Applying Fuzzy weights to Triple Inner Dependence AHP

Shin-ichi Ohnishi Takahiro Yamanoi

Faculty of Engineering
Hokkai-Gakuen University
Sapporo, Japan

email: {ohnishi, yamanoi}@hgu.jp

Abstract **- Analytic Hierarchy Process (AHP) is major method for decision making, and inner dependence AHP is used for cases in which criteria or/and alternatives are not independent enough. Using the original AHP or inner dependence AHP may cause results that cannot have enough reliability because of the inconsistency of the comparison matrix. In such cases, fuzzy representation for weighting criteria or/and alternatives using results from sensitivity analysis is useful. In the previous papers, we defined local weights of criteria and alternatives and overall weights for double inner dependence AHP (among criteria and among alternatives, respectively) via fuzzy sets. In this paper, we extend these weighs to those of triple inner dependence structure AHP (among 2 levels of criteria and among alternatives, respectively).**

Keywords - AHP; fuzzy sets; sensitivity analysis.

I. INTRODUCTION

The Analytic Hierarchy Process (AHP) proposed by T.L. Saaty in 1977 [1] is widely used in decision making, because it reflects humans feelings naturally. A normal AHP assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. The inner dependence method AHP [3] is used to solve this problem even for criteria or alternatives having dependence.

A comparison matrix may not have enough consistency when AHP is used because, for instance, a problem may contain too many criteria or alternatives for decision making, meaning that answers from decision-makers, i.e., comparison matrix components, do not have enough reliability, they are too ambiguous or too fuzzy [3]. To avoid this problem, we usually have to revise again or abandon the data, but it takes a lot of time and it is expensive [1][2].

Then, we consider that weights should also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights using fuzzy sets. Our research first applied sensitivity analysis to inner dependence AHP to analyze how much the components of a pairwise comparison matrix influence the weights and consistency of a matrix [4]. This may enable us to show the magnitude of fuzziness in weights. We previously proposed new representation for criteria and alternatives weights for inner dependence, as L-R fuzzy numbers [5]. In the next step, we address the double inner dependence structure [7]. Then, we consider composition of weights to obtain over all alternative weights for double inner dependence structure, using results from sensitivity analysis and fuzzy operations. At last, we apply these fuzzy weighs to a result of triple inner dependence (among 2 levels of criteria and among alternatives respectively) when comparison matrices in all levels do not have enough consistency.

In Sections 2 and 3, we introduce the inner dependence AHP, consistency index, and sensitivity analyses for AHP. Then, in Section 4, we define fuzzy weights, and Section 5 is a summary.

II. CONSISTENCY AND INNER DEPENDENCE

A. Process of Normal AHP

(Process 1) Representation of structure by a hierarchy. The problem under consideration can be represented in a hierarchical structure. At the middle levels, there are multiple criteria. Alternative elements are put at the lowest level of the hierarchy.

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is created from a decision maker's answers. Let n be the number of elements at a certain level, the upper triangular components of the matrix a_{ij} ($i< j = 1,\dots,n$) are 9, 8, .. , 2, 1, 1/2, …, or 1/9. These denote intensities of importance from element i to j. The lower triangular components a_{ji} are described with reciprocal numbers, for diagonal elements, let $a_{ii} = 1$.

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grades of importance among each element, are calculated from the pairwise comparison matrix. The eigenvector that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in the paper.

(Process 4) Priority of an alternative by a composition of weights. With repetition of composition of weights, the overall weights of the alternative, which are the priorities of the alternatives with respect to the overall objective, are finally found**.**

B. Consistency

Since components of the comparison matrix are obtained by comparisons between two elements, coherent consistency is not guaranteed. In AHP, the consistency of the comparison matrix A is measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{1}$$

where n is the order of comparison matrix A, and λ_A is its maximum eigenvalue (Frobenius root).

If the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. The comparison matrix is consistent if the following holds.

$$\text{C.I.} \le 0.1 \tag{2}$$

C. Inner Dependence Structure

The normal AHP ordinarily assumes independence among criteria and alternatives, although it is difficult to choose enough independent elements. The dependency means some kind of interaction among the elements. Inner dependence AHP [2] is used to solve this type of problem even for criteria or alternatives having dependence.

In the method, using a dependency matrix $F=\{f_{ij}\}$, we can calculate modified weights $\boldsymbol{w}^{(n)}$ as follows,

$$\boldsymbol{w}^{(n)} = F\boldsymbol{w} \tag{3}$$

where $\boldsymbol{w}$ represents weights from independent criteria or alternatives, i.e., normal weights of normal AHP and dependency matrix F is consist of eigenvectors of influence matrices showing dependency among criteria or alternatives.

If there is dependence in both lower levels, i.e., not only among criteria but also among alternatives, we call such kind of structure "double inner dependence". In the double inner dependence structure, we have to calculate modified weights of criteria and alternatives, $\boldsymbol{w}^{(n)}$ and $\boldsymbol{u}_i^{(n)}$. Then we composite these 2 modified weights to obtain overall weights of alternative k, $v_k^{(n)}$ as follow:

$$v_k^{(n)} = \sum_{i}^{m} w_i^{(n)} u_{ik}^{(n)} \tag{4}$$

where m is the number of criteria.

Also, using the same steps again, we can composite weights of "triple inner dependence" structure, in the case when there is dependency in the 3 lower levels, i.e., not only among alternatives and 1 level criteria but also 2 levels of criteria.

III. SENSITIVITY ANALYSES

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence its consistency or the weights. In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change the structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds. From Perron-Frobenius theorem, the following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let $A = (a_{ij})$, $(i, j = 1,\dots, n)$ denote a comparison matrix and let $A(\varepsilon) = A + \varepsilon D_A$, $D_A=(a_{ij}d_{ij})$ denote a matrix that has been perturbed. Let λ_A be the Frobenius root of A, $\boldsymbol{w}$ be the eigenvector corresponding to λ_A, and $\boldsymbol{v}$ be the eigenvector corresponding to the Frobenius root of A'. Then, a Frobenius root $\lambda(\varepsilon)$ of $A(\varepsilon)$ and a corresponding eigenvector $\boldsymbol{w}(\varepsilon)$ can be expressed as follows*

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{5}$$

$$\boldsymbol{w}(\varepsilon) = \boldsymbol{w} + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{6}$$

where

$$\lambda^{(1)} = \frac{\boldsymbol{v}' D_A \boldsymbol{w}}{\boldsymbol{v}' \boldsymbol{w}}, \tag{7}$$

$\boldsymbol{w}^{(1)}$ is an n-dimension vector that satisfies

$$(A - \lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}\ , \tag{8}$$

where $\boldsymbol{o}(\varepsilon)$ denotes an n-dimension vector in which all components are $o(\varepsilon)$.

About a fluctuation of the consistency index, the following corollaries hold.

Corollary 1 *Using appropriate g_{ij}, we can represent the consistency index C.I.(ε) of the perturbed comparison matrix $A(\varepsilon)$ as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_{i}^{n}\sum_{j}^{n} g_{ij} d_{ij} + o(\varepsilon). \tag{9}$$

To see g_{ij} in the equation (9) in Corollary 1, we can know how the components of a comparison matrix impart influence on its consistency.

Corollary 2 *Using appropriate $h_{ij}^{(k)}$, we can represent the fluctuation $\boldsymbol{w}^{(1)}=(w_k^{(1)})$ of the weight (i.e., the eigenvector corresponding to the Frobenius root) as follows*

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \qquad (10)$$

Then, we can evaluate how the components of a comparison matrix impart influence on the weights, to see $h_{ij}^{(k)}$ in the equation (10).

Proofs of these corollaries are shown in [4].

IV. Fuzzy Weights Representations

When a comparison matrix has poor consistency (i.e., 0.1<C.I.<0.2), comparison matrix components are considered to be fuzzy because they are results from human fuzzy judgment. Weights should therefore be treated as fuzzy numbers [5][6].

Definition 1 (fuzzy weight) Let $w_k^{(n)}$ be a crisp weight of criterion or alternative k of inner dependence model, and g_{ij} | $h_{ij}^{(k)}$| denote the coefficients found in Corollary 1 and 2. If 0.1<C.I.<0.2, then a fuzzy weight $\tilde{w}_k$ is defined by

$$\tilde{w}_k = (w_k, \alpha_k, \beta_k)_{LR} \qquad (11)$$

$$\alpha_k = \mathrm{C.I.}\sum_i^n \sum_j^n s(-, h_{kij}) g_{ij} \mid h_{kij} \mid, \qquad (12)$$

$$\beta_k = \mathrm{C.I.}\sum_i^n \sum_j^n s(+, h_{kij}) g_{ij} \mid h_{kij} \mid, \qquad (13)$$

For double inner dependence structure, we can define and calculate modified fuzzy local weights of a criteria $\tilde{\boldsymbol{w}}^{(n)} = (\tilde{w}_i^{(n)})$, $i = 1, \ldots, n$ and also weights of alternatives $\tilde{\boldsymbol{u}}_i^{(n)} = (\tilde{u}_{ik}^{(n)})$, $k = 1, \ldots, m$ with only respect to criterion i using an dependence matrix F_C, F_A, as follows

$$\boldsymbol{w}^{(n)} = (w_i^{(n)}) = F_C \boldsymbol{w} \qquad (14)$$

$$\boldsymbol{u}_i^{(n)} = (u_{ik}^{(n)}) = F_A \boldsymbol{u}_i \qquad (15)$$

$\boldsymbol{w}$ is crisp weights of criteria and $\boldsymbol{u}_i$ is crisp local alternative weights with only respect to criterion *i*. $\alpha_i, \beta_i, \alpha_{ik}, \beta_{ik}$ are calculated by fuzzy multiple operations, using (3) and Definition 1.

For triple inner dependence structure, we can also define overall weights of alternatives $\tilde{y}_k^{(n)}$, as follows:

$$\tilde{y}_k^{(n)} = \sum_j^l x_i^{(n)} \tilde{v}_{jk}^{(n)} \qquad (16)$$

V. Conclusions

There are many cases in which data of AHP does not have enough reliability. We show the possibility to apply fuzzy weight representation to triple inner dependence. Our approach can show how to represent weights and will be efficient to investigate how the result of AHP has fuzziness when data is not sufficiently consistent.

References

[1] T. L. Saaty, The Analytic Hierarchy Process. McGraw-Hill, New York, 1980.

[2] T. L. Saaty, Inner and Outer Dependence in AHP, University of Pittsburgh, 1991

[3] S. Ohnishi, D. Dubois, H. Prade, and T. Yamanoi, "A Fuzzy Constraint-based Approach to the Analytic Hierarchy Process," Uncertainty and Intelligent Information Systems, June 2008, pp.217-228.

[4] S. Ohnishi, H. Imai, and M. Kawaguchi, "Evaluation of a Stability on Weights of Fuzzy Analytic Hierarchy Process using a sensitivity analysis," J. Japan Soc. for Fuzzy Theory and Sys., 9(1), Jan. 1997, pp.140-147.

[5] S. Ohnishi, T. Yamanoi, and H. Imai, "A Fuzzy Weight Representation for Inner Dependence AHP," Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol.15, No.3, June 2011, pp. 329-335.

[6] S. Ohnishi, T. Furukawa, and T. Yamanoi, " Compositions of Fuzzy Weights for Double Inner Dependence AHP," COGNITIVE2013, May 2013, pp. 83-86.

 ISBN: 978-1-61208-408-4