Metacognitive Support of Mathematical Abstraction Processes

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***Abstract*—A significant and distinctive feature of human beings is the ability of performing abstraction operations, e.g., when forming categories of objects or even consciouly creating abstract objects as it is typical in mathematics. Although the possible range of corresponding abilities is certainly pre-determined by individual genetic factors, a high-level abstraction performance will typically be achieved gradually by an intensive practice in solving abstraction prone problems. On the other hand, mathematical abstraction is often considered to be a serious obstacle in mathematics education. This raises the question whether there are some basic principles of abstraction that could be taught on a metacognitive level in order to support the progress in abstraction abilities. The paper presents a concept of a corresponding teaching experiment. We hope it will provide more effective teaching as well as a better understanding of cognitive processes underlying mathematical abstraction.**

Keywords*–*Abstraction; Mathematics Education; Metacognition.

I. Introduction

Basic mathematics courses belong to the greatest challenges for first year university students from many many disciplines. The author's long run experience in conducting such courses at the University of Paderborn indicates that one main reason for that is the lack of appropriate study and working techniques. As a remedy, we created a system of in-teaching *metacognitive* support instruments [1] by means of which first improvements could already be achieved [2] [3]. Even with this, we often see refusal or even fear of the perceived *abstractness* of mathematics. Moreover, many of the beginning students are quite unfamiliar with any kind of abstractness. Hence, coping with mathematics becomes particularly hard for them. This raises the question how to facilitate the "access to abstraction" for them.

It is impossible to rise this question without referring to the aspect of time, because good abstraction abilities are typically achieved "by doing", i.e., by solving problems that require – or at least promote – a certain level of abstraction. Even mathematicians develop their abstraction skills within a lengthy process of education and mathematical work. However, in our basic courses for non-mathematicians, there is not enough time to re-run along this path. As an alternative, we propose to support some basic aspects of abstraction on a *metacognitive* level, by explicitly "teaching abstraction principles", with the objective to accelerate the process of acquiring abstraction skills. In order to derive such rules, we discuss several aspects of abstraction. A generally adopted hypothesis is that abstraction operations are organized hierarchically. Piaget [4] has described that, and how, this hierarchy is run through in children's development of mathematical thinking. The hierarchical nature of abstraction was also emphasized by Dubinsky [5], [6] and Arnon et al. [7]. In contrast to the forementioned ones the approach pursued here aims to additionally support the construction of several layers of abstraction by *explicit metacognitive instruction*. Although this work is still in an early stage, we hope that it shall yield not only better teaching instruments but a better understanding of the underlying cognitive processes as well.

The paper is organized as follows: In Section II, we highlight the need of abstraction in economics education. The nature of abstraction and its "economics" is discussed in Sections III, and IV. The following section deals with operational aspects of abstraction. Section VI gives an outlook of a forthcoming teaching project and possible applications of the results.

II. Is Abstraction Educationally Needed?

It is often believed that abstraction is a matter of "pure mathematics" rather than of its applications. However, practically this is not true. Especially in economics, there is a particular demand of "abstraction" at least along three different lines. First, fundamental economic phenomena are explained with the help of abstract mathematical concepts. Look, e.g., at a preference relation as described here:

$$\underline{x} \preceq \underline{y} \quad :\Longleftrightarrow \quad 2x_1 + 3x_2 \leq 2y_1 + 3y_2. \tag{1}$$

The students must be able to read, understand, and handle symbolic expressions like this. Second, modern economics is interested in qualitative results that are valid under quite general assumptions. Accordingly, these results rely on abstract qualitative properties of the underlying models. And third, by employing modern and sophisticated results of mathematics, economics adopt the abstraction level of mathematics itself. This confirms that Devlin's [8] statement "The main benefit of learning and doing mathematics is not the specific content; rather it's the fact that it develops the ability to reason precisely and analytically about formally defined abstract structures" holds true for modern economics, as well as for other sciences.

III. What is Abstraction?

So far, "abstraction" was used in quite general way. For the purposes of this paper, we shall describe some specific aspects of interest and put them in the general context.

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A. General Aspects

Everybody knows somehow and from somewhere "what is abstraction", as this word became present in a lot of domains within the last two centuries. A common feature of many conceptions of "abstraction" refers to the latin word *abstrahere* in the philosophical sense of omitting unessential details of an object in the process of inductive thinking, resulting in a new – or simpler – entity, as it was described first by Aristoteles. The large number of publications on this subject indicates that "abstraction" is a rather rich and complex notion. It is neither possible nor the purpose of this paper to give a full account to all essential aspects of it. Rather we shall concentrate on some aspects that may be essential both from the cognitive point of view and for teaching mathematics.

B. Abstraction as Mental Processes

Henceforth, we shall use "abstraction" in the narrow sense to denote individual mental processes. Typically, these processes result in *abstract objects* or, more precisely, in new – and simpler – mental representations of previously present mental objects or their relations, respectively, or even in the creation of new mental objects. In particular, abstraction can lead to a re-structured organization of mental knowledge structures (Hershkowitz et al. [9]). In a wide sense, we understand "abstraction" also as mental processes of understanding and exploiting already existing abstract objects and concepts.

It is obvious that abstraction plays a prominent role in those brain domains that are responsible for conscious thinking and human language processing, but it is also quite reasonable to assume that abstraction mechanisms already work in more basic layers of the brain's functional architecture, in particular, when processing sensomotoric informations. Here, one of the most basic operations is visual pattern recognition, possibly followed by identifying simultaneously occuring similar patterns. The occurence of patterns – or patterns of patterns – is processed further by higher cognitive layers. A particular task of these higher layers is to define categories of perceived objects, like "animal", "cat" vs. "dog", etc. This task is highly abstractive as it requires to detect essential common features and to neglect non-essential features of the objects. Note that whether some features are "essential" or not depends on the underlying cognitive purpose. A further abstraction step is performed by creating category labels, and yet another by handling category labels instead of a variety of objects itself. From there, a much higher level of abstraction is achieved by including structural relations between categories or labels, respectively. Altogether, it appears that abstraction processes are organized within a complex architecture that mirrors the functional brain architecture itself.

C. Mathematical Abstraction

When talking about mathematical abstraction we confine ourselves to abstraction processes connected with "understanding mathematics" or "doing mathematics", respectively. This means that the objects of cognition themselves are representations of mathematical objects or relations. An early attempt to give a formal description of mathematical abstraction is due to Rinkens [10], where abstraction is understood as a (non-injective) mapping. Here, we have to be more specific w.r.t. the teaching objectives. We want to distinguish between *receptive*, *applicative* and *creative* abstraction. By *receptive* abstraction we refer to individual brain activities that provide "understanding" of abstract concepts that have been defined beforehand by other individuals. To the opposite, *creative* abstraction is concerned with the construction of new mental representations without external inspiration. *Applied* abstraction means to employ abstract objects and relations, regardless whether these have been created by other individuals or not. Accordingly, enhancing receptive abstraction is the primary concern of teaching, where active and creative abstraction play an important role in problem solving, which comes into the focus in the advanced stages of teaching.

Although complex, there are some particular aspects of abstraction that can be isolated. We shall consider the following activities as basic aspects of abstraction:

- *encapsulation:*
  i.e., to see a number of objects as a whole entitiy, e.g., to see
  $$e^{\frac{4x^2}{23x+17}} \quad \text{as} \quad e^{\boxed{\text{something}}} \tag{2}$$
- *symbolization:*
  i.e., introducing abstract referents (indices) for patterns like expressions, relations, statements etc.; e.g.,
  $$e^{\frac{4x^2}{23x+17}} = e^{\boxed{\text{a}}}, \tag{3}$$
- *analogization:*
  i.e., identifying common features in different objects or domains and creating a new object out of them, e.g., identifying the common property of squares, rectangles, rhombus, etc., as being a quadrangle
- *class formation:*
  i.e., encapsulation of a number of analogized objects, e.g., forming the class (or set) of quadrangles
- *structural synthesis:*
  e.g., grouping separate objects x and y to a pair (x, y) being considered as a new object

The following activities work upon a certain stock of pre-established abstract objects:

- *object embedding:*
  i.e., seeing a particular object as an element of an appropriate category (set) in order to use category properties rather than individual properties, e.g., as here:
  $$e^{\frac{4x^2}{23x+17}} = e^{\varphi(x)} \tag{4}$$
  In the example, the left hand superscript expression is interpreted as evaluation of some differentiable function φ; hence, results for the whole class can be applied (e.g., the chain rule of differentiation).
- *switching embedding levels:*
  i.e., embedding/outbedding in nested structures; e.g., the changes of focus between a set and its elements
- *structure-object interchange:*
  that is, rendering structures, i.e., relations between different objects, to encapsulated objects of consideration
- *recursion:*
  i.e., establishing recursive structures within problems or within problem solving strategies; e.g., when trying to simplify the expression
  $$A \cap (B \cup (A \cap (B \cup (A \cap (B \cup (A \cap B)))))) \tag{5}$$

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This enumeration is by no means complete, but may suffice for the purpose of this paper.

IV. The Economics of Abstraction

As already mentioned, a significant feature of creative abstraction is to omit "unessential" details of the object under consideration. However, what is "unessential" can vary heavily with the underlying cognitive task. This can be observed in a veriety of domains and is particularly true in mathematics. For example, the set of the real numbers, equipped with the usual addition and multiplication, represents different abstract objects at the same time, e.g., a vector space, a ring, a field, etc. Which property is "essential" clearly depends on the problem under consideration. Typically, the choice of the appropriate abstraction will ease the solution of a problem – the problem can be solved with less mental effort, within less time, with deeper insight in its nature, etc. Sometimes, it is even impossible to solve a given problem without appropriate abstraction. So far, this phenomenon is clearly a social experience of the mathematical community, but on the other hand, it can be re-experienced by each individual that deals with mathematical problems. Hence, our hypothesis is: *A latent aversion against abstraction can be reduced by the individual experience of "economic benefits" when using abstraction.*

V. Operational Aspects of Abstraction

For the purposes of the project, we have to confine ourselves to selected aspects of abstraction. Our selection takes into account the needs of abstraction within our math course as described above, the degree of operationability, and the degree of observability. Recall that we want to support problem understanding and solving processes with the help of *metacognitive* abstraction rules. These can be understood as rules that guide and structure the *working process* rather than providing particular abstraction results. From this point of view, we shall concentrate on such aspects of abstraction that appear to be in reach of such metacognitive rules. Examples of such aspects are

- encapsulation/analogization/symbolization
- structuring
- recursion techniques and
- qualitative reasoning.

To illustrate the idea of abstraction meta-rules suppose that the student's problem under consideration is given by some text, formula or so, henceforth called the *document*. The first of the forementioned abstraction aspects is closely related to the visual input. Hence, we support it by the following meta-rules:

(a) *Provide a clear* visual *organization of the document.*
(b) *Identify large substructures.*
If appropriate *put them into containers/symbolize them.*
(c) *Identify similar patterns.*
If appropriate *symbolize them.*
(d) *Identify repetition indicators w.r.t. tasks / structures.*
Try to use *one* solution for all repeated tasks and *one* principle to work with repeated structures.

For example, consider this task for students:

Task 1: *Determine the operating minimum, given the following cost functions:* 1) $K_1(x) := 4x^2 + 15\ x + 42, x \geq 0$, 2) $K_2(x) := 242x^2 + 72x + 117, x \geq 0$, ... 5) $K_5(x) := 25x^2 + 5x + 242, x \geq 0$.

Obviously, there are at least three different levels of abstraction on which this task could be fulfilled. We call the least one *level*

(0) Without any experience in abstraction-aided working, the students would tend to solve each of the problems 1 to 5 individually, using only numerical computations. This would imply to perform the corresponding ansatzes and solving techniques altogether five times, and probably some of the students would try to facilitate the computation somehow "on the way".

We claim that by respecting the above rules progress to a higher abstraction level could be promoted. Indeed, a better visual organization of the task according to rule (a) might already change the document as follows:

Task 1: *Determine the operating minimum, given the following cost functions:*

1.	$K_1(x) := \quad 4\ x^2 + 15\ x + \ \ 42,\ x \geq 0$
2.	$K_2(x) := 242\ x^2 + 72\ x + 117,\ x \geq 0$
...	...
5.	$K_5(x) := \ \ 25\ x^2 + \ \ 5\ x + 242,\ x \geq 0.$

From here, looking both at the five *repetitions* as proposed by rule (d) and at *similar patterns* as proposed by rule (c), the students might more easily see the uniform structure

$$K_{\square}(x) := \square\ x^2 + \square\ x + \square,\ x \geq 0, \tag{6}$$

where the gray boxes symbolize containers with different contents. According to (d), we recommend to find a unified solution from here. Thus, it is appropriate to follow (c) and to symbolize the contents of the boxes as

$$K_{\square}(x) := a\ x^2 + b\ x + c\ x \geq 0. \tag{7}$$

Thus, we reach *abstraction level*

(1) The problem can be solved *at once* in a symbolic manner, yielding a result in terms of the parameters a, b and c. Then, the desired five numerical results can easily be obtained by just plugging in the appropriate numbers.

Note that working on level 1 rather than on level 0 is quite obviously advantageous; it pays in time savings, less error sensitivity, qualitative insights, and also aesthetics. All these advantages can be experienced by the students themselves and they might also stimulate them to try such an approach again, when solving other problems.

Analogous meta-rules can be formulated for structuring and recursion techniques, although there we shall need and exploit additional syntactical guidelines. But what about qualitative reasoning? This refers to abstraction level

(2) This level of abstraction is achieved when referring to general classes of functions that are of economic relevance. The students might observe that *each K is a neoclassic cost function*. Thus, the operating minimum

– as the minimum average variable costs - is nothing but the limit of the average variable costs as $x \downarrow 0$. Now it is quite easy to obtain the same results as above.

Clearly, to step here from level (1) is quite complex and requires a solid theoretical background. It is clear that to work on this level cannot – and shall not – be trained before this solid theoretical background was laid out. But provided this was done, a corresponding meta-rule could be

- *Try to work in economic categories rather than with numeric examples.*

To follow this rule, the students need a very clear overview over the mathematical tools at their disposal. This overview is supported by the toolbox concept as described in [2].

VI. The Project

The forementioned meta-rules can only brought to life by an intense training that shows how to use them and how they can help to re-structure ones own work in order to gain more progress within the same time. We intend to test and to improve corresponding training measures within a voluntary project group. These measures should

- positively change the students' attitude towards abstraction
- increase the aceptance of (at least passive) abstraction
- enhance the ability of active abstraction
- enrich the regular teaching process.

The project group shall be constituted by random choice from a set of voluntary applicants, hence there shall be an untreated control group as well. The only incentive for participating shall be the perspective of being able to cope better with mathematics, but no examen credits shall be promised. Before and after the series of training units we shall perform guideline based interviews as well as observed and videotaped working sessions. Through appropriatelydesigned tasks, it shall be observed whether the students become more apt to understand and use abstract approaches than before. The training sessions shall focus on the different aspects of abstraction, as mentioned above. Task 1 might serve as a possible example: First, before the training starts, the students are asked to solve a task of this kind by their own. Their approaches and solutions are observed and video documented. After that, we introduce the meta-rules and explain how they work in this and other examples. It will be important to address the benefits of using abstract approaches as well. At the end of the training sessions, the students shall be given another set, and again their approaches and solutions are documented. Ideally, there shall be a tendency to work on a (slightly) higher abstraction level as at the beginning of the training.

VII. Conclusion

In large and heterogeneous basic mathematics courses students need support to manage mathematical abstraction. We described some particular aspects of mathematical abstraction that, so our hypothesis, can be trained with the help of metacognitive rules. Some examples of corresponding metacognitive abstraction rules are provided. Further we presented a framework for an appropriate field study in order to investigate the possible effects of a metacognitive-rule based training. Performing such a field study as well as adjusting the training instruments is subject to future work.

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