Lyapunov-based Control Theory of Closed Quantum Systems

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Abstract—**According to whether the internal Hamiltonians are strong regular and/or the control Hamiltonians are full connected, the quantum systems can be considered as ideal closed quantum systems or the quantum systems in degenerate cases. In this paper, we propose a unique formation of quantum Lyapunov-based control method, which is suitable for both ideal closed quantum systems and the systems in degenerate cases. This Lyapunov-based control method of closed quantum systems with unique formation is realized by means of introducing implicit Lyapunov functions into the control laws, which make the control system become strong regular and full connection. The proposed Lyapunov-based control theory can transfer from arbitrary initial states to arbitrary final states in the way of asymptotic stability. The paper gives the complete design procedure of control laws. At last, a numerical experiment of the state transfer between two mixed states in degenerate cases is given to demonstrate the effectiveness of the Lyapunov-based control theory proposed.**

Keywords- ideal closed quantum systems; quantum systems in degenerate cases; implicit Lyapunov-based control method.

I. INTRODUCTION

From the perspective of system control, a quantum system can be considered as a closed or an open quantum system. The closed quantum system is an isolated system or without interaction with the environment. The majority of actual quantum systems are open quantum systems. However, the closed quantum systems have their own characteristics, namely, they are simpler to be analyzed and studied, and the research results of closed quantum systems are the foundations of open quantum systems. The role of a closed system in quantum systems is similar to that of the system which is a linear, definite and time-invariant in macroscopic systems. Even so, the control task of state transfer in closed quantum systems is quite difficult because there are eigenstates, superposition states and mixed states, in which only the eigenstate corresponds to the classical state in macroscopic systems, while other two states do not exist in the macroscopic world.

The solutions of the control problems obtained by means of the system control theory are generally the control laws in an N dimensional quantum system, which can be easily applied to the high dimensional quantum systems without increasing control cost and design difficulty. Therefore, the closed quantum system control theory has a guiding significance for the realization of the actual experiments, especially for complex quantum systems. In the last 30 years, the control theory of quantum systems has developed rapidly. Many quantum control methods have been developed, such as coherent control [1]-[3], Bang-bang control and geometrical control [4][5], dynamical decoupling control [6]-[8], sliding mode control [9][14], robust control [10], optimal control [11]-[14], Lyapunov-based control [15]-[18], feedback control [19]-[21]. Among all the quantum control theories, optimal quantum control is the most widely used in quantum system control. Like the optimal quantum control method, the Lyapunov-based quantum control is also a very powerful control method. By means of the Lyapunov stability theorem, this control method designs an asymptotically stable controller by making the first time derivative of the Lyapunov function constructed not great than zero. Unlike the way it is being used in the macroscopic engineering field, in which the controller is only required to be designed as a stable one, the Lyapunov-based control method used in quantum fields should be designed as a convergent one in order to guarantee the control system to reach the target state with 100% probability. This is because the variable controlled in quantum systems is usually the density matrix, which is a probability. A general model of an N dimensional closed quantum system can be descried by the Liouville equation: $i\dot{\rho}(t)=[H_0+\sum_{k=1}^{r}H_k u_k(t),\rho(t)]$, in which $\rho(t)$ is the density matrix; H_0 is the internal Hamiltonian; $u_k(t)$ are external control fields; H_k are the control Hamiltonians. The eigenvalue (or spectrum) of the internal Hamiltonian

$H_0 = diag(\lambda_1, \lambda_2, ..., \lambda_N)$ of the system, in which $\lambda_j, j = 1,2,...,N$ indicate the energy levels of the system, while $\omega_{jl} = \lambda_j - \lambda_l$ are the transitions (Bohr) frequencies between the energy levels of the system. We define a non-degenerate quantum system if all the energy levels of a quantum system are not the same and a quantum system without degenerate transition, which means all Bohr frequencies are not the same. A quantum system is called full connection if $(H_k)_{jk} \neq 0, j < l \in \{=1,2,...,N\}$ for $\forall k \in \{1,2,...,N\}$ holds. Based on the Lyapunov stability theorem, the analytical expressions of control laws can be designed by means of the construction of a suitable Lyapunov function $V(\rho)$, and under the condition of $\dot{V}(\rho) < 0$. The Lyapunov function is not unique. A general form of Lyapunov function is $V(\rho) = tr(P\rho)$, in which P is a positive definite Hermitian operator to be determined, which is one part of control laws design. P can be regarded as an imaginary mechanical value of the system. In mathematics, $V(\rho) = tr(P\rho)$ is a trace calculation. In physics, $V(\rho)$ is an expectation value of Hermitian operator P. By calculating the first order derivatives of $V(\rho)$ for the time, one can obtain $\dot{V}(\rho)$=-$i$tr([P,H_0],P)-$i\sum_{k=1}^{r}$ tr([P,H_k]ρ)u_k. Because the first term in the right side of $\dot{V}(\rho)$ is independent of the control laws this term can be eliminated by $[P,H_0]$=0, which also provides a condition of designing P. When H_0 is non-degenerate, P is a diagonal matrix. The control laws can be obtained by letting $\dot{V}(\rho) = 0$, and the expressions of control laws are $u_k = i\varepsilon_k tr([P,H_k]\rho), k = 1,2,...r$, in which ε_k is used to regulate the amplitude of the control laws. According to the LaSalle invariant set principle, the control system can be guaranteed that any trajectory converges to a maximum invariant set.

Now that the control laws is obtained by $\dot{V}(\rho) = 0$, besides the target state, generally there are many other states which can also make $\dot{V}(\rho) = 0$, all of which are the state points of the Lyapunov function $V(\rho)$. The number of these state points is even un-numerical in the cases when the target state is a supposition state or mixed state. In order to make the control system converge to the desired target state, one must add the constraint conditions to narrow the invariant set. For the different kinds of the target state, the conditions the system needs to meet are different. Generally speaking, the convergence conditions of a quantum system by using the Lyapunov control method based on the average value of an imaginary mechanical quantity P are three points, which are the requirements of internal Hamiltonian, control Hamiltonians, and target state, respectively. These three conditions are:

i) The internal Hamiltonian is strongly regular, i.e., the transition energies between two different levels are clearly identified;
ii) The control Hamiltonians are full connected, i.e., any two levels are directly coupled [18];
iii) The target state must be diagonal, which makes $[\rho_f, H_0] = 0$ hold.

Fig. 1 is an example that satisfies the conditions i) and ii).

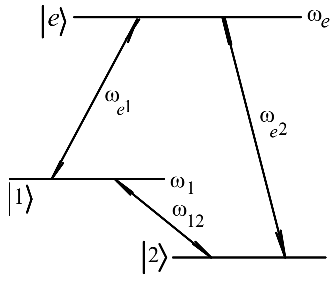

Figure 1. Quantum system that satisfies the conditions i) and ii).

The system which satisfies above mentioned conditions i) and ii) is called ideal quantum system. Under the conditions i) and ii), condition iii) is the condition of the state transfer of closed quantum systems from arbitrary initial state to an arbitrary diagonal target state, which can be an eigenstate, supposition state, or mixed state.

However, many quantum systems in practice do not satisfy the conditions i) or/and ii). For example,

$$H_0 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}$$

or/and

$$H_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Fig. 2 shows that the V-type and Λ - type quantum systems we often encounter in practice do not satisfy conditions i) and ii). Because the convergence conditions obtained are so extremely rigorous, the designed control laws have little practical application value.

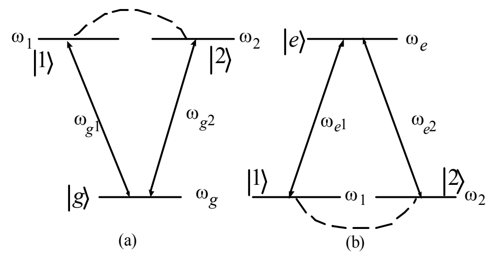

Figure 2. (a) V-type; (b) Λ -type quantum systems.

In order to obtain Lyapunov-based quantum control methods which have practical application value, one needs to solve the problems which appear in the quantum systems in degenerate cases, so as to establish the Lyapunov quantum control theory. Up to now, many researches have been done. Zhao et al. utilized an implicit Lyapunov control to solve the problem of convergence for the single control Hamiltonian systems governed by the Schrödinger equation [22]. We once investigated the implicit Lyapunov control method of multi-control Hamiltonian systems governed by the Schrödinger equation based on the state distance and state error, both of which are only suitable for the control of the pure states [23]. We also studied the implicit Lyapunov quantum control method based on the imaginary mechanical quantity for pure states and mixed states, in which the stricter convergence proof was given [24].

In this paper we propose a unified formulation of Lyapunov control theory for closed quantum systems. The basic idea is: for a quantum system which does not satisfy the convergent conditions i), ii) or/and iii), we introduce an implicit Lyapunov function into the quantum control design method based on the imaginary mechanical quantity in Liouville equation, in order to make the designed control system satisfy three convergent conditions. The Lyapunov quantum control theory proposed here can be used in both degenerate cases and ideal quantum systems, which can transfer the state from an arbitrary initial state to an arbitrary target state. The "arbitrary" here means eigenstate, superposition state or mixed state. The proposed control method in this paper is a unique formation of quantum Lyapunov-based control method , which has important significance.

The rest of the paper is structured as follows: Section II is the Lyapunov-based quantum control theory, in which implicit Lyapunov functions are introduced, as well as the procedure of control designs of η_k , $\gamma_k(t)$, $v_k(t)$ and $P_{\eta\gamma}$ in detail. Section III is the numerical simulation, and Section IV is the conclusion.

II. LYAPUNOV-BASED QUANTUM CONTROL THEORY

A. Implicit Lyapunov Functions

Consider the N-level closed quantum systems governed by the following quantum Liouville equation which may be in degeneration cases:

$$i\dot{\rho}(t) = [H_0 + \sum_{k=1}^{r} H_k u_k(t), \rho(t)] \quad (1)$$

where $\rho(t)$ is the density operator; H_0 is the internal Hamiltonian; H_k , $k = 1,2...,r$, are control Hamiltonian; and $u_k(t),(k = 1,\cdots,r)$ are scalar and real total control laws.

The way to solve the degeneration problems is to introduce the implicit Lyapunov functions as the control disturbances such that the system with additional control disturbances may satisfy those convergence conditions. A completely unified designing method of control laws is proposed here. The control laws are composed of three parts:

$$u_k(t) = \gamma_k(t) + v_k(t) + \eta_k \quad (2)$$

in which $\gamma_k(t)$ are designed to make the system (1) satisfy the convergence conditions i) and ii); $v_k(t)$ are the control laws designed to transfer any initial state to the invariant set; η_k are used to make the target state commute with the internal Hamiltonian H_0, i.e., $[\rho_f, H_0] = 0$, so as to make the control system be able to converge to the desired target state.

The Lyapunov function is constructed as:

$$V(\rho) = tr(P_{\eta\gamma}\rho) \quad (3)$$

where $P_{\eta\gamma} = f(\eta_1,\cdots,\eta_r,\gamma_1(t),\cdots,\gamma_r(t))$ are functional of η_k and $\gamma_k(t)$, $(k = 1,2...,r)$ and positive definite.

Eq. (3) is called the implicit Lyapunov function based on the average value of an imaginary mechanical quantity. The function of (1) is used to design control laws (2), in which η_k will be designed in the case $[\rho_f, H_0] \neq 0$, which does not satisfy the condition iii). $\gamma_k(t)$ will be designed in the cases when condition i) or/and ii) are not satisfied. $v_k(t)$ is used to design the control laws of transferring the state from an arbitrary initial state to an arbitrary target state.

Next, we'll give in detail the design procedures and the explanations of how these three parts play roles in control laws.

B. Control Design of η_k

In the procedure of designing η_k, first, check whether the target state ρ_f commutes with the internal Hamiltonian H_0, and one can know what type the target state is. If the target state does not commute with the internal Hamiltonian, this results in $[\rho_f, H_0] = D,\quad D \neq 0$. The supposition state and some non-diagonal mixed state are in such cases. Then a set of appropriate constant values η_k need to be introduced into the control laws. Then, $H_{0\eta} = H_0 + \sum_{k=1}^{r} H_k\eta_k$, $(k = 1,2...,r)$ will be considered to be the new internal Hamiltonian. Last, design η_k in order to make

$$[\rho_f, H_{0\eta}] = 0, H_{0\eta} = H_0 + \sum_{k=1}^{r} H_k\eta_k \quad (4)$$

hold.

For the special case when the target state commute with the internal Hamiltonian H_0, that is, $[\rho_f, H_0] = 0$, one can set

$\eta_k = 0$, which is the quantum system with the target state being eigenstates or some mixed states which commute with H_0. After introducing and designing the constant values η_k, the target state ρ_f will become commute with $H_{0\eta}$ in (4).

C. Control Design of $\gamma_k(t)$

There are two objectives of designing $\gamma_k(t)$, one is to make $H_{0\eta\gamma} = H_{0\eta} + \sum_{k=1}^{r} H_k \gamma_k(t)$ such that $H_{0\eta\gamma}$ is strongly regular. Denote eigenstate of $H_{0\eta\gamma}$ as $|\phi_{1\eta,\gamma}\rangle,\cdots,|\phi_{N\eta,\gamma}\rangle$. The control Hamiltonian with $H_{0\eta\gamma}$ is $H_{k\eta\gamma}$: $H_{k\eta\gamma} = U_{\eta\gamma}{}^{\dagger} H_k U_{\eta\gamma}$, in which $U_{\eta\gamma} = \left(|\phi_{1\eta,\gamma}\rangle,\cdots,|\phi_{N\eta,\gamma}\rangle\right)$. Another objective is to make the $H_{k\eta\gamma}$ be full connected. To achieve these two objectives, $\gamma_k(t)$ can be designed as

$$\gamma_k(t) = \gamma(t) = \begin{cases} F(s), & k = k_1,\cdots,k_m \\ 0, & k \neq k_1,\cdots,k_m (1 \le k_1,\cdots,k_m \le r) \end{cases} \tag{5}$$

in which F is the function of s, and satisfies $F(0) = 0$, $F(s) > 0$, and $F'(s) > 0$, which means $F(s)$ is a monotonic increasing function.

Usually, the simplest $F(s)$ can be constructed as: $s = V(\rho) - V(\rho_f)$, where $C > 0$, and $C \in R$. Combining with (3) $\gamma_k(t)$ can be designed as

$$\gamma_k(t) = \gamma(t) = C \cdot (tr(P_{\eta\gamma}\rho) - tr(P_{\eta\gamma}\rho_f)) \tag{6}$$

D. Control Design of $v_k(t)$

The role of control laws $v_k(t)$ is to ensure $\dot{V}(t) \le 0$. $v_k(t)$, $k = 1,\cdots,r$ are designed to ensure the first time derivative of Lyapunov function (3) is not greater than zero, from which we can obtain:

$$v_k(t) = K_k f_k\left(itr([P_{\eta\gamma}, H_{k\eta\gamma}]\rho)\right) \tag{7}$$

where K_k are constants and $K_k > 0$, $k = 1,\cdots,r$, $H_{k\eta\gamma} = U_{\eta\gamma}{}^{\dagger} H_k U_{\eta\gamma}$, $U_{\eta\gamma} = \left(|\phi_{1\eta,\gamma}\rangle,\cdots,|\phi_{N\eta,\gamma}\rangle\right)$ and $y_k = f_k(x_k),(k = 1,2,\cdots,r)$ are monotonic increasing functions which are through the coordinate origin of the plane $x_k - y_k$.

In fact, LaSalle invariant principle can only guarantee the control system to converge to the invariant set, but not guarantee to converge to the target state. In order to make the control system converge to the target state, we still need to deal with another problem: the number of critical states in the invariant set, i.e., the number of the states which satisfy $\dot{V}(t) = 0$. For a closed quantum system, only when the target state commutes with the internal Hamiltonian, the number of critical states in the invariant set is at most $N!$. There are un-numerical critical states in the invariant set when the target does not commute with the internal Hamiltonian. This problem can be solved in two ways: one is to make the un-numerical critical states in the invariant set become numerical ones by introducing a set of constant values η_k into the control laws; another is to make the target state be the minimum value of the Lyapunov function (3) by designing the imaginary mechanical quantity .

The control laws (2) designed by (6) and (7) can only guarantee $\dot{V}(t) \le 0$. In order to ensure $\dot{V}(t) < 0$, we provide another condition

$$V(\rho_f) < V(\rho_{other}) \tag{8}$$

which means the value of the Lyapunov function at the target state is less than the values of Lyapunov function at all other states.

The role of $P_{\eta\gamma}$ in (6) is to make the control system converge to the target state ρ_f. In order to do so, on one hand, we need to design $P_{\eta\gamma}$ to make the condition (8) hold, where ρ_{other} represents any other critical states in the invariant set except the target state. On the other hand, the condition $[P_{\eta\gamma}, H_{0\eta\gamma}] = 0$ must hold, which means that $P_{\eta\gamma}$ and $H_{0\eta\gamma}$ have the same eigenstates $|\phi_{1\eta,\gamma}\rangle,\cdots,|\phi_{N\eta,\gamma}\rangle$. We design the eigenvalues of $P_{\eta\gamma}$ to be constant, denoted by $P_1, P_2,\cdots,P_N$, and design $P_{\eta\gamma}$ as

$$P_{\eta\gamma} = \sum_{j=1}^{N} P_j |\phi_{j\eta,\gamma}\rangle\langle\phi_{j\eta,\gamma}| \tag{9}$$

In order to make (8) hold, we design P_j as follows: If $(\rho_{f\eta})_{ii} < (\rho_{f\eta})_{jj}, 1 \le i,j \le N$, design $P_i > P_j$;

if $(\rho_{f\eta})_{ii} = (\rho_{f\eta})_{jj}, 1 \le i,j \le N$, design $P_i \neq P_j$; else if $(\rho_{f\eta})_{ii} > (\rho_{f\eta})_{jj}, 1 \le i,j \le N$, design $P_i < P_j$,

then $V(\rho_{f\eta}) < V(\rho_{other})$ holds, where $(\rho_{f\eta})_{ii}$ is the (i,i)-th element of $\rho_{f\eta} = U_\eta^{\dagger}\rho_f U_\eta$; $U_\eta = (|\phi_{1,\eta}\rangle,\cdots,|\phi_{N,\eta}\rangle)$; $|\phi_{1,\eta}\rangle,\cdots,|\phi_{N,\eta}\rangle$ are the eigenstates of $H_{0\eta} = H_0 + \sum_{k=1}^{r} H_k \eta_k$.

For the above deduction, refer to the proof of Theorem 2 in [24].

Based on LaSalle's invariance principle, the convergence of the control system with above control laws designed by $u_k(t) = \gamma_k(t) + v_k(t) + \eta_k$ in (2), we proven the following theorem: Consider the control system (1) and the constructed Lyapunov function (3), under the action of control laws (2), in which $\gamma_k(t)$ is designed by (6); $v_k(t)$ is designed by (7); η_k is used to make (4) hold, $P_{\eta\gamma}$ is designed as (9), which can make the control system satisfy:

i) $\omega_{l,m,\eta\gamma} \neq \omega_{i,j,\eta\gamma}$, $(l,m) \neq (i,j)$, $i,j,l,m \in \{1,2,\cdots,N\}$, $\omega_{l,m,\gamma} = \lambda_{l,\eta\gamma} - \lambda_{m,\eta\gamma}$, where $\lambda_{l,\eta\gamma}$ is the l-th eigenvalue of $H_{0\eta\gamma} = H_0 + \sum_{k=1}^{r} H_k(\eta_k + \gamma_k(t))$ corresponding to the eigenstate $\left|\phi_{l,\eta\gamma}\right\rangle$;

ii) $\forall j \neq l$, for $k = 1,\cdots,r$, there exists at least a $(H_{k\eta\gamma})_{jl} \neq 0$, where $(H_{k\eta\gamma})_{jl}$ is the (j,l)-th element of $H_{k\eta\gamma} = U_{\eta\gamma}{}^{\dagger} H_k U_{\eta\gamma}$ with $U_{\eta\gamma} = \left(\left|\phi_{1,\eta\gamma}\right\rangle,\cdots,\left|\phi_{N,\eta\gamma}\right\rangle\right)$;

iii) $[P_{\eta\gamma}, H_{0\eta\gamma}] = 0$; For any $l \neq j, (1 \le l, j \le N)$, $(P_{\eta\gamma})_{ll} \neq (P_{\eta\gamma})jj$ holds, where $(P_{\eta\gamma})_{ll}$ is the (l,l)-th element of $P_{\eta\gamma}$, and the control system will converge toward the invariant set E:

$$E = \left\{\rho_\gamma(t_0) \middle| \left(U_{\eta\gamma}{}^{\dagger}\rho_\gamma(t_0)U_{\eta\gamma}\right)_{ij} = 0, \gamma = \gamma(\rho_\gamma(t_0)), t_0 \in R\right\} \tag{10}$$

The proof of the theorem is similar to the proof in [24], and we will not repeat it here.

By designing the control laws proposed in this paper, a quantum system in degenerate case can become an ideal quantum system, which satisfies three convergent conditions of state transfer. In fact, the proposed control designed method in this paper is also suitable for the state transfer of ideal quantum systems, so up to now we establish a complete Lyapunov - based closed quantum control theory, which is not only suitable for quantum systems in non-degenerate, but also suitable for the quantum systems in degenerate cases.

III. NUMERICAL SIMULATION

In this section, we perform an experiment to design a specific controller to transfer a state to a superposition state by using the implicit Lyapunov control based on the average value of an imaginary mechanical quantity.

Consider a 3-level quantum system, whose internal Hamiltonian is non-strong regular, and the control Hamiltonians are not full connected:

$$H_0 = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{11}$$

In the numerical simulation experiment, the initial state ρ_0 is a mixed state which does not commute with the internal Hamiltonian and the target ρ_f is a mixed state which commutes with the internal Hamiltonian:

$$\rho_0 = \begin{bmatrix} 0.1 & 0.1 & 0.04 \\ 0.1 & 0.5 & 0.08 \\ 0.04 & 0.08 & 0.4 \end{bmatrix} \qquad \rho_f = diag\{0.5687, 0.3562, 0.075\} \tag{12}$$

According to the design ideas proposed in this paper, the control laws are $u_1(t) = \gamma_1(t) + v_1(t)$, in which $v_1(t)$ is designed as:

$$v_1(t) = K_1\left(itr([P_{\gamma_1}, H_1]\rho)\right) \tag{13}$$

in which K_1 is the gain of $v_1(t)$, and $K_1 > 0$.

The implicit function $\gamma_1(t)$ is designed as:

$$\gamma_1(\rho) = M_1 \cdot (tr(P_{\gamma_1}\rho) - tr(P_{\gamma_1}\rho_f)) \tag{14}$$

where M_1 is the gain of $\gamma_1(t)$, and $M_1 > 0$.

According to the design method of the imaginary mechanical quantity in (9), design the eigenvalues of $P_{\gamma 1}$ are:

$$P_1 < P_2 < P_3 , \quad P_{\gamma_1} = \sum_{j=1}^{3} P_j \left|\phi_{j,\gamma_1}\right\rangle \tag{15}$$

where $\left|\phi_{j,\gamma_1}\right\rangle$ is the eigenstates of $H_0 + \sum_{k=1}^{r} H_k \gamma_k(t)$.

In the simulation experiment, the simulation step is set to be 0.01 a.u., and control duration is 300 a.u.. The parameters used in experiment are: $M_1 = 0.1$, $K_1 = 0.34$, $P_1 = 0.01$, $P_2 = 2$ and $P_3 = 2.9$. The results of numerical simulating experiments are shown in Fig. 3 and Fig. 4. Fig. 3 represents the evolution curves of density metrics, in which ρ_{ii} is the diagonal elements of ρ . Fig. 4 shows the control curves of the $\gamma_1(t), v_1(t)$ and $u_1(t)$.

From Fig. 3 and Fig. 4, one can see that at the time 300 a.u., $\rho_{11} = 0.56811$, $\rho_{22} = 0.35215$, $\rho_{33} = 0.07973$, and transfer probability is 99.53% , which verifies the effectiveness of the proposed method in this paper.

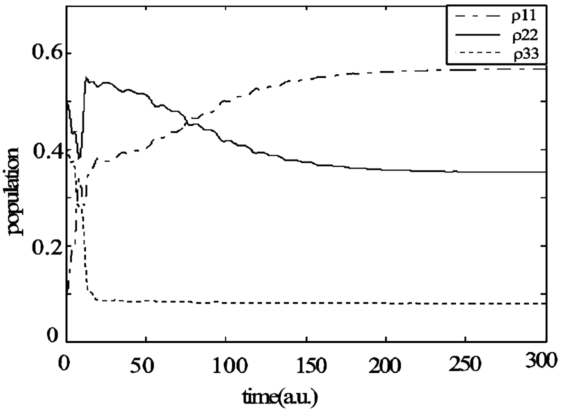

Figure 3 Evolution curves of density metrics ρ_{ii} .

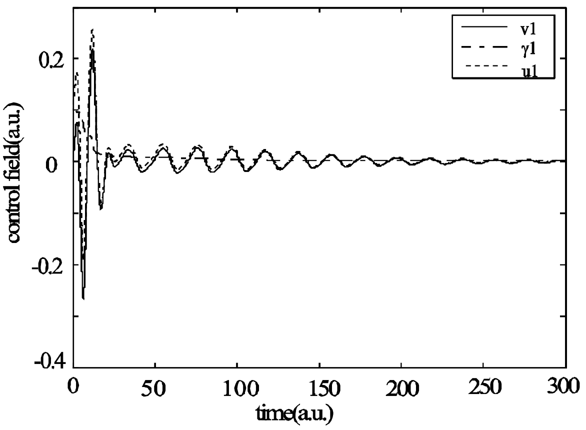

Figure 4 Control fields of the control system.

IV. CONCLUSION

This paper proposed a complete design procedure of control laws for closed quantum systems in degenerate cases. The proposed control design method is also suitable for ideal quantum systems. Based on the Lyapunov-based control theory of quantum systems proposed in this paper, the state transfer task of closed quantum systems from arbitrary initial state to arbitrary final state can be completed, and the Lyapunov-based control theory of closed quantum systems has been established.

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REFERENCES

[1] C. Altafini, "Coherent control of open quantum dynamical systems", *Physical Review A,* 70, pp. 062321-1-062321-8 , 2004

[2] W. S. Warren, H. Rabitz, and M. Dahleh, " Coherent control of quantum dynamics: The dream is alive", *Science*, 259, pp. 1581-1589 , 1993

[3] S. Cong and Y. Lou, "Coherent control of Spin 1/2 quantum systems using phases", *Control Theory and Applications*, vol. 25, Feb. 2008, 187-192

[4] W. C. Cao, X. S. Liu, and H. B. Bai, *Science in China Series G:* Physics, Mechanics *&* Astronomy, vol. 51, Jan. 2008, pp. 29-37.

[5] S. Cong, J. Wen, and X. Zou. "Comparison of time optimal control for two level quantum systems *", Journal of Systems Engineering and Electronics*, vol. 15, Jan. 2014, pp. 95-103.

[6] G. S. Uhrig, "Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences". *Physical Review Letters,* vol. 98, pp. 100504-1 - 100504-4. 2007

[7] L. Viola, E. Knill and S. Lloyd, "Dynamical Decoupling of Open Quantum Systems", *Physical Review Letters*, vol. 82, Dec. 1999, pp. 2417-2421.

[8] S. Cong, L. Chan, and J. Liu, "An Optimized Dynamical Decoupling Strategy To Suppress Decoherence", *International Journal of Quantum Information*, vol. 9, pp. 1599-1615 , 2011

[9] D. Dong and I. R. Petersen, "Sliding mode control of quantum systems", *New Journal of Physics*, vol. 11, pp. 105033-1 -105033-18, 2009

[10] R J. Matthew, "Control Theory: From Classical to Quantum Optimal, Stochastic, and Robust Control", *Notes for Quantum Control Summer School*, Caltech, August, 2005.

[11] D. D'Alessandro, "The optimal control problem on SO(4) and its applications to quantum control", *IEEE Transaction on Automatic Control*, vol. 47, Jan. 2002, pp. 87–92.

[12] N. Khaneja, R. Brockett, and S. J. Glaser, "Time optimal control in spin systems", *Physical Review A*, vol. 63, March 2001, pp. 032308-1-032308-13.

[13] J. P. Palao and R. Kosloff, "Optimal control theory for unitary transformations", Phys. Rev. A, 2003, 68, 062308.

[14] S. Grivopoulos, *Optimal control of quantum systems.* Santa Barbara: University of California, 2005.

[15] S. Grivopoulos and B. Bamieh, "Lyapunov-based control of quantum systems", Proceedings of the 42nd IEEE Conference on Decision and Control, 2003, Maui, USA: IEEE, pp. 434−438.

[16] S. Kuang and S. Cong. "Lyapunov control methods of closed quantum systems". *Automatica*, vol. 44, Jan. 2008, pp. 98-108.

[17] M. Mirrahimi(a), P. Rouchon, and G. Turinici. "Lyapunov control of bilinear Schrödinger equations". *Automatica*, vol. 41, pp. 1987-1994. 2005

[18] X. Wang and S. Schirmer, "Analysis of Effectiveness of Lyapunov Control for Non-generic Quantum States", *IEEE Transactions on Automation Control*, vol. 55, June 2010, pp. 1406-1411.

[19] C. Altafini. "Feedback stabilization of isospectral control systems on complex flag manifolds: application to quantum ensembles". *IEEE Transactions on Automatic Control*, vol 52, Nov. 2007, pp. 2019-2028.

[20] A. C. Doherty and K. Jacobs, *"Feedback control of quantum systems using continuous state estimation*", *Physical Review A*, 60, pp. 2700-2711, 1999

[21] S. Lloyd and J. J. E. Slotine. "Quantum feedback with weak measurements", Phys. Rev. A, vol. 62, pp. 012307-012311 , 2000

[22] S. W. Zhao, H. Lin, J. T. Sun, and Z. G. Xue. "An implicit Lyapunov control for finite-dimensional closed quantum systems". *International Journal of Robust and Nonlinear control,* vol. 22, Issue 11, pp. 1212-1228., 2012

[23] F. Meng, S. Cong, and S. Kuang, "Implicit Lyapunov control of multi-control Hamiltonian systems based on state distance", *Proceedings of the 10th World Congress on Intelligent Control and Automation* (*WCICA),* pp. 5127-5132, 2012

[24] S. Cong, F. Meng, and S. Kuang, "Quantum Lyapunov control based on the average value of an imaginary mechanical quantity*", Proceedings of the 19th World Congress The International Federation of Automatic Control (IFAC)*, Cape Town, South Africa, pp. 9991-9997, 2014

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