb{R}^{n\times n}$ the given matrix, and let $k = 1,\ldots,n$ denote the number of eigenvalues that have to be multiplied. We denote with $\Omega_k(B)$ the matrix function whose eigenvalues are all the possible sums of k eigenvalues of B, i.e.,

$$\mathrm{spec}(\Omega_k(B)) \left\{ \sum_{i=1}^{k} \lambda_{z_i}(U),\ z \in \mathcal{I}_k \right\} \tag{6}$$

where $\mathcal{I}_k$ is the set of k-tuples in $\{1,\ldots,n\}$ defined by

$$\begin{array}{rcl} \mathcal{I}_k & = & \{(z_1,\ldots,z_k):\ z_i \in \{1,\ldots,n\}, \\ & & z_i < z_{i+1}\ \forall i = 1,\ldots,k-1\}. \end{array} \tag{7}$$

The matrix function $\Omega_k(B)$ can be built for any positive integer n and for any $k \in \{1,\ldots,n\}$ following the idea described by Bellman [9].

The second step of the proposed approach is to build a modified Routh-Hurwitz table. Specifically, let $B \in \mathbb{R}^{n\times n}$ and $\theta \in \mathbb{C}$. Let us define

$$f(\theta, B) = \det(\theta I - B) \tag{8}$$

which is a polynomial in θ. We denote with $g_{i,j}(B)$ the (i,j)-th entry of the table obtained for $f(\theta, B)$ under the following constraints:

1) $g_{i,j}(B)$ is a polynomial in B;
2) $g_{i,1}(B) > 0$ if and only if B is Hurwitz.

The third step of the proposed approach is to exploit convex optimization. Specifically, let $w > 0$, and for $k = 1,\ldots,n$ let us define

$$h_{i,k}(p,w) = g_{2i,1}\left(\Omega_k(A(p)) - wI\right) \tag{9}$$

which is a polynomial in p. Let $m_{i,k}(p,w)$ be the homogeneous polynomial in p with the minimum degree satisfying

$$m_{i,k}(p,w) = h_{i,k}(p,w) \quad \forall p \in \mathcal{S}. \tag{10}$$

Let $d_{i,k}$ denote such a degree. Then, a condition for establishing that w is an upper bound of μ^* can be obtained by looking for a scalar $\varepsilon > 0$ such that

$$m_{i,k}(p^2,w) - \varepsilon \|p\|_2^{2d_{i,k}} \text{ is SOS } \forall i,k \tag{11}$$

where SOS stands for sum of squares of polynomials.

The condition (11) amounts to solving a convex optimization problem because establishing whether a polynomial is SOS amounts to establishing feasibility of an LMI; see, for instance, [10] and references therein. It can be shown that the condition (11) is sufficient for establishing that w is an upper bound of μ^*. Moreover, it can also be shown that this condition is also necessary by suitably increasing the degree of the polynomial in (11) following the ideas in [11].

IV. Example

Let us consider for simplicity (1)–(3) with

$$A(p) = p_1 \begin{pmatrix} 3.4 & 2.9 \\ -1.6 & -1.6 \end{pmatrix} + p_2 \begin{pmatrix} -2.9 & -4.1 \\ 4.5 & 0.3 \end{pmatrix}.$$

We test the condition (11) for different values through bisection, finding that the best upper bound guaranteed by this condition is

$$\hat{\mu} = 2.457$$

(the condition (11) is equivalent to an LMI with 4 scalar variables and can be solved in less than one second on standard personal computers). Brute force search shows that this upper bound is tight, i.e., $\mu^* = \hat{\mu}$. Indeed,

$$p = (0.785, 0.215)' \quad \Rightarrow \quad A(p) = \hat{\mu}.$$

V. Conclusion

A novel approach has been proposed for establishing upper bounds of the largest topological entropy in continuous-time polytopic systems. The proposed approach can be easily implemented with standard software, moreover the numerical example has shown that the computational burden can be significantly low and that the upper bounds can be non-conservative. Future work will explore the use of the proposed approach for control design.

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References

[1] R. Bowen, "Entropy for group endomorphisms and homogeneous space," Transactions of the American Mathematical Society, vol. 153, 1971, pp. 401–414.

[2] G. N. Nair, F. Fagnani, S. Sampieri, and R. J. Evans, "Feedback control under data rate constraints: an overview," Proceeding of IEEE vol. 95, 2007, pp. 108–137.

[3] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," IEEE Transactions on Automatic Control, vol. 52, no. 8, 2007, pp. 1391–1403.

[4] M. Fu and L. Xie, "The sector bound approach to quantized feedback control," IEEE Transactions on Automatic Control, vol. 50, no. 11, 2005, pp. 1698–1711.

[5] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation," IEEE Transactions on Automatic Control, vol. 58, no. 3, 2013, pp. 554–568.

[6] G. Chesi, "Measuring the instability in continuous-time linear systems with polytopic uncertainty," in IEEE Conference on Decision and Control, Florence, Italy, 2013, pp. 1131–1136.

[7] G. Chesi, "LMI-based computation of the instability measure of continuous-time linear systems with a scalar parameter," in IEEE Canadian Conference on Electrical and Computer Engineering, Toronto, Canada, 2014, pp. 374–379.

[8] J. P. LaSalle and S. Lefschetz, Stability by Lyapunov's Direct Method with Applications. New York: Academic Press, 1961.

[9] R. Bellman, Introduction to Matrix Analysis. New York: McGraw-Hill, 1974.

[10] G. Chesi, "LMI techniques for optimization over polynomials in control: a survey," IEEE Transactions on Automatic Control, vol. 55, no. 11, 2010, pp. 2500–2510.

[11] G. Chesi, "On the non-conservatism of a novel LMI relaxation for robust analysis of polytopic systems," Automatica, vol. 44, no. 11, 2008, pp. 2973–2976.

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